Concentration phenomena for a FitzHugh-Nagumo neural network

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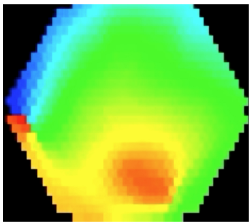
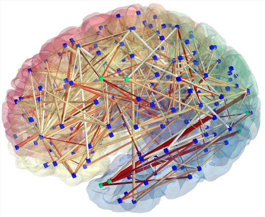
Modèles et méthodes pour les équations cinétiques

Overview

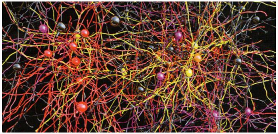
We consider that neural networks are particle systems. It is possible to describe them through 3 distinct scales :



Macroscopic scale



Mesoscopic scale



Microscopic scale

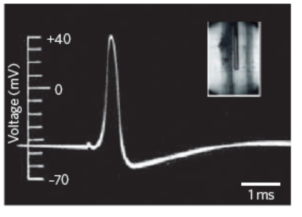
GOAL :

Derive information at the macroscopic scale from the mesoscopic scale

Microscopic scale and mean field limit

Behavior of a neuron

• We focus on the dynamics of the voltage through the membrane of a neuron. Experiments showcase 2 main features



Hodgkin & Huxley, '52.

(*i*) Delay when responding to an external input.

(*ii*) Self-regulation.

• Hodgkin & Huxley obtained a precise but mathematically complicated model.

• We will use a simplified version that captures its main features : FitzHugh-Nagumo's model for a neuron

FitzHugh-Nagumo's neuron

A neuron i is modeled by $(v_t^i, w_t^i) \in \mathbb{R}^2$ (voltage & delay)

$$\begin{cases} dv_t^i = \left(N(v_t^i) - w_t^i + I_{\text{ext}}\right) dt + \sqrt{2} dB_t, \\ dw_t^i = A\left(v_t^i, w_t^i\right) dt, \end{cases}$$

- A is an affinity while N is non-linear, typically $N(v) = v - v^3$.
- I_{ext} is the current resulting from interaction with other neurons

$$I_{\text{ext}} = -\frac{1}{n}\sum_{j=1}^{n} \Phi(x_i, x_j)(v^i - v^j),$$

where $x_i \in K \subset \mathbb{R}^d$ is the location of neuron i.

- We obtain the following microscopic model, where $1 \leq i \leq N$

$$\begin{cases} dv_t^i = \left(N(v_t^i) - w_t^i - \frac{1}{n}\sum_{j=1}^{n} \Phi(x_i, x_j)(v_t^i - v_t^j)\right) dt + \sqrt{2} dB_t^i, \\ dw_t^i = A(v_t^i, w_t^i) dt. \end{cases} \tag{1}$$

Mean field limit

It was proved that in the mean field limit $N \to +\infty$ the microscopic system is described by

FitzHugh-Nagumo's mean field equation

$$\begin{cases} \partial_t f = -\partial_v \left(\left(N(v) - w - \mathcal{K}_\Phi(f) \right) f \right) - \partial_w \left(A(v,w) f \right) + \partial_v^2 f, \\ f(0,\cdot) = f_0, \end{cases}$$

where $f(t,x,v,w)$ is the probability distribution of finding neurons with a potential $v \in \mathbb{R}$, adaptation variable $w \in \mathbb{R}$, at time $t \geq 0$ and position $x \in K$ within the network, and $\mathcal{K}_\Phi(f)$ is the non-local term due to interactions between neurons

$$\mathcal{K}_\Phi(f)(x,v) = \int_{K \times \mathbb{R}^2} \Phi(x,x')(v - v') f(x',v',w') dx' dv' dw'.$$

Mean field limit

References

- J. Baladron, D. Fasoli, O. Faugeras and J. Touboul. *Mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons* (2012).
- M. Bossy, O. Faugeras and D. Talay. *Clarification and complement to "mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons"* (2015).
- S. Mischler, C. Quiñinao and J. Touboul. *On a kinetic FitzHugh-Nagumo model of neuronal network* (2015).
- E. Luçon and W. Stannat. *Mean-field limit for disordered diffusions with singular interactions* (2014).
- J. Crevat. *Mean-field limit of a spatially-extended FitzHugh-Nagumo neural network*(2019).

Concentration phenomena

Modeling assumptions

We decompose the interaction kernel Φ as follows

$$\Phi(x, x') = \Psi(x, x') + \frac{1}{\epsilon}\delta_0(x - x'),$$

where
- Ψ is "more regular" and accounts for Weak/Long range interactions,
- dirac mass δ_0 accounts for Strong/Local interactions with strength $\epsilon > 0$.

GOAL :

Study the system in the regime of Strong/Local interactions, that is when $\epsilon \to 0$.

References :

Bressloff (03), Luçon/Stannat (14).

Strong interactions & concentration phenomenon

- With this ansatz the equation rewrites

$$\partial_t f^\epsilon = -\,\partial_v\left(\left(N(v) - w - \frac{\rho_0^\epsilon}{\epsilon}(v - \mathcal{V}^\epsilon) - \mathcal{K}_\Psi(f^\epsilon)\right) f^\epsilon\right) \qquad (2)$$

$$-\,\partial_w\left(A(v,w)f^\epsilon\right) + \partial_v^2 f^\epsilon,$$

where

$$\rho_0^\epsilon(x) = \int_{\mathbb{R}^2} f^\epsilon dvdw \quad \text{and} \quad \mathcal{V}^\epsilon(t,x) = \frac{1}{\rho_0^\epsilon}\int_{\mathbb{R}^2} vf^\epsilon dvdw.$$

- Multiplying the equation by $|v - \mathcal{V}^\epsilon|^2$ and integrating yields

$$\int_{\mathbb{R}^2} |v - \mathcal{V}^\epsilon(t)|^2 f^\epsilon dvdw \underset{\epsilon\to 0}{=} O(\epsilon).$$

Hence, f^ϵ is expected to concentrate around $\mathcal{V}^\epsilon$

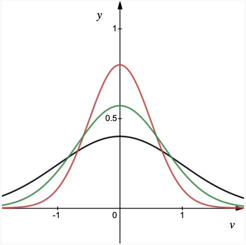
$$f^\epsilon(t,v,w) \underset{\epsilon\to 0}{=} \delta_{\mathcal{V}^\epsilon(t)}(v) \otimes F^\epsilon(t,w) + \sqrt{\epsilon},$$

where $F^\epsilon(t,w) = \int_{\mathbb{R}} f^\epsilon(t,v,w)dv$, in some probability space.

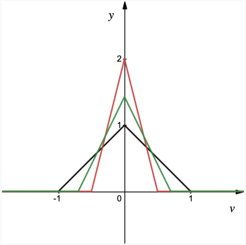
Concentration’s profile

Concentration's profile

Our goal is to determine the profile of this concentration.



Concentration with Gaussian profile



Concentration with triangular profile

Here are plots of

$$y = \frac{1}{\sqrt{\epsilon}} g\left(\frac{v}{\sqrt{\epsilon}}\right),$$

for $\sqrt{\epsilon} = 1; 0.7; 0.5$ and g a gaussian profile (fig. 1) and triangular profile (fig. 2).

Goal of the presentation

- Consider the following re-scaled version g^ϵ of f^ϵ :

$$f^\epsilon(t,v,w) = \frac{1}{\epsilon^\alpha} g^\epsilon\left(t, \frac{v - \mathcal{V}^\epsilon(t)}{\epsilon^\alpha}, w - \mathcal{W}^\epsilon(t)\right), \tag{3}$$

where $\alpha > 0$ needs to be determined, and

$$\mathcal{W}^\epsilon = \frac{1}{\rho_0^\epsilon} \int_{\mathbb{R}^2} w f^\epsilon dv dw.$$

- GOAL :

proving that g^ϵ converges and compute the limit.

Formal derivation of the concentration's profile

We obtain the equation on g^ϵ changing variables in the equation on f^ϵ

$$\partial_t g^\epsilon = -\frac{1}{\epsilon^\alpha}\partial_u\left[\left(N(\mathcal{V}^\epsilon+\epsilon^\alpha u)-N(\mathcal{V}^\epsilon)-(\Psi *_r \rho_0^\epsilon)\,\epsilon^\alpha u-\omega-\mathcal{E}\right)g^\epsilon\right] -\partial_\omega\left[A_0\left(\epsilon^\alpha u,\omega\right)g^\epsilon\right]+\frac{1}{\epsilon^{2\alpha}}\partial_u\left(\rho_0^\epsilon\epsilon^{2\alpha-1}ug^\epsilon+\partial_u g^\epsilon\right),$$

where $A_0 = A - c$ and $\mathcal{E}$ is an error term.

- It is natural to take $\alpha = 1/2$. Indeed, we obtain

$$\partial_t g^\epsilon = -\frac{1}{\sqrt{\epsilon}}\partial_u\left[\left(N(\mathcal{V}^\epsilon+\sqrt{\epsilon}u)-N(\mathcal{V}^\epsilon)-(\Psi *_r \rho_0^\epsilon)\sqrt{\epsilon}u-\omega-\mathcal{E}\right)g^\epsilon\right] -\partial_\omega\left[A_0\left(\sqrt{\epsilon}u,\omega\right)g^\epsilon\right]+\frac{1}{\epsilon}\partial_u\left(\rho_0^\epsilon ug^\epsilon+\partial_u g^\epsilon\right). \tag{4}$$

Formal derivation

- Considering the stiffer term in the former equation, we expect

$$g^\epsilon(t,x,u,\omega) \underset{\epsilon\to 0}{=} \mathcal{M}_{\rho_0^\epsilon}(u) \otimes G^\epsilon(t,x,\omega) + O\left(\sqrt{\epsilon}\right),$$

where $\mathcal{M}_{\rho_0^\epsilon}(u) = \sqrt{\frac{\rho_0^\epsilon}{2\pi}} \exp\left(-\frac{\rho_0^\epsilon u^2}{2}\right)$ and $G^\epsilon(t,x,\omega) = \int_{\mathbb{R}} g^\epsilon du.$

- Furthermore, since G^ϵ solves

$$\partial_t G^\epsilon - b\partial_\omega (xG^\epsilon) = -a\sqrt{\epsilon}\partial_\omega \left(\int_{\mathbb{R}} ug^\epsilon du\right), \qquad (5)$$

and

$$\int_{\mathbb{R}} ug^\epsilon du \underset{\epsilon\to 0}{=} \int_{\mathbb{R}} u\mathcal{M}_{\rho_0^\epsilon} \otimes G^\epsilon du + O(\sqrt{\epsilon}) = O(\sqrt{\epsilon}),$$

it is expected that

$$G^\epsilon(t,x,\omega) \underset{\epsilon\to 0}{=} G(t,x,\omega) + O(\epsilon),$$

where G solves

$$\partial_t G - b\partial_\omega (\omega G) = 0.$$

Main result

Coming back to the solution f^ϵ to the mean field equation, this yields

Théorème

Under some assumptions on f_0^ϵ, there exists a constant $C > 0$ such that

$$\sup_{x\in K} W_2\left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0}\mathcal{M}_{\rho_{\mathbf{0}}/\epsilon}(\cdot - \mathcal{V}) \otimes F\right) \leq Ce^{Ct}\epsilon,$$

for all $\epsilon > 0$ and $t \geq 0$, where the macroscopic system $(\mathcal{V}, F)$ solves the following coupled reaction-diffusion/transport equation

$$\begin{cases} \partial_t \mathcal{V} = N(\mathcal{V}) - \mathcal{W} - \left(\Psi *_r \rho_0(x)\mathcal{V} - \Psi *_r (\rho_0\mathcal{V})(x)\right), \\ \partial_t F + \partial_w \left(A(\mathcal{V}, w)F\right) = 0, \\ \rho_0(x)\mathcal{W} = \int_{\mathbb{R}} wFdw. \end{cases} \tag{6}$$

Here, W_2 stands for the Wasserstein distance of order 2.

Key arguments & Comments

• Since $W_2(\mathcal{M}_{\rho_{\mathbf{0}}/\epsilon}, \delta_0) = \sqrt{\epsilon/\rho_0}$, we recover

$$\sup_{x \in K} W_2 \left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0} \delta_{\mathcal{V}} \otimes F \right) \underset{\epsilon \to 0}{\sim} \sqrt{\epsilon},$$

Key argument for the proof :

• Uniform estimates (in time and ϵ) for moments and some relative energy using confining properties of A and N (uniformity in time) and concentrating properties of the stiffer term $-\frac{1}{\epsilon}(v - \mathcal{V}^\epsilon)$ (uniformity in ϵ).

• Coupling method in order to estimate the Wasserstein distance between g^ϵ and $M \otimes G$

References :

Fournier/Perthame (2019).

Current work

- Obtaining similar results following a Hamilton-Jacobi method.
- Obtaining a strong convergence result.
- Numerical analysis of the model.

Thanks

Thank you for your attention !