

Concentration phenomena for a FitzHugh-Nagumo neural network

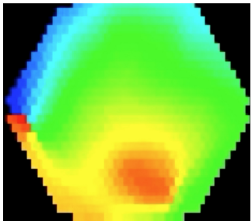
Alain Blaustein, joint work with Francis Filbet and Emeric Bouin
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Institut de Mathématiques de Toulouse

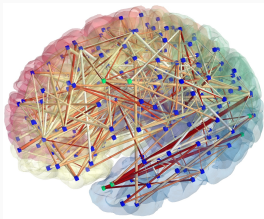
Introduction

Overview

We consider that **neural networks** are **particle systems**. It is possible to describe them through **3 distinct scales** :



Macroscopic scale



Mesoscopic scale



Microscopic scale

GOAL : Quantitative analysis of the asymptotic regime

" (mesoscopic scale) $_{\epsilon}$ $\xrightarrow{\epsilon \rightarrow 0}$ macroscopic scale "

Behavior of a *single* neuron

- We focus on the **voltage** through the membrane of a neuron.
- **Hodgkin & Huxley (52')** : precise but complicated model [gif] .
- **FitzHugh-Nagumo** : **simplified** model which captures the **main features**

FitzHugh-Nagumo's model

$$\begin{cases} dv_t = (N(v_t) - w_t + I_{\text{ext}}) dt + \sqrt{2}dB_t, \\ dw_t = A(v_t, w_t) dt, \end{cases}$$

- **2 equations** for **periodic behavior** (v_t : voltage, w_t : adaptation variable) .
- **Confining assumptions** to ensure **spikes** :

$$A(v, w) = a v - b w + c, \quad " N(v) = v - v^3 " .$$

- **Noise** to take into account random fluctuations.

Microscopic description

FitzHugh-Nagumo neural network of size n

For i between 1 and n :

$$\begin{cases} dv_t^i = (N(v_t^i) - w_t^i + I_{\text{ext}}^i) dt + \sqrt{2}dB_t^i, \\ dw_t^i = A(v_t^i, w_t^i)dt. \end{cases}$$

- Neurons interact following Ohm's law

$$I_{\text{ext}}^i = -\frac{1}{n} \sum_{j=1}^n \Phi(x_i, x_j)(v_t^i - v_t^j).$$

- The conductance $\Phi(x_i, x_j)$ between neuron i and j depend on their spatial location x_i and x_j .

FitzHugh-Nagumo's mean-field equation

$$\partial_t f + \nabla_{(v,w)} \cdot \left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Phi(f) \\ A(v, w) \end{pmatrix} f \right] - \partial_v^2 f = 0,$$

- $f(t, x, v, w)$ is the probability of finding neurons at time $t \geq 0$ and position $x \in K$, with potential $v \in \mathbb{R}$ and adaptation variable $w \in \mathbb{R}$.
- $\mathcal{K}_\Phi(f)$ is the **non-local term** due to **interactions** between neurons

$$\mathcal{K}_\Phi(f)(x, v) = \int_{K \times \mathbb{R}^2} \Phi(x, x')(v - v') f(x', v', w') dx' dv' dw'.$$

References

- E. Luçon and W. Stannat (14').
- S. Mischler, C. Quiñinao and J. Touboul (15').
- J. Crevat (19').

Regime of strong interactions

Weak-Long / Strong-Short decomposition

$$\Phi(x, x') = \underbrace{\Psi(x, x')}_{\text{weak-long range interactions}} + \underbrace{\frac{1}{\epsilon} \delta_0(x - x')}_{\text{strong-short range interactions}} .$$

The mean-field equation rewrites

$$\partial_t f^\epsilon + \nabla_{(v,w)} \cdot \left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v, w) \end{pmatrix} f^\epsilon \right] - \partial_v^2 f^\epsilon = \frac{\rho_0^\epsilon}{\epsilon} \partial_v [(v - \mathcal{V}^\epsilon) f^\epsilon],$$

where

$$\rho_0^\epsilon(x) = \int_{\mathbb{R}^2} f^\epsilon dv dw \quad \text{and} \quad \mathcal{V}^\epsilon(t, x) = \frac{1}{\rho_0^\epsilon} \int_{\mathbb{R}^2} v f^\epsilon dv dw.$$

Main goal

Analysis of the regime of **Strong/Local interactions**, that is when $\epsilon \rightarrow 0$.

Formal derivation

Strong interactions and concentration phenomenon

$$(1) : \partial_t f^\epsilon + \nabla_{(v,w)} \cdot \left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v,w) \end{pmatrix} f^\epsilon \right] - \partial_v^2 f^\epsilon = \frac{\rho_0^\epsilon}{\epsilon} \partial_v [(v - \mathcal{V}^\epsilon) f^\epsilon],$$

- f^ϵ converges to a local equilibrium : $\int |v - \mathcal{V}^\epsilon|^2 \times (1) dv dw$

$$W_2^2 \left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0^\epsilon} \delta_{\mathcal{V}^\epsilon} \otimes F^\epsilon \right) = \int_{\mathbb{R}^2} |v - \mathcal{V}^\epsilon|^2 f^\epsilon dv dw \underset{\epsilon \rightarrow 0}{=} O(\epsilon),$$

where W_2 is the Wasserstein distance of order 2. In the end, we obtain ¹

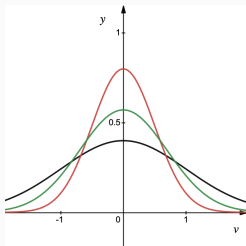
$$f^\epsilon(t, x, v, w) \underset{\epsilon \rightarrow 0}{=} \delta_{\mathcal{V}(t,x)}(v) \otimes F(t, x, w) + O(\sqrt{\epsilon}),$$

where (\mathcal{V}, F) satisfies

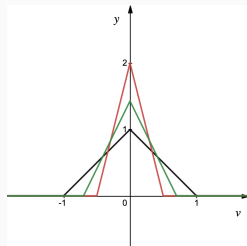
$$\left\{ \begin{array}{l} \partial_t \mathcal{V} = N(\mathcal{V}) - \mathcal{W} - (\Psi *_r \rho_0(x) \mathcal{V} - \Psi *_r (\rho_0 \mathcal{V}))(x), \\ \partial_t F + \partial_w (A(\mathcal{V}, w) F) = 0, \\ \rho_0(x) \mathcal{W} = \int_{\mathbb{R}} w F dw. \end{array} \right. \quad (1)$$

Concentration's profile

What is the **profile of concentration** ?



Concentration with **Gaussian profile**



Concentration with **triangular profile**

Here are plots of

$$y = \frac{1}{\sqrt{\epsilon}} g\left(\frac{v}{\sqrt{\epsilon}}\right),$$

for $\sqrt{\epsilon} = 1; 0.7; 0.5$ and g a gaussian profile (fig. 1) and triangular profile (fig. 2).

Formal derivation of the profile

- It is driven by diffusion term with respect to the voltage variable v

$$\partial_t f^\epsilon + \nabla_{(v,w)} \cdot \left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v, w) \end{pmatrix} f^\epsilon \right] = \partial_v \left[\frac{\rho_0^\epsilon}{\epsilon} (v - \mathcal{V}^\epsilon) f^\epsilon + \partial_v f^\epsilon \right],$$

- f^ϵ converges to the local equilibrium :

$$f^\epsilon(t, x, v, w) \underset{\epsilon \rightarrow 0}{\sim} \mathcal{M}_{\rho_0^\epsilon/\epsilon}(v - \mathcal{V}^\epsilon) \otimes F^\epsilon(t, x, w),$$

$$\text{where } \mathcal{M}_{\rho_0^\epsilon/\epsilon}(v - \mathcal{V}^\epsilon) = \sqrt{\frac{\rho_0^\epsilon}{2\pi\epsilon}} \exp\left(-\frac{\rho_0^\epsilon}{2\epsilon}(v - \mathcal{V}^\epsilon)^2\right).$$

Goal

Rigorously prove that the **profile is Gaussian** with **quantitative estimates**.

Hamilton-Jacobi approach

Hopf-Cole transform of f^ϵ

- Consider the Hopf-Cole transform² ϕ^ϵ of f^ϵ

$$f^\epsilon(t, x, v, w) = \sqrt{\frac{\rho_0}{2\pi\epsilon}} \exp\left(\frac{1}{\epsilon} \phi^\epsilon(t, x, v, w)\right).$$

We prove

Theorem (E. Bouin and A.B.³)

Suppose that in $L_{loc}^\infty(K \times \mathbb{R}^2)$

$$\phi_0^\epsilon(x, v, w) \underset{\epsilon \rightarrow 0}{=} -\frac{\rho_0(x)}{2} |v - \mathcal{V}_0(x)|^2 + O(\epsilon),$$

Then it holds in $L_{loc}^\infty(\mathbb{R}^+ \times K \times \mathbb{R}^2)$

$$\phi^\epsilon(t, x, v, w) \underset{\epsilon \rightarrow 0}{=} -\frac{\rho_0(x)}{2} |v - \mathcal{V}(t, x)|^2 + O(\epsilon)^4.$$

2. Barles, Mirrahimi, Perthame (09)
3. In preparation
4. Mirrahimi, Roquejoffre (15)

Strategy : we write

$$\phi^\epsilon(t, x, v, w) = -\frac{\rho_0^\epsilon}{2} |v - \mathcal{V}^\epsilon(t, x)|^2 + \epsilon \phi_1^\epsilon(t, x, v, w).$$

The correction ϕ_1^ϵ solves

$$H_1^\epsilon[\phi_1^\epsilon] + \frac{1}{\epsilon} J_1^\epsilon[\phi_1^\epsilon] = 0.$$

We look for **sub/super-solution** for the operator $H_1^\epsilon + \frac{1}{\epsilon} J_1^\epsilon$ under the form $\phi_- = \phi_1 - \phi$, where $J_1^\epsilon[\phi_1] = 0$ then apply a comparison principle.

Kinetic approach

Formal derivation

- We consider a re-scaled version g^ϵ of f^ϵ

$$f^\epsilon(t, x, v, w) = \frac{1}{\theta^\epsilon} g^\epsilon \left(t, x, \frac{v - \mathcal{V}^\epsilon}{\theta^\epsilon}, w - \mathcal{W}^\epsilon \right).$$

Suppose $\theta^\epsilon = \sqrt{\epsilon}$. Changing variables in the equation on f^ϵ it yields

Equation on the profile

$$\partial_t g^\epsilon + \nabla_{(v,w)} \cdot [b_0^\epsilon g^\epsilon] = \frac{1}{\epsilon} \partial_v [\rho_0^\epsilon v g^\epsilon + \partial_v g^\epsilon],$$

where b_0^ϵ depends on $1/\sqrt{\epsilon}$ and f^ϵ . Therefore, we expect

$$g^\epsilon(t, x, v, w) \underset{\epsilon \rightarrow 0}{=} \mathcal{M}_{\rho_0^\epsilon}(v) \otimes G^\epsilon(t, x, w) + O(\sqrt{\epsilon}).$$

Strategy

proving that g^ϵ converges to $\mathcal{M}_{\rho_0} \otimes G$.

Weak convergence result

Weak convergence result

Theorem (F. Filbet and A.B.⁵)

Under suitable assumptions on f_0^ϵ , there exists $C > 0$ such that

$$\sup_{x \in K} W_2 \left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0} \mathcal{M}_{\rho_0/\epsilon}(\cdot - \mathcal{V}) \otimes F \right) \leq C \left(e^{Ct} \epsilon + e^{-\rho_0^\epsilon t/\epsilon} \right),$$

for all $\epsilon > 0$ and $t \geq 0$.

Here, W_2 stands for the Wasserstein distance of order 2.

Key arguments of the proof :

- Uniform moment estimates.
- Analytic coupling method⁶ in order to estimate the Wasserstein distance between g^ϵ and $\mathcal{M} \otimes G$ (G satisfies (1) after changing variables) .

5. Concentration phenomena in Fitzhugh-Nagumo's equations : A mesoscopic approach, arXiv :2201.02363

6. Fournier, Perthame (20).

Strong convergence results

Towards strong convergence : time dependent θ^ϵ

- We consider a re-scaled version g^ϵ of f^ϵ

$$f^\epsilon(t, x, v, w) = \frac{1}{\theta^\epsilon} g^\epsilon \left(t, x, \frac{v - \mathcal{V}^\epsilon}{\theta^\epsilon}, w - \mathcal{W}^\epsilon \right).$$

- $\theta^\epsilon = \sqrt{\epsilon}$ induces that at time $t = 0$, it holds

$$f_0^\epsilon(x, v, w) = \frac{1}{\sqrt{\epsilon}} g_0^\epsilon \left(x, \frac{v - \mathcal{V}_0^\epsilon}{\sqrt{\epsilon}}, w - \mathcal{W}_0^\epsilon \right).$$

- Therefore, we impose $\theta^\epsilon(t = 0) = 1$. The only suitable choice is

$$\theta^\epsilon(t, x) = \sqrt{\epsilon} \left[1 + \underbrace{e^{-2\rho_0^\epsilon(x)t/\epsilon} (\epsilon^{-1} - 1)}_{\text{exponentially decaying remainder}} \right]^{\frac{1}{2}}.$$

The equation on g^ϵ rewrites

$$\partial_t g^\epsilon + \nabla_{(v,w)} \cdot [b_0^\epsilon g^\epsilon] = \frac{1}{|\theta^\epsilon|^2} \partial_v [\rho_0^\epsilon v g^\epsilon + \partial_v g^\epsilon].$$

Strong convergence result

Theorem (A.B.⁷)

Under suitable assumptions on f_0^ϵ , there exists $C > 0$ such that

$$\int_0^t \|f^\epsilon - f\|_{L_x^\infty L_{(v,w)}^1}(s) ds \leq C e^{Ct} \sqrt{\epsilon},$$

for all $\epsilon > 0$ and $t \geq 0$, where the limit f is given by

$$f(t, x, v, w) = \mathcal{M}_{\rho_0 |\theta^\epsilon|^{-2}}(v - \mathcal{V}) \otimes F,$$

where (\mathcal{V}, F) solves (1).

7. Large coupling in a FitzHug-Nagumo neural network : quantitative and strong convergence results, arXiv :2203.14558v2.

Key arguments

- Relative entropy estimate yields $g^\epsilon \underset{\epsilon \rightarrow 0}{\sim} \mathcal{M}_{\rho_0^\epsilon} \otimes G^\epsilon + O(\sqrt{\epsilon})$ in L^1 .
- Proving that G^ϵ converges towards G :

(i) We work on the re-scaled version H^ϵ

$$H^\epsilon = \int_{\mathbb{R}} g^\epsilon \left(t, x, v, w - \epsilon^{3/2} v \right) dv.$$

- (ii) L^1 -equicontinuity estimates for g^ϵ yield $H^\epsilon \underset{\epsilon \rightarrow 0}{=} G^\epsilon + O(\sqrt{\epsilon})$ in L^1 .
- (iii) Then we prove $H^\epsilon \underset{\epsilon \rightarrow 0}{=} G + O(\sqrt{\epsilon})$ in L^1 .

H^ϵ converges towards G : simplified example

- We consider the diffusive scaling for Fokker-Planck equation

$$\partial_t g^\epsilon + \frac{1}{\epsilon} v \cdot \nabla_x g^\epsilon = \frac{1}{\epsilon^2} \nabla_v \cdot [v g^\epsilon + \nabla_v g^\epsilon],$$

and we prove

$$G^\epsilon = \int_{\mathbb{R}} g^\epsilon dv \xrightarrow{\epsilon \rightarrow 0} G, \quad \text{where} \quad \partial_t G = \Delta_x G.$$

- Define

$$H^\epsilon(t, x) = \int_{\mathbb{R}} g^\epsilon(t, x - \epsilon v, v) dv.$$

- H^ϵ SOLVES the limiting equation

$$\partial_t H^\epsilon = \Delta_x H^\epsilon.$$

- Therefore, it is sufficient to prove $H^\epsilon \sim G^\epsilon$ (i.e. equicontinuity for g^ϵ).

Conclusion :

- L^∞ convergence estimates following a [Hamilton-Jacobi](#) approach.
- [Weak convergence](#) result with the (formal) optimal convergence rate.
- L^1 convergence result with deteriorated convergence rate.
- We also prove a convergence result in (inverse Gaussian) weighted L^2 spaces and recover the optimal rate by propagating regularity.