Concentration phenomena for a FitzHugh-Nagumo neural network

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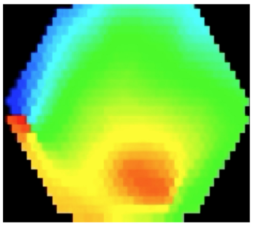
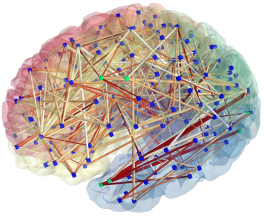
Overview

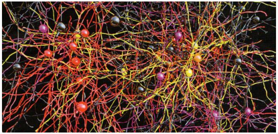
We consider that neural networks are particle systems. It is possible to describe them through 3 distinct scales :



Macroscopic scale



Mesoscopic scale



Microscopic scale

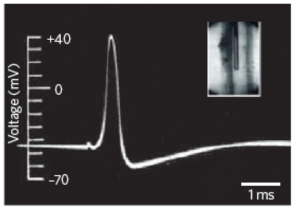
GOAL :

Derive information at the macroscopic scale from the mesoscopic scale

Microscopic scale and mean field limit

Behavior of a neuron

• We focus on the dynamics of the voltage through the membrane of a neuron. Experiments showcase 2 main features



Hodgkin & Huxley, '52.

(i) Delay when responding to an external input.

(ii) Self-regulation.

• Hodgkin & Huxley obtained a precise but mathematically complicated model.

• We will use a simplified version that captures its main features : FitzHugh-Nagumo's model for a neuron

FitzHugh-Nagumo's neuron

A neuron i is modeled by $(v_t^i, w_t^i) \in \mathbb{R}^2$ (voltage & delay)

$$\begin{cases} dv_t^i = \left(N(v_t^i) - w_t^i + I_{\text{ext}}\right) dt + \sqrt{2} dB_t, \\ dw_t^i = A\left(v_t^i, w_t^i\right) dt, \end{cases}$$

- A is an affinity while N is non-linear, typically $N(v) = v - v^3$.
- I_{ext} is the current resulting from interaction with other neurons

$$I_{\text{ext}} = -\frac{1}{n} \sum_{j=1}^{n} \Phi(x_i, x_j)(v^i - v^j),$$

where $x_i \in K \subset \mathbb{R}^d$ is the location of neuron i.

- We obtain the following microscopic model, where $1 \leq i \leq N$

$$\begin{cases} dv_t^i = \left(N(v_t^i) - w_t^i - \frac{1}{n} \sum_{j=1}^{n} \Phi(x_i, x_j)(v_t^i - v_t^j) \right) dt + \sqrt{2} dB_t^i, \\ dw_t^i = A(v_t^i, w_t^i) dt. \end{cases} \tag{1}$$

Mean field limit

It was proved that in the mean field limit $N \to +\infty$ the microscopic system is described by

FitzHugh-Nagumo's mean field equation

$$\begin{cases} \partial_t f = -\partial_v \left(\left(N(v) - w - \mathcal{K}_\Phi(f) \right) f \right) - \partial_w \left(A(v,w) f \right) + \partial_v^2 f, \\ f(0,\cdot) = f_0, \end{cases}$$

where $f(t,x,v,w)$ is the probability distribution of finding neurons with a potential $v \in \mathbb{R}$, adaptation variable $w \in \mathbb{R}$, at time $t \geq 0$ and position $x \in K$ within the network, and $\mathcal{K}_\Phi(f)$ is the non-local term due to interactions between neurons

$$\mathcal{K}_\Phi(f)(x,v) = \int_{K \times \mathbb{R}^2} \Phi(x,x')(v-v') f(x',v',w') dx' dv' dw'.$$

Mean field limit

References

- J. Baladron, D. Fasoli, O. Faugeras and J. Touboul. *Mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons* (2012).
- M. Bossy, O. Faugeras and D. Talay. *Clarification and complement to "mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons"* (2015).
- S. Mischler, C. Quiñinao and J. Touboul. *On a kinetic FitzHugh-Nagumo model of neuronal network* (2015).
- E. Luçon and W. Stannat. *Mean-field limit for disordered diffusions with singular interactions* (2014).
- J. Crevat. *Mean-field limit of a spatially-extended FitzHugh-Nagumo neural network*(2019).

Concentration phenomena

Modeling assumptions

We decompose the interaction kernel Φ as follows

$$\Phi(x, x') = \Psi(x, x') + \frac{1}{\epsilon}\delta_0(x - x'),$$

where
- Ψ is "more regular" and accounts for Weak/Long range interactions,
- dirac mass δ_0 accounts for Strong/Local interactions with strength $\epsilon > 0$.

GOAL :

Study the system in the regime of Strong/Local interactions, that is when $\epsilon \to 0$.

References :

Bressloff (03), Luçon/Stannat (14).

- With this ansatz the equation rewrites

$$\partial_t f^\epsilon = -\,\partial_v \left(\left(N(v) - w - \frac{\rho_0^\epsilon}{\epsilon}(v - \mathcal{V}^\epsilon) - \mathcal{K}_\Psi(f^\epsilon) \right) f^\epsilon \right) \qquad (2)$$

$$-\,\partial_w \left(A(v,w) f^\epsilon \right) + \partial_v^2 f^\epsilon,$$

where

$$\rho_0^\epsilon(x) = \int_{\mathbb{R}^2} f^\epsilon dv dw \quad \text{and} \quad \mathcal{V}^\epsilon(t,x) = \frac{1}{\rho_0^\epsilon} \int_{\mathbb{R}^2} v f^\epsilon dv dw.$$

- Multiplying the equation by $|v - \mathcal{V}^\epsilon|^2$ and integrating yields

$$\int_{\mathbb{R}^2} \left| v - \mathcal{V}^\epsilon(t) \right|^2 f^\epsilon dv dw \underset{\epsilon \to 0}{=} O(\epsilon).$$

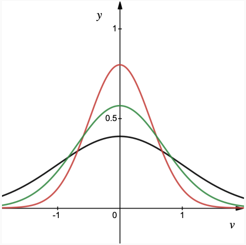
Hence, f^ϵ is expected to concentrate around $\mathcal{V}^\epsilon$

$$f^\epsilon(t,v,w) \underset{\epsilon \to 0}{=} \delta_{\mathcal{V}^\epsilon(t)}(v) \otimes F^\epsilon(t,w) + \sqrt{\epsilon},$$

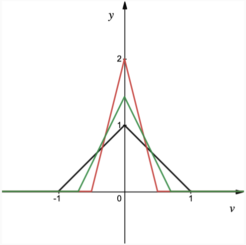
where $F^\epsilon(t,w) = \int_{\mathbb{R}} f^\epsilon(t,v,w) dv$.

Concentration's profile

Our goal is to determine the profile of this concentration.



Concentration with Gaussian profile



Concentration with triangular profile

Here are plots of

$$y = \frac{1}{\sqrt{\epsilon}} g\left(\frac{v}{\sqrt{\epsilon}}\right),$$

for $\sqrt{\epsilon} = 1; 0.7; 0.5$ and g a gaussian profile (fig. 1) and triangular profile (fig. 2).

Goal of the presentation

- Consider the following re-scaled version g^ϵ of f^ϵ :

$$f^\epsilon(t, v, w) = \frac{1}{\epsilon^\alpha} g^\epsilon \left(t, \frac{v - \mathcal{V}^\epsilon(t)}{\epsilon^\alpha}, w - \mathcal{W}^\epsilon(t) \right), \tag{3}$$

where $\alpha > 0$ needs to be determined, and

$$\mathcal{W}^\epsilon = \frac{1}{\rho_0^\epsilon} \int_{\mathbb{R}^2} w f^\epsilon dv dw.$$

- GOAL :

proving that g^ϵ converges and compute the limit.

Formal derivation of the concentration's profile

We obtain the equation on g^ϵ changing variables in the equation on f^ϵ

$$\begin{aligned}\partial_t g^\epsilon = \quad & -\frac{1}{\epsilon^\alpha}\partial_u \left[\left(N(\mathcal{V}^\epsilon + \epsilon^\alpha u) - N(\mathcal{V}^\epsilon) - (\Psi *_r \rho_0^\epsilon)\,\epsilon^\alpha u - \omega - \mathcal{E}\right) g^\epsilon\right] \\ & -\partial_\omega \left[A_0\left(\epsilon^\alpha u, \omega\right) g^\epsilon\right] + \frac{1}{\epsilon^{2\alpha}}\partial_u \left(\rho_0^\epsilon \epsilon^{2\alpha-1} u g^\epsilon + \partial_u g^\epsilon\right),\end{aligned}$$

where $A_0 = A - c$ and $\mathcal{E}$ is an error term.

- It is natural to take $\alpha = 1/2$. Indeed, we obtain

$$\begin{aligned}\partial_t g^\epsilon = \quad & -\frac{1}{\sqrt{\epsilon}}\partial_u \left[\left(N(\mathcal{V}^\epsilon + \sqrt{\epsilon} u) - N(\mathcal{V}^\epsilon) - (\Psi *_r \rho_0^\epsilon)\sqrt{\epsilon} u - \omega - \mathcal{E}\right) g^\epsilon\right] \\ & -\partial_\omega \left[A_0\left(\sqrt{\epsilon} u, \omega\right) g^\epsilon\right] + \frac{1}{\epsilon}\partial_u \left(\rho_0^\epsilon u g^\epsilon + \partial_u g^\epsilon\right).\end{aligned} \tag{4}$$

Formal derivation

- Considering the stiffer term in the former equation, we expect

$$g^{\epsilon}(t,x,u,\omega) \underset{\epsilon\to 0}{=} \mathcal{M}_{\rho_0^{\epsilon}}(u) \otimes G^{\epsilon}(t,x,\omega) + O\left(\sqrt{\epsilon}\right),$$

where $\mathcal{M}_{\rho_0^{\epsilon}}(u) = \sqrt{\frac{\rho_0^{\epsilon}}{2\pi}} \exp\left(-\frac{\rho_0^{\epsilon} u^2}{2}\right)$ and $G^{\epsilon}(t,x,\omega) = \int_{\mathbb{R}} g^{\epsilon} du.$

- Furthermore, since G^{ϵ} solves

$$\partial_t G^{\epsilon} - b\partial_{\omega}\left(xG^{\epsilon}\right) = -a\sqrt{\epsilon}\partial_{\omega}\left(\int_{\mathbb{R}} ug^{\epsilon} du\right), \tag{5}$$

and

$$\int_{\mathbb{R}} ug^{\epsilon} du \underset{\epsilon\to 0}{=} \int_{\mathbb{R}} u\mathcal{M}_{\rho_0^{\epsilon}} \otimes G^{\epsilon} du + O(\sqrt{\epsilon}) = O(\sqrt{\epsilon}),$$

it is expected that

$$G^{\epsilon}(t,x,\omega) \underset{\epsilon\to 0}{=} G(t,x,\omega) + O\left(\epsilon\right),$$

where G solves

$$\partial_t G - b\partial_{\omega}\left(\omega G\right) = 0.$$

Main result

Main result

Coming back to the solution f^ϵ to the mean field equation, this yields

Théorème

Under some assumptions on f_0^ϵ, there exists a constant $C > 0$ such that

$$\sup_{x\in K} W_2\left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0}\mathcal{M}_{\rho_{\mathbf{0}}/\epsilon}(\cdot - \mathcal{V}) \otimes F\right) \leq Ce^{Ct}\epsilon,$$

for all $\epsilon > 0$ and $t \geq 0$, where the macroscopic system $(\mathcal{V}, F)$ solves the following coupled reaction-diffusion/transport equation

$$\begin{cases} \partial_t \mathcal{V} = N(\mathcal{V}) - \mathcal{W} - \left(\Psi *_r \rho_0(x)\mathcal{V} - \Psi *_r (\rho_0 \mathcal{V})(x)\right), \\ \partial_t F + \partial_w \left(A(\mathcal{V}, w)F\right) = 0, \\ \rho_0(x)\mathcal{W} = \int_{\mathbb{R}} wF dw. \end{cases} \tag{6}$$

Here, W_2 stands for the Wasserstein distance of order 2.

Key arguments of the proof

• Uniform estimates (in time and ϵ) for moments and some relative energy using confining properties of A and N (uniformity in time) and concentrating properties of the stiffer term $-\frac{1}{\epsilon}(v - \mathcal{V}^\epsilon)$ (uniformity in ϵ).

• Coupling method in order to estimate the Wasserstein distance between g^ϵ and $M \otimes G$

References :

Fournier/Perthame (2019).

Consequences

- Convergence of the macroscopic system at order ϵ

$$\left\| \left(W_2\left(\frac{1}{\rho_0^\epsilon} F^\epsilon, \frac{1}{\rho_0} F \right), \mathcal{V}^\epsilon - \mathcal{V} \right) \right\|_{L^\infty(K)} \leq C e^{Ct} \epsilon.$$

- We deduce an "equivalent" at order 0

$$\frac{\sqrt{\epsilon}}{C} \leq \sup_{x \in K} W_2\left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0} \delta_{\mathcal{V}} \otimes F \right) \leq C\sqrt{\epsilon},$$

for all $t \in [0, T]$, where C may depend on T.

Current work

- Obtaining similar results following a Hamilton-Jacobi method.
- Obtaining a strong convergence result.
- Numerical analysis of the model.

Thanks

Thank you for your attention !