THÈSE

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Étude numérique et analytique de modèles issus des neurosciences et de la physique

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Résumé

Ce travail est consacré à l'analyse théorique et numérique d'équations aux dérivées partielles issues de la théorie cinétique et des neurosciences.
D'une part, on étudiera le modèle de Vlasov-Poisson-Fokker-Planck qui décrit la distribution cinétique d'un nuage d'électrons sujet aux collisions avec un bain ionique ainsi qu'aux interactions coulombiennes entre particules chargées. Pour commencer, on se penchera sur le régime parabolique, dans lequel la distribution cinétique des électrons converge vers une distribution macroscopique. Nous démontrerons des taux optimaux permettant de caractériser cette convergence. Puis, on complètera ce résultat théorique par l'analyse numérique du modèle. Dans un premier temps, nous proposerons une méthode numérique traitant le cas d'un champ électrique appliqué. Notre approche consistera à adapter au cadre discret les techniques d'hypocoercivité jusqu'alors classiques pour l'étude du modèle continu. Cela nous permettra de démontrer quantitativement que notre méthode reproduit le comportement du modèle continu dans le régime parabolique ainsi que sur des temps longs, uniformément par rapport aux paramètres de discrétisation. Enfin, nous proposerons une méthode permettant de traiter le couplage avec un champ électrique auto-consistant. Dans le cas d'un couplage linéaire, nous démontrerons des résultats similaires au cas du champ appliqué. Dans le cas d'un couplage non-linéaire, nous mènerons des simulations numériques variées visant à illustrer des phénomènes non-linéaires tels que l'écho plasma.
D'autre part, on se concentrera sur un modèle de type champ moyen décrivant la distribution des potentiels membranaires dans un réseau neuronal de FitzHugh-Nagumo. On proposera des méthodes permettant de caractériser quantitativement le comportement du réseau lorsque les interactions entre neurones proches sont intenses. Plus précisément, on montrera que dans ce régime, les neurones voisins tendent à aligner leur potentiel de membrane, conduisant à la concentration de la distribution des potentiels en chaque point d'espace. Nous caractériserons le profil de concentration de la distribution des potentiels en suivant trois méthodes. La première nous permettra d'obtenir un résultat de convergence faible, la seconde un résultat de convergence forte et la dernière un résultat de convergence uniforme. On attachera une attention particulière au caractère quantitatif de nos résultats et on discutera leur optimalité en comparant ces différentes approches.

Mots-clés: équations de champs moyen, théorie cinétique, neurosciences, analyse numérique, FitzHug-Nagumo, Vlasov-Poisson-Fokker-Planck

Abstract

This manuscript is devoted to the theoretical and numerical analysis of partial differential equations arising in kinetic theory and neuroscience.
On the one hand, we study the Vlasov-Poisson-Fokker-Planck model which describes the kinetic distribution of an electron cloud subject to collisions with an ionic bath as well as to Coulombic interactions between charged particles. To begin with, we focus on the parabolic regime, in which the kinetic distribution of electrons converges to a macroscopic distribution. We will prove optimal rates which characterize this convergence. Then, we complement this theoretical result with the numerical analysis of the model. First, we propose a numerical method treating the case of an applied electric field. Our approach consists in adapting to the discrete framework hypocoercivity techniques which have been classically used to study the continuous model. These techniques will allow us to demonstrate quantitatively our method's ability to reproduce the behavior of the continuous model in both parabolic and long time regimes and uniformly with restect to discretization parameters. Finally, we propose a method to treat the coupling with a self-consistent electric field. For a linear coupling, we demonstrate similar results to the case of the applied field. In the case of non-linear coupling, we carry out various numerical simulations to illustrate non-linear phenomena such as plasma echo.
On the other hand, we focus on a model describing a spatially structured network of FitzHugh-Nagumo neurons. We propose methods to quantitatively characterize the behavior of the network when the interactions between nearby neurons are intense. More precisely, we show that in this regime, neighboring neurons tend to align their membrane potential, leading to the concentration of the membrane potential distribution at each spatial location. We characterize the concentration profile of the distribution following three methods. The first one leads to a weak convergence result, the second to a strong convergence result and the last to a uniform convergence result. We pay particular attention to the quantitative character of our results and we discuss their optimality by comparing these different approaches.

Keywords: Mean field equations, Kinetic theory, Neuroscience, Numerical analysis, FitzHug-Nagumo, Vlasov-Poisson-Fokker-Planck

Introduction

1 Description statistique de systèmes physiques et biologiques

Au milieu du XIX-ème siècle, Lord Kelvin, J.C. Maxwell et L. Boltzmann bâtissaient les fondements de la théorie cinétique des gaz. Celle-ci avait pour but d'expliquer le comportement des gaz perceptible à l'échelle macroscopique par un modèle simple de la matière à l'échelle atomique. Dans cette première partie, nous proposerons un cadre abstrait pour expliquer les problématiques auxquelles répond la physique statistique. De plus, nous verrons comment la vision de la matière à l'échelle atomique proposée par Lord Kelvin, J.C. Maxwell et L. Boltzmann permet de construire des modèles capables de décrire un grand nombre de systèmes physiques (gaz, plasmas, systèmes de galaxies...) et biologiques (colonies de bactéries, cortex cérébral...). C'est dans ce cadre général que s'inscrira notre étude.

1.1 L'échelle microscopique

Lord Kelvin, J. C. Maxwell et L. Boltzmann ont été les premiers à décrire un gaz comme un système composé d'un grand nombre de particules en interactions. La dynamique de chaque particule est prescrite par une équation différentielle que nous écrirons de façon générique

$$\mathrm{d}Z_t \,=\, F(Z_t)\,\mathrm{d}t\,,$$

où Z_t est le vecteur d'état : ses composantes comprennent l'ensemble des quantités physiques décrivant la particule à un temps donné t. Par exemple, cette équation peut être obtenue en appliquant les lois de Newton à une particule ponctuelle n'étant soumise à aucune force extérieure. Dans ce cas, la particule est décrite par le couple position-vitesse $Z_t = (X_t, V_t)$ évoluant selon l'équation particulière suivante

$$\begin{cases} \mathrm{d}X_t \,=\, V_t\,\mathrm{d}t\,, \\ \mathrm{d}V_t \,=\, 0\,. \end{cases}$$

Dans les systèmes complexes évoqués plus haut, les particules sont en interactions les unes avec les autres. Cela se traduit par un terme supplémentaire dans l'équation d'évolution :

si notre système est composé de m particules, chacune décrite par un vecteur d'état Z^i pour i entre 1 et m, on obtient

$$\mathrm{d}Z_t^i \,=\, F(Z_t^i)\,\mathrm{d}t \,+\, \left(\sum_{j=1}^{m} G^m(Z_t^i, Z_t^j)\right)\mathrm{d}t\,, \quad 1 \leq i \leq m\,.$$

Ce système d'équations différentielles, dont la taille est proportionnelle au nombre de particules composant le système, constitue une description **microscopique** du système d'intérêt : le modèle prend en compte la dynamique individuelle de chaque particule. Dans les exemples évoqués plus haut, le nombre de particules composant le système est typiquement très grand, de l'ordre du nombre d'Avogadro : $m \sim 10^{24}$ pour un gaz ou un plasma. Étant donné ces ordres de grandeur, l'étude directe du système d'équations différentielles est hors de portée, aussi bien du point de vue analytique que numérique. C'est pour surmonter cette difficulté que nous introduirons l'échelle **mésoscopique**.

1.2 L'échelle mésoscopique

L'idée de génie formalisée par Lord Kelvin, Maxwell et L. Boltzmann consiste à substituer à la description de chaque particule du système, une description statistique de l'ensemble. Dans notre cadre, cela correspond à étudier la mesure empirique associée au m-uplet $(Z^1, ..., Z^m)$, c'est-à-dire

$$f^m(t, \boldsymbol{z}) \,=\, \frac{1}{m}\sum_{j=1}^{m} \delta_{Z_t^j}(\boldsymbol{z})\,,$$

où $\delta_{Z_t^j}$ désigne la masse de Dirac centrée en Z_t^j. De plus, on considère la renormalisation suivante du modèle microscopique

$$\mathrm{d}Z_t^i \,=\, F(Z_t^i)\,\mathrm{d}t \,+\, \left(\frac{1}{m}\sum_{j=1}^{m} G(Z_t^i, Z_t^j)\right)\mathrm{d}t\,, \quad 1 \leq i \leq m\,.$$

Cela correspond à supposer que les interactions binaires sont d'intensité inversement proportionnelle au nombre de particules. Dans le cas où les particules interagissent *via* leur masse selon la loi universelle de gravitation, cette renormalisation peut être justifiée en considérant que les corps sont de masse inversement proportionnelle à la masse totale du système. Lorsque l'équation d'évolution de chaque particule est déterministe, on peut vérifier formellement que la mesure empirique f^m est solution de l'équation suivante

$$\partial_t f \,+\, \nabla_{\boldsymbol{z}} \cdot \left(\left(F(\boldsymbol{z}) \,+\, \mathcal{G}\left[f\right](t, \boldsymbol{z})\right) f\right) \,=\, 0\,,$$

où le terme $\mathcal{G}\left[f\right]$ est le pendant continu du terme d'interaction discret

$$\mathcal{G}\left[f\right](t, \boldsymbol{z}) \,=\, \int G(\boldsymbol{z}, \boldsymbol{z}')\, f(t, \boldsymbol{z}')\,\mathrm{d}\boldsymbol{z}'\,.$$

Différence fondamentale entre cette équation et le modèle microscopique, cette équation ne décrit pas la dynamique individuelle des particules mais leur répartition **statistique** dans l'espace des phases ; en effet la quantité $f(t, \boldsymbol{z})$ représente la densité de particules à l'instant $t \geq 0$ ayant pour vecteur d'état $\boldsymbol{z}$. Cette classe de modèles est connue sous le nom d'"équation **mésoscopique**" ou de "**champ-moyen**". Lorsque l'on s'intéresse à des particules décrites par le couple position-vitesse, on parle d'équation cinétique.

1.3 L'échelle macroscopique

L'échelle mésoscopique constitue une description fine puisqu'il est possible de définir les quantités macroscopiques associées à un système à partir de sa distribution statistique. Considérons par exemple un système décrit par la distribution cinétique $f(t, \boldsymbol{x}, \boldsymbol{v})$ de ses particules au temps $t \geq 0$, ayant pour position $\boldsymbol{x}$ et vitesse $\boldsymbol{v}$: on définit la densité macroscopique $n(t, \boldsymbol{x})$ de particules au point d'espace $\boldsymbol{x}$ de la manière suivante

$$n(t, \boldsymbol{x}) = \int f(t, \boldsymbol{x}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \,,$$

ainsi que vitesse et température moyennes au point $\boldsymbol{x}$ de manière analogue. Il apparaît que ces quantités macroscopiques constituent une description moins complète que celle procurée par la distribution f.

Dans certains régimes, la donnée unique des quantités macroscopiques est suffisante pour caractériser complètement la distribution f, dont on dit alors qu'elle est à l'**équilibre thermodynamique local**. Les conditions physiques sous lesquelles f est à l'équilibre thermodynamique local sont réunies lorsque le système étudié est un fluide. En effet, les collisions entre particules sont bien plus fréquentes dans un fluide que lorsque la matière est plus diluée, comme c'est le cas d'un gaz. Cette fréquence importante assure que la distribution des vitesses soit d'une forme donnée, typiquement une distribution Gaussienne, appelée Maxwellienne dans le cadre de la théorie cinétique.

Lorsque le système étudié est un fluide, il est donc possible de réduire le modèle cinétique à un modèle fermé décrivant l'évolution des quantités macroscopiques : densité spatiale, vitesse et température moyenne. On obtient ainsi les équations classiques de la mécanique des fluides.

Établir un lien rigoureux entre les descriptions mésoscopiques et macroscopiques d'un système constitue l'un des grands enjeux de la théorie cinétique. Cette démarche a été initiée au début du XX-ème siècle dans le but d'axiomatiser la physique. Les questions abordées dans la suite de ce manuscrit s'inscrivent dans cette lignée : nous démontrerons des résultats permettant d'établir le lien quantitatif entre les descriptions mésoscopique et macroscopique de systèmes variés.

2 L'échelle mésoscopique en physique des plasmas

Cette section est dédiée à la motivation ainsi qu'à l'étude d'un modèle cinétique classique en physique des plasmas, celui de Vlasov-Poisson-Fokker-Planck (VPFP). Dans un premier temps, nous présenterons un contexte général dans lequel s'inscrit notre étude en introduisant un modèle décrivant un plasma bi-espèces (ions-électrons). Puis, nous identifierons un régime physique d'intérêt dans lequel il est possible de découpler la dynamique de chaque espèce, justifiant ainsi la réduction du modèle bi-espèce à l'étude de deux modèles uni-espèce, un pour les ions et l'autre pour les électrons. Après avoir discuté certains aspects de l'analyse du modèle pour les ions, nous nous pencherons sur l'analyse de la dynamique des électrons, qui correspond au modèle de VPFP. Nous aborderons notamment la question de son comportement en temps long, de ses limites macroscopiques et nous proposerons des méthodes numériques originales capables de reproduire fidèlement et efficacement la dynamique du modèle continu dans ces régimes.

2.1 Modélisation d'un plasma

L'étude des plasmas a été initiée par le physicien Irving Langmuir à la fin des années 1920. I. Langmuir a été le premier à utiliser le terme *plasma* pour décrire le quatrième état de la matière, obtenu lorsqu'un gaz est suffisamment chaud pour permettre l'ionisation des particules, le rendant ainsi conducteur du courant électrique. La prise en compte les effets électro-magnétiques est donc primordiale pour modéliser un plasma de façon satisfaisante. Pour cela, il est nécessaire de considérer des modèles comptant au minimum deux espèces de charges opposées. Lorsque l'étude des plasmas est tournée vers des applications telles que la fusion nucléaire, les conditions extrêmes, nécessitant d'être réunies pour que celle-ci se produise, doivent être prises en compte dans la modélisation. Par exemple, il est estimé qu'une température d'environ 10^8 degrés Celsius est nécessaire pour rendre la fusion thermonucléaire exploitable sur Terre. Dans cette optique, une description cinétique du système est nécessaire pour décrire la dynamique hors équilibre induite par ces conditions extrêmes. Pour conclure, on considèrera aussi les effets dus aux collisions entre particules, qui jouent un rôle important dans la dynamique de ces systèmes.

Modèle pour un plasma bi-espèce

Commençons par décrire le modèle qui servira de point de départ à notre étude. Nous considérons un plasma bi-espèces composé d'anions ou électrons de charge négative $q < 0$ et de masse m_e, et de cations ou ions de charge positive $-q$ et de masse m_i. L'espèce chargée positivement (resp. négativement) sera représentée par sa distribution cinétique $f_i\left(t, \boldsymbol{x}, \boldsymbol{v}\right)$ (resp. $f_e\left(t, \boldsymbol{x}, \boldsymbol{v}\right)$) décrivant la densité de particules à la position $\boldsymbol{x} \in \mathbb{T}^d$ et ayant une vitesse $\boldsymbol{v} \in \mathbb{R}^d$ au temps $t \geq 0$. La prise en compte des effets mentionnés plus haut nous mène au système d'équations suivant

$$\begin{cases} \partial_t f_i + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f_i - \dfrac{q}{m_i} \mathbf{E} \cdot \nabla_{\boldsymbol{v}} f_i = \dfrac{1}{\tau_{ii}} Q_{ii} (f_i, f_i) + \dfrac{1}{\tau_{ie}} Q_{ie} (f_i, f_e) \,, \\ \partial_t f_e + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f_e + \dfrac{q}{m_e} \mathbf{E} \cdot \nabla_{\boldsymbol{v}} f_e = \dfrac{1}{\tau_{ee}} Q_{ee} (f_e, f_e) + \dfrac{1}{\tau_{ei}} Q_{ei} (f_e, f_i) \,, \\ \mathbf{E} = -\nabla_{\boldsymbol{x}} \phi \,, \quad -\varepsilon_0 \Delta_{\boldsymbol{x}} \phi = q (n_e - n_i) \,, \quad n_\alpha = \displaystyle\int f_\alpha \, \mathrm{d}\boldsymbol{v} \,, \quad \alpha \in \{i, e\} \,. \end{cases}$$

Dans le modèle précédent, les phénomènes électrostatiques induits par les forces coulombiennes sont pris en compte par le champ électrique $\mathbf{E}$ obtenu par le couplage des équations cinétiques avec une équation de Poisson impliquant les densités macroscopiques des particules n_α ainsi que la constante ε_0 désignant la permittivité du vide. Par ailleurs, les opérateurs $Q_{\alpha\beta}$ pour $\alpha, \beta \in \{i, e\}$ décrivent l'effet des collisions entre les particules α et β sur l'espèce α ; ils sont pondérés par les coefficients $\tau_{\alpha\beta}$ appelés temps moyen de libre parcours et décrivant le temps séparant deux collisions entre particules α et β. On supposera que les opérateurs de collision vérifient les invariants collisionnels usuels

$$\int \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} Q_{\alpha\alpha}(f_\alpha, f_\alpha) \, \mathrm{d}\boldsymbol{v} = 0 \,,$$

ainsi que

$$\int Q_{ie}(f_i, f_e) \, \mathrm{d}\boldsymbol{v} = \int Q_{ei}(f_e, f_i) \, \mathrm{d}\boldsymbol{v} = 0 \,,$$

et

$$\int \begin{pmatrix} v \\ |v|^2 \end{pmatrix} \left(\frac{m_i}{\tau_{ie}} Q_{ie}(f_i, f_e) + \frac{m_e}{\tau_{ei}} Q_{ei}(f_e, f_i) \right) \mathrm{d}\boldsymbol{v} = 0 \,.$$

Ces propriétés permettent d'assurer le caractère conservatif du système avec la conservation de la charge,

$$\int q \, f_\alpha(t, \boldsymbol{x}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} = \int q \, f_\alpha(0, \boldsymbol{x}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \,, \quad \alpha \in \{e, i\} \,,$$

de la quantité de mouvement

$$\int \boldsymbol{v} \, (m_i \, f_i + m_e \, f_e) \, (t, \boldsymbol{x}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} = \int \boldsymbol{v} \, (m_i \, f_i + m_e \, f_e) \, (0, \boldsymbol{x}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \,,$$

ainsi que de l'énergie totale

$$\mathcal{E}(t) = \mathcal{E}(0) \,,$$

avec

$$\mathcal{E}(t) = \frac{1}{2} \int |\boldsymbol{v}|^2 \, (m_i \, f_i + m_e \, f_e)) \, (t, \boldsymbol{x}, \boldsymbol{v}) \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} + \frac{\varepsilon_0}{2} \int |\mathbf{E}(t, \boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} \,.$$

Enfin, nous précisons que ne seront considérées ici que des données initiales vérifiant la condition de compatibilité suivante, qui assure la neutralité globale du plasma

$$\int q \, f_i(0, \boldsymbol{x}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} = \int q \, f_e(0, \boldsymbol{x}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \,.$$

Paramètres physiques et adimensionnement du modèle

Dans certaines configurations physiques, il n'est pas nécessaire de considérer une description cinétique couplée de chacune des deux espèces composant le plasma. L'objectif de cette partie est d'identifier les paramètres physiques définissant les régimes d'évolution dans lesquels il est possible de réduire le modèle bi-espèce présenté plus haut. Dans cette optique, nous allons procéder à l'adimensionnement de l'équation : pour chaque quantité G on notera $\bar{G}$ la valeur caractéristique de G dans le régime d'intérêt.

Nous commençons par effectuer l'analyse dimensionnelle des termes associés au transport des particules sur la base de deux hypothèses. D'une part, nous considérons que le plasma est quasi-neutre. Comme les deux espèces sont de charge égale $|q|$, cela revient à supposer que les densités spatiales de chaque espèce sont du même ordre

$$\bar{n}_i = \bar{n}_e = \bar{n} .$$

Notre seconde hypothèse correspond au régime des plasmas chauds, dans lequel les températures caractéristiques de chaque espèce sont du même ordre de grandeur

$$\bar{T}_e = \bar{T}_i = \bar{T} .$$

Soulignons que dans le cadre d'une description cinétique du système, vitesses microscopiques (ou thermiques) et températures caractéristiques de chaque espèce sont liées par la constante de Boltzmann k_B *via* la relation suivante

$$m_i \, \bar{v}_i^2 = k_B \, \bar{T}_i \quad \text{et} \quad m_e \, \bar{v}_e^2 = k_B \, \bar{T}_e .$$

Ainsi, en introduisant la quantité, sans dimension, décrivant le rapport de masse ions/électrons

$$\varepsilon^2 := \frac{m_e}{m_i} ,$$

on observe que l'hypothèse des plasmas chauds induit une différence d'ordre ε entre les vitesses thermiques des deux espèces

$$\bar{\boldsymbol{v}}_e = \frac{1}{\varepsilon} \, \bar{\boldsymbol{v}}_i .$$

Un autre paramètre central dans la description d'un plasma est la longueur de Debye λ_D, qui peut être interprétée comme le rayon caractéristique autour duquel la quasi-neutralité du plasma n'est plus respectée

$$\lambda_D = \sqrt{\frac{\varepsilon_0 \, k_B \, \bar{T}}{\bar{n} \, q^2}} .$$

Pour conclure l'analyse dimensionnelle des termes de transport, nous introduisons le paramètre adimensionné η décrivant le rapport entre énergies thermique et électrique

$$\eta = \frac{k_B \, \bar{T}}{q \, \bar{\phi}} .$$

Passons maintenant à l'analyse dimensionnelle des termes de collision. Comme expliqué dans [12, 121], les temps de collisions caractéristiques inter- et intra-espèces $\tau_{\alpha\beta}$ sont liés entres eux *via* le rapport de masse ε entre les espèces selon la règle suivante

$$\frac{\bar{\tau}_{ee}}{\bar{\tau}_{ei}} = 1\,, \quad \frac{\bar{\tau}_{ee}}{\bar{\tau}_{ii}} = \varepsilon\,, \quad \frac{\bar{\tau}_{ee}}{\bar{\tau}_{ie}} = \varepsilon^2\,.$$

De plus, les noyaux de collisions ont la dimension suivante

$$\bar{Q}_{\alpha\beta} = \bar{f}_\alpha\,.$$

On peut maintenant ré-écrire le modèle cinétique sous forme adimensionné. Pour cela on choisit la longueur caractéristique d'observation, notée $\bar{\boldsymbol{x}}$, et on se place à l'échelle de la vitesse des ions en posant

$$\bar{t} = \frac{\bar{\boldsymbol{x}}}{\bar{\boldsymbol{v}}_i}\,.$$

Cela nous permet de définir le paramètre adimensionné donné par le rapport entre la longueur de Debye et la longueur caractéristique d'observation

$$\lambda = \frac{\lambda_D}{\bar{\boldsymbol{x}}}\,,$$

ainsi que le rapport entre le temps moyen de libre parcours des électrons et notre échelle en temps

$$\tau = \frac{\bar{\tau}_{ee}}{\bar{t}}\,.$$

Enfin, les distributions adimensionnées f'_α et n'_α sont définies par

$$\begin{cases} f_\alpha(t,\boldsymbol{x},\boldsymbol{v}) = \dfrac{\bar{n}_\alpha}{\bar{v}_\alpha^3} f'_\alpha\left(t\,\bar{t}^{-1}, \boldsymbol{x}\,\bar{\boldsymbol{x}}^{-1}, \boldsymbol{v}\,\bar{\boldsymbol{v}}_\alpha^{-1}\right)\,, \\ n_\alpha(t,\boldsymbol{x},\boldsymbol{v}) = \bar{n}\, n'_\alpha\left(t\,\bar{t}^{-1}, \boldsymbol{x}\,\bar{\boldsymbol{x}}^{-1}\right) \end{cases}$$

pour $\alpha \in \{e, i\}$. On obtient alors l'équation sur f'_α en effectuant le changement de variable $(t,\boldsymbol{x},\boldsymbol{v}) \to \left(t\,\bar{t}^{-1}, \boldsymbol{x}\,\bar{\boldsymbol{x}}^{-1}, \boldsymbol{v}\,\bar{\boldsymbol{v}}_\alpha^{-1}\right)$ dans l'équation vérifiée par f_α et on aboutit au système suivant où l'on a omis les primes dans un souci de clarté,

$$\begin{cases} \partial_t f_i + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f_i - \eta\,\mathbf{E}\cdot\nabla_{\boldsymbol{v}} f_i = \dfrac{\varepsilon}{\tau}\left(Q_{ii}(f_i,f_i) + \varepsilon\, Q_{ie}(f_i,f_e)\right)\,, \\ \partial_t f_e + \dfrac{1}{\varepsilon}\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f_e + \dfrac{1}{\varepsilon}\eta\,\mathbf{E}\cdot\nabla_{\boldsymbol{v}} f_e = \dfrac{1}{\tau}\left(Q_{ee}(f_e,f_e) + Q_{ei}(f_e,f_i)\right)\,, \\ \mathbf{E} = -\nabla_{\boldsymbol{x}}\phi\,, \quad -\lambda^2\,\eta\,\Delta_{\boldsymbol{x}}\phi = n_e - n_i\,, \quad n_\alpha = \displaystyle\int f_\alpha\,\mathrm{d}\boldsymbol{v}\,, \quad \alpha\in\{i,e\}\,. \end{cases} \tag{1}$$

Nous invitons le lecteur souhaitant plus de détails au sujet des paramètres plasmas à consulter les ouvrages [12, 69, 121].

Dynamique des ions dans le régime des électrons sans masse

Le modèle (1) est très coûteux en termes de simulation numérique car c'est un système de deux équations couplées, chacune posée sur l'espace des phases $(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{T}^d \times \mathbb{R}^d$ de dimension $2d$. Cependant, dans certaines situations pertinentes du point de vue de la physique, le modèle se réduit à des systèmes simplifiés. Nous présenterons ici l'une de ces situations d'intérêt, appelée régime des électrons sans masse, dans laquelle il est possible de remplacer la distribution cinétique des électrons par une distribution de Maxwell-Boltzmann. Comme son nom l'indique, le régime des électrons sans masse consiste à prendre le ratio de masse électrons-ions petit devant 1, c'est-à-dire

$$\varepsilon^2 \ll 1\,.$$

Du point de vue de la modélisation, cela revient à supposer que les électrons évoluent plus rapidement que les ions. Ainsi, en se plaçant dans le référentiel des ions, on peut considérer que les électrons sont à l'état d'équilibre. Plusieurs études [4, 109] sont dédiées à l'analyse formelle de la limite $\varepsilon \to 0$: elles montrent que sous l'hypothèse $\tau = \tau_0\,\varepsilon$, les électrons suivent un processus de thermalisation décrit mathématiquement par la convergence de leur distribution vers l'équilibre de Maxwell-Boltzmann suivant

$$f_e\,(t,\boldsymbol{x},\boldsymbol{v}) \underset{\varepsilon\to 0}{\longrightarrow} \exp\left(-\,\eta\,\frac{\phi^0(t,\boldsymbol{x})}{T_e^0(t)}\right)\frac{1}{\sqrt{2\,\pi\,T_e^0(t)}^{d}}\,\exp\left(-\frac{1}{2\,T_e^0(t)}\,|v|^2\right)\,,$$

où le potentiel électrique ϕ^0 limite est obtenu en résolvant l'équation prescrivant la dynamique des ions

$$\begin{cases} \partial_t\, f_i^0 \,+\, \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\, f_i^0 \,-\, \eta\,\mathbf{E}^0\cdot\nabla_{\boldsymbol{v}}\, f_i^0 \,=\, \dfrac{1}{\tau_0}Q_{ii}\left(f_i^0, f_i^0\right)\,, \\[2ex] \mathbf{E}^0 \,=\, -\,\nabla_{\boldsymbol{x}}\,\phi^0\,, \quad -\,\lambda^2\,\eta\,\Delta_{\boldsymbol{x}}\,\phi^0 \,=\, e^{-\,\eta\frac{\phi_0}{T_e}} - n_i^0\,, \quad n_i^0 \,=\, \displaystyle\int f_i^0\,\mathrm{d}\boldsymbol{v}\,, \end{cases} \tag{2}$$

tandis que la température limite des électrons T_e^0 est obtenue de manière à assurer la conservation de l'énergie totale du système

$$\mathcal{E}^0(t) \,=\, \mathcal{E}^0(0)\,,$$

où l'énergie est définie par

$$\mathcal{E}^0(t) \,=\, \frac{d\,M}{2}\,T_e^0(t) \,+\, \frac{1}{2}\int|\boldsymbol{v}|^2\,f_i^0(t,\boldsymbol{x},\boldsymbol{v})\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v} \,+\, \frac{\lambda^2\,\eta^2}{2}\int\left|\mathbf{E}^0(t,\boldsymbol{x})\right|^2\,\mathrm{d}\boldsymbol{x}\,,$$

tandis que M désigne la masse totale des électrons

$$M \,=\, \int\exp\left(-\,\eta\,\frac{\phi^0(t,\boldsymbol{x})}{T_e^0(t)}\right)\mathrm{d}\boldsymbol{x}\,.$$

On obtient donc, dans la limite des électrons sans masse, le modèle (2) composé d'une seule équation cinétique décrivant la dynamique des ions. Ce modèle a fait l'objet de plusieurs travaux. On mentionne [32, 130], dans lesquels est démontrée l'existence globale de solutions, ainsi que [129], dans lequel M. Griffin-Pickering et M. Iacobelli étudient la limite quasi-neutre $\lambda \to 0$ de (2) et dérivent le modèle à partir d'un système de particules.

2.2 Dynamique des électrons

Dans la section précédente, nous avons vu que lorsque le ratio de masse électrons-ions est petit, les électrons suivent le processus de thermalisation correspondant à la convergence de leur densité vers une distribution de Maxwell-Boltzmann. Dans cette section, nous étudierons en détail ce processus, dont nous verrons que, dans certaines hypothèses, il est naturellement décrit par le comportement du système de Vlasov-Poisson-Fokker-Planck. Cela nous conduira à nos problématiques centrales.

Le modèle de Vlasov-Poisson-Fokker-Planck

Comme expliqué plus haut, dans le régime des électrons sans masse, les électrons sont beaucoup plus rapides que les ions. Ainsi, dans le référentiel des électrons, on peut considérer que les ions sont immobiles. Dès lors, nous traiterons tout au long de cette partie l'espèce ionique comme un bain stationnaire dans lequel évoluent les électrons. Pour cela, nous supposerons que les ions ont une distribution f_i indépendante du temps ainsi qu'une vitesse macroscopique uniformément nulle. Par souci de simplification, nous supposerons que la température $T_0 > 0$ des ions est homogène. Sous ces hypothèses, il est possible de supprimer la première ligne du système (1) et de modéliser le terme de collision inter-espèces Q_{ei} par l'opérateur de Fokker-Planck suivant

$$Q_{ei}\,(f_e, f_i) \,=\, \nabla_{\boldsymbol{v}} \cdot [\, \boldsymbol{v}\, f_e \,+\, T_0\, \nabla_{\boldsymbol{v}}\, f_e\,]\,.$$

Toujours dans un souci de simplification, nous négligerons les collisions intra-espèces et fixerons $\lambda = \eta = 1$. On notera $f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{v})$ la distribution cinétique des électrons dans l'espace des phases $(t, \boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d$, solution de

$$\begin{cases} \partial_t\, f^\varepsilon \,+\, \dfrac{1}{\varepsilon}\, \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}}\, f^\varepsilon \,+\, \dfrac{1}{\varepsilon}\, \mathbf{E}^\varepsilon \cdot \nabla_{\boldsymbol{v}}\, f^\varepsilon \,=\, \dfrac{1}{\tau(\varepsilon)}\, \nabla_{\boldsymbol{v}} \cdot [\, \boldsymbol{v}\, f^\varepsilon \,+\, T_0\, \nabla_{\boldsymbol{v}}\, f^\varepsilon\,]\,, \\[2ex] \mathbf{E}^\varepsilon \,=\, -\,\nabla_{\boldsymbol{x}}\, \phi^\varepsilon\,, \quad -\,\Delta_{\boldsymbol{x}}\, \phi^\varepsilon \,=\, \rho^\varepsilon - \rho_i\,, \quad \rho^\varepsilon \,=\, \displaystyle\int_{\mathbb{R}^d} f^\varepsilon\, \mathrm{d}\boldsymbol{v}\,, \\[2ex] f^\varepsilon(0, \boldsymbol{x}, \boldsymbol{v}) \,=\, f^\varepsilon_0(\boldsymbol{x}, \boldsymbol{v})\,. \end{cases} \tag{3}$$

On complète le système (3) par la condition de moyenne nulle, permettant d'assurer l'unicité du potentiel électrique ϕ^ε

$$\int_{\mathbb{T}^d} \phi^\varepsilon\, \mathrm{d}\boldsymbol{x} \,=\, 0\,.$$

Comme vu ci-dessus, $\tau(\varepsilon) > 0$ désigne le rapport entre le temps d'observation et celui de libre parcours des électrons dans le bain ionique.

Collisions, irréversibilité et hypocoercivité

Dans ce paragraphe, seront présentés les enjeux de l'analyse du modèle de Vlasov-Poisson-Fokker-Planck. Une propriété essentielle du modèle (3) réside dans son irréversibilité. Cette caractéristique est due aux collisions entre électrons et bain ionique, modélisées par l'opérateur de Fokker-Planck qui apparaît dans le membre de droite de la première équation du système (3). Cet opérateur tend à ramener les électrons vers l'équilibre thermodynamique. L'irréversibilité se traduit par le fait qu'une quantité décrivant un système décroit au cours du temps ; dans notre cas, la dissipation de l'énergie libre le long des trajectoires de (3) est régie par l'estimée capitale suivante

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f^\varepsilon, f_\infty) \;=\; -\frac{1}{\tau(\varepsilon)}\,\mathcal{I}(f^\varepsilon, f_\infty), \tag{4}$$

où l'énergie libre $\mathcal{H}(f^\varepsilon, f_\infty)$ et la dissipation d'entropie $\mathcal{I}(f^\varepsilon, f_\infty)$ sont données par

$$\left\{\begin{aligned} &\mathcal{H}(f^\varepsilon, f_\infty) \;:=\; \int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\ln\left(f^\varepsilon\right) + \frac{1}{2\,T_0}|\boldsymbol{v}|^2\right) f^\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v} \;+\; \frac{1}{2\,T_0}\,\|\,\mathbf{E}^\varepsilon\,\|_{L^2}^2\;,\\ &\mathcal{I}(f^\varepsilon, f_\infty) \;:=\; 4\,T_0\int_{\mathbb{T}^d\times\mathbb{R}^d}\left|\partial_v\sqrt{\frac{f^\varepsilon}{\mathcal{M}}}\right|^2\mathcal{M}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,, \end{aligned}\right.$$

où $\mathcal{M}$ désigne la distribution Maxwellienne de température T_0

$$\mathcal{M}(\boldsymbol{v}) \;=\; (2\pi\,T_0)^{-\frac{d}{2}}\,\exp\left(-\frac{|\boldsymbol{v}|^2}{2\,T_0}\right)\,.$$

La relation (4) est centrale dans l'étude du modèle de Vlasov-Poisson-Fokker-Planck : elle mesure la vitesse de retour des solutions f^ε de (3) vers l'équilibre thermodynamique. Il est en effet possible d'interpréter l'énergie libre $\mathcal{H}(f^\varepsilon, f_\infty)$ comme une mesure de la distance de f^ε à l'équilibre puisque l'on a

$$\left\{\begin{aligned} &\mathcal{H}(f^\varepsilon, f_\infty) \;:=\; \int_{\mathbb{T}^d\times\mathbb{R}^d}\ln\left(\frac{f^\varepsilon}{f_\infty}\right) f^\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v} \;+\; \frac{1}{2\,T_0}\,\|\,\mathbf{E}^\varepsilon \;-\; \mathbf{E}_\infty\,\|_{L^2}^2\;,\\ &\mathcal{I}(f^\varepsilon, f_\infty) \;:=\; 4\,T_0\int_{\mathbb{T}^d\times\mathbb{R}^d}\left|\partial_v\sqrt{\frac{f^\varepsilon}{f_\infty}}\right|^2 f_\infty\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,, \end{aligned}\right.$$

où f_∞ est l'équilibre suivant de (3)

$$f_\infty(\boldsymbol{x},\boldsymbol{v}) \;=\; \rho_\infty(\boldsymbol{x})\,\mathcal{M}(\boldsymbol{v})\,, \tag{5}$$

tandis que ρ_∞ vérifie l'équation de Maxwell-Boltzmann suivante

$$\begin{cases} \rho_\infty \;=\; \exp\left(-\dfrac{\phi_\infty}{T_0}\right) , \\ -\Delta_{\boldsymbol{x}}^2 \phi_\infty \;=\; \rho_\infty - \rho_i \, , \end{cases}$$

complétée par la condition

$$\int_{\mathbb{T}^d} \exp\left(-\frac{\phi_\infty}{T_0}\right) \mathrm{d}\boldsymbol{x} \;=\; \int_{\mathbb{T}^d} \rho_i \, \mathrm{d}\boldsymbol{x} \, .$$

Par la suite, on notera $\mathbf{E}_\infty := -\nabla_{\boldsymbol{x}} \phi_\infty$ le champ électrique à l'équilibre.
Autre point crucial dans la compréhension de la dynamique de (3) : l'irréversibilité induite par les collisions est un processus microscopique. Cela se traduit mathématiquement par le fait que la dissipation d'entropie $\mathcal{I}(f^\varepsilon, f_\infty)$ ne constitue qu'une mesure partielle de la distance entre f^ε et l'équilibre f_∞. Plus précisément, $\mathcal{I}(f^\varepsilon, f_\infty)$ mesure la distance entre f^ε et son **équilibre local** défini par $\rho^\varepsilon \mathcal{M}$, comme le garantit l'inégalité fonctionnelle suivante

$$\| f^\varepsilon(t) \; - \; \rho^\varepsilon(t) \, \mathcal{M} \|^2_{L^1\left(\mathbb{T}^d \times \mathbb{R}^d\right)} \; \lesssim \; \mathcal{I}(f^\varepsilon(t), f_\infty) \, .$$

Ainsi, en intégrant (4) par rapport au temps, on obtient une mesure quantitative en termes de $\tau(\varepsilon)$ de la distance entre f^ε et son équilibre local

$$\| f^\varepsilon \; - \; \rho^\varepsilon \, \mathcal{M} \|^2_{L^2\left(\mathbb{R}^+ , \, L^1\left(\mathbb{T}^d \times \mathbb{R}^d\right)\right)} \; \lesssim \; \tau(\varepsilon) \, \mathcal{H}(f_0^\varepsilon, f_\infty) \, . \tag{6}$$

Cependant, l'estimation précédente ne fournit pas d'indication sur la dynamique de la densité macroscopique ρ^ε. Un des enjeux de notre analyse, et plus généralement de la théorie cinétique, consiste donc à comprendre comment les mécanismes d'irréversibilité microscopiques affectent la dynamique macroscopique du système. Les modèles concernés sont dotés de la structure caractéristique suivante :

(i) une estimation décrivant la dissipation d'une quantité (par exemple une énergie),

(ii) et permettant de prédire la dynamique microscopique du système (représentée par la distribution des vitesses dans notre cas) et non sa dynamique globale.

Au début des années 2000, C. Villani a introduit la notion **d'hypocoercivité** pour décrire cette structure et a développé des outils permettant de faire le lien entre les mécanismes d'irréversibilité microscopiques et la dynamique macroscopique des systèmes hypocoercifs. Soulignons que tous nos résultats s'interprètent dans ce langage.

Cadre de l'étude

Venons en à présent au cadre spécifique de notre travail. Nous étudierons la limite $\varepsilon \to 0$ de (3) en considérant les régimes collisionnels suivants

$$\varepsilon^2 \; \lesssim \; \tau(\varepsilon) \; \lesssim \; \varepsilon \, .$$

Notons que les deux cas extrêmes $\tau(\varepsilon) \in \{\varepsilon\,,\,\varepsilon^2\}$ couvrent l'ensemble des comportements possibles.

Dans le cas où $\tau(\varepsilon) = \varepsilon$, la limite $\varepsilon \to 0$ correspond au comportement en temps long de (3). Ce choix est consistant avec la limite des électrons sans masse du modèle bi-espèce (1) présentée plus haut : lorsque $t/\varepsilon \to +\infty$, la distribution f^ε converge vers l'équilibre de Maxwell-Boltzmann f_∞.

Le cas où $\tau(\varepsilon) = \varepsilon^2$ correspond quant à lui au changement d'échelle parabolique, aussi appelé changement d'échelle diffusif. Cette asymptotique est plus riche que la précédente car, bien que la distribution des vitesses soit aussi Maxwellienne conformément à l'estimation (6)

$$f^\varepsilon(t,\boldsymbol{x},\boldsymbol{v}) \underset{\varepsilon\to 0}{\sim} \rho^\varepsilon(t,\boldsymbol{x})\,\mathcal{M}(\boldsymbol{v})\,,$$

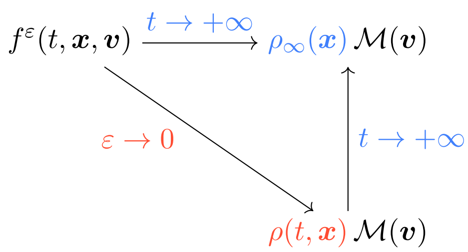
la distribution macroscopique limite $\rho = \lim_{\varepsilon\to 0}\rho^\varepsilon$ n'est pas stationnaire. En effet, nous démontrerons que celle-ci résout l'équation non linéaire de dérive-diffusion suivante

$$\begin{cases} \partial_t\,\rho + \nabla_{\boldsymbol{x}}\cdot\left[\,\mathbf{E}\,\rho - T_0\,\nabla_{\boldsymbol{x}}\rho\,\right] = 0\,, \\ \\ -\Delta_{\boldsymbol{x}}\,\phi = \rho - \rho_i\,, \quad \mathbf{E} = -\nabla_{\boldsymbol{x}}\,\phi\,. \end{cases} \tag{7}$$

Le régime parabolique est caractérisé par le fait que la distribution en vitesse y atteint son équilibre Maxwellien contrairement à la distribution spatiale, dont la dynamique est plus lente. Ce régime permet donc d'observer des variations du champ électrique. Il décrit des échelles de temps antérieures au régime $\tau(\varepsilon) = \varepsilon$ puisque dans la limite $t \to +\infty$, on retrouve l'équilibre de Maxwell-Boltzmann

$$\rho(t,\boldsymbol{x}) \underset{t\to\infty}{\longrightarrow} \rho_\infty(\boldsymbol{x})\,.$$

On peut ainsi résumer la dynamique du modèle par le diagramme suivant



Notre travail portera sur les enjeux tant théoriques que numériques de ces différentes asymptotiques. En Section 0.2.3, nous présenterons une approche aboutissant à des résultats quantitatifs décrivant la convergence forte du modèle dans le régime parabolique. En Section 0.2.4, nous proposerons et analyserons une méthode numérique pour le modèle de Vlasov-Fokker-Planck avec un champ extérieur appliqué. À l'aide d'estimations quantitatives valables uniformément par rapport aux paramètres de discrétisation, nous montrerons que solutions numériques et continues ont un comportement identique dans tout les régimes évoqués plus haut. Enfin, dans la Section 0.2.5 nous proposerons une méthode numérique pour le modèle non linéaire de Vlasov-Poisson-Fokker-Planck (3). Nous montrerons notamment des taux de retour exponentiels vers l'équilibre dans le cas d'un couplage linéaire avec l'équation de Poisson. Cette dernière étude sera également axée sur la simulation et l'expérimentation numériques. Nous proposerons par exemple des tests permettant d'observer des phénomènes physiques non linéaires tels que l'écho plasma ou la formation de patterns périodiques, phénomènes dont nous étudierons la stabilité dans différents régimes collisionnels.

2.3 Analyse du régime parabolique (Chapitre 1)

Dans cette section, nous présenterons un résultat quantitatif de convergence forte valable en toute dimension qui permet de caractériser le comportement du modèle de Vlasov-Poisson-Fokker-Planck dans le régime parabolique. Ce régime correspond à la limite $\varepsilon \to 0$ dans le cas où $\tau(\varepsilon) = \varepsilon^2$; le système (3) s'écrit donc

$$\begin{cases} \partial_t f^\varepsilon + \dfrac{1}{\varepsilon} \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f^\varepsilon + \dfrac{1}{\varepsilon} \mathbf{E}^\varepsilon \cdot \nabla_{\boldsymbol{v}} f^\varepsilon = \dfrac{1}{\varepsilon^2} \nabla_{\boldsymbol{v}} \cdot [\, \boldsymbol{v} f^\varepsilon + \nabla_{\boldsymbol{v}} f^\varepsilon \,] \,, \\[2ex] \mathbf{E}^\varepsilon = -\nabla_{\boldsymbol{x}} \phi^\varepsilon \,, \quad -\Delta_{\boldsymbol{x}} \phi^\varepsilon = \rho^\varepsilon - \rho_i \,, \quad \rho^\varepsilon = \displaystyle\int_{\mathbb{R}^d} f^\varepsilon \, \mathrm{d}\boldsymbol{v} \,, \\[2ex] f^\varepsilon(0, \boldsymbol{x}, \boldsymbol{v}) = f_0^\varepsilon(\boldsymbol{x}, \boldsymbol{v}) \,, \end{cases} \tag{8}$$

où l'espace des phases est décrit par $(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{K}^d \times \mathbb{R}^d$ avec $\mathbb{K} \in \{\mathbb{T}, \mathbb{R}\}$ et $d \geq 1$; on a fixé $T_0 = 1$ sans perte de généralité. Commençons par rappeler que les collisions des électrons avec le bain ionique tendent à ramener f^ε vers son équilibre local. Cette convergence est assurée par l'estimation (6), qui dans le régime parabolique se ré-écrit

$$\| f^\varepsilon - \rho^\varepsilon \mathcal{M} \|^2_{L^2\left(\mathbb{R}^+, L^1\left(\mathbb{K}^d \times \mathbb{R}^d\right)\right)} \lesssim O\left(\varepsilon^2\right) , \tag{9}$$

dès lors que l'on considère des données initiales ayant une énergie libre finie. Ainsi, l'enjeu de l'analyse mathématique du régime parabolique consiste à démontrer la convergence de la distribution macroscopique ρ^ε vers la distribution limite ρ solution de (7).
Notre analyse s'inscrit dans la lignée des travaux de F. Poupaud et J. Soler qui sont les premiers à démontrer un résultat de convergence forte en temps court

Théorème 1 ([192])**.** *Fixons $\mathbb{K}^d = \mathbb{R}^d$ et considérons une suite $(f^\varepsilon)_{\varepsilon > 0}$ de solutions de* (8) *dont les données initiales vérifient*

$$\sup_{\varepsilon > 0} \left(\mathcal{H}(f_0^\varepsilon, f_\infty) + |||f_0^\varepsilon|||_p \right) \leq m \,,$$

pour une constante $m > 0$, un exposant $p > d$ tandis que la norme $|||\cdot|||_p$ est définie par

$$|||f^\varepsilon|||_p^p := \int_{\mathbb{R}^d \times \mathbb{T}^d} \left| \frac{f^\varepsilon}{\mathcal{M}} \right|^p \mathcal{M} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \,.$$

Il existe $T^\star > 0$ dépendant de m tel qu'à extraction près, on ait

$$f^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \rho \mathcal{M}\,, \quad \textit{fortement dans} \quad L^r\left([0,T^\star], L^q\left(\mathbb{R}^d \times \mathbb{R}^d\right)\right)\,,$$

où ρ est solution de (7)*, avec $1 \leq r < +\infty$ et $1 \leq q < p$.*

Le point crucial de leur démonstration consiste à estimer la quantité $|||\cdot|||_p$ le long des trajectoires de (8), ce qui permet notamment de contrôler $\mathbf{E}^\varepsilon$ dans L^∞ ainsi que ρ^ε dans L^p conformément aux inégalités fonctionnelles suivantes

$$\|\mathbf{E}^\varepsilon\|_{L^\infty} \;\leq\; C_p\,\|\rho^\varepsilon\|_{L^p} \;\leq\; C_p |||f^\varepsilon|||_p\,, \quad \forall\, p > d\,. \tag{10}$$

La première inégalité est obtenue grâce à la régularisation elliptique due à l'équation de Poisson dans (8), la seconde, quant à elle, résulte d'une inégalité de Jensen par rapport à la mesure $\mathcal{M}\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$. L'estimation de la contribution non linéaire du champ électrique dans les variations de $|||f^\varepsilon|||_p$ présentée dans [192] conduit à l'inégalité différentielle suivante

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\,|||f^\varepsilon|||_p^p \;\;\leq\;\; C_p\,\|\mathbf{E}^\varepsilon\|_{L^\infty}^2\;|||f^\varepsilon|||_p^p\,, \tag{11}$$

qui, conformément à l'inégalité fonctionnelle (10), assure

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\,|||f^\varepsilon|||_p^p \;\;\leq\;\; C_p\,|||f^\varepsilon|||_p^{p+2}\,,$$

et permet de contrôler $|||f^\varepsilon|||_p$ seulement sur des **temps courts**, ce qui constitue une restriction majeure du Théorème 1. En couplant ces estimations avec un lemme de moyenne, repris des travaux du premier auteur avec F. Golse [123], F. Poupaud et J. Soler obtiennent la compacité forte de la suite $(\rho^\varepsilon)_{\varepsilon>0}$ et passent à la limite dans le terme non linéaire sans difficulté grâce au contrôle du champ $\mathbf{E}^\varepsilon$ dans L^∞. Les auteurs démontrent par ailleurs un résultat de **compacité faible global en temps** dans le cas $d = 2$. C'est la structure particulière du potentiel Coulombien en cette dimension qui leur permet de passer à limite dans le terme non linéaire. Toujours en dimension $d = 2$, T. Goudon [125] généralise le résultat précédent dans le cas attractif des interactions Newtoniennes : il démontre un critère de non-explosion explicite sous lequel la compacité faible de la suite $(f^\varepsilon)_{\varepsilon>0}$ est valable globalement en temps.

La compacité forte et globale en temps de la suite $(f^\varepsilon)_{\varepsilon>0}$ fut ultérieurement obtenue par N. Masmoudi et M.L. Tayeb [165] dans le cas où l'opérateur de collision est de type Boltzmann linéarisé puis par N. El Ghani et N. Masmoudi [103] dans le cas de l'opérateur de Fokker-Planck. Ces résultats s'appuient sur les "lemmes de moyennes" introduits par F. Golse, P.- L. Lions, B. Perthame et R. Sentis puis développés par R. J. DiPerna, P.-L. Lions et Y. Meyer [122, 96], qui permettent d'obtenir la compacité forte des quantités macroscopiques associées aux solutions d'équations de transport. Les lemmes de moyenne sont valables pour des solutions dont seule l'énergie libre est finie, ce qui permet de s'affranchir de l'estimation de $|||f^\varepsilon|||_p$, valable uniquement sur des temps courts. Toutefois, considérer des solutions aussi peu régulières nécessite de donner un sens au terme non linéaire dans (8). À cette fin, les auteurs utilisent des techniques de renormalisation, initialement introduites par R. J. Diperna et P.-L. Lions pour démontrer l'existence globale de solutions à des systèmes de type Vlasov-Poisson [95]. Cette méthode a permis à N. El Ghani et N. Masmoudi de démontrer le résultat suivant

Théorème ([103])**.** *Fixons* $\mathbb{K} = \mathbb{R}$ *et considérons une suite* $(f^\varepsilon)_{\varepsilon>0}$ *de solutions de* (8) *dont les données initiales vérifient*

$$\sup_{\varepsilon>0} \mathcal{H}(f_0^\varepsilon, f_\infty) \,<\, +\infty\,.$$

Pour tout temps $T > 0$*, la suite* $(f^\varepsilon)_{\varepsilon>0}$ *vérifie*

$$f^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \rho\,\mathcal{M}\,, \quad \text{fortement dans} \quad L^1\left([0,T], L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)\right)\,,$$

où ρ *est solution de* (7)*.*

Les techniques développées dans [165, 103] se sont imposées dans la littérature. Leur robustesse a permis de généraliser l'analyse du régime parabolique dans le cas d'espèces multiples [212], du couplage avec les équations de Maxwell [102] ainsi que pour des plasmas fortement magnétisés [137].

Basée sur une méthode de compacité, cette approche a cependant pour inconvénient de ne pas fournir d'estimation d'erreur par rapport au paramètre d'échelle ε. Précisément, notre contribution consiste à mettre en œuvre des méthodes plus explicites en vue d'obtenir des taux de convergence. Ainsi, le Chapitre 1 de ce manuscrit est consacré à la preuve d'un résultat assurant la convergence **forte** de la solution f^ε de (8) avec des **taux** formellement optimaux et sans restriction sur la dimension d ; nous résumons ce résultat dans le théorème suivant

Théorème 2 ([22])**.** *Pour tout exposant* $p > d$ *posons*

$$\beta \,=\, \frac{p-d}{p-1}\,,$$

et considérons une suite $(f^\varepsilon)_{\varepsilon>0}$ *de solutions de* (8) *dont les données initiales vérifient*

$$\sup_{\varepsilon>0}\left(|||f_0^\varepsilon|||_p + \sup_{|\boldsymbol{x}_0|\,\leq\,1}\left(|\boldsymbol{x}_0|^{-\beta}\,|||\tau_{\boldsymbol{x}_0} f_0^\varepsilon \,-\, f_0^\varepsilon|||_p\right) \,+\, \varepsilon^{-\beta}\,\|\rho_0^\varepsilon - \rho_0\|_{L^p}\right) + \infty\,,$$

où $|||\cdot|||_p$ *est la norme introduite dans le Théorème 1. Pour tout temps* $T > 0$*, il existe des constantes* C_T *et* ε_T *telles que*

$$\left(\int_0^T |||f^\varepsilon(t) \,-\, \rho(t)\,\mathcal{M}|||_2^2\,\mathrm{d}t\right)^{1/2} \,\leq\, C_T\,\varepsilon^\beta\,, \qquad \forall\,\varepsilon\leq\varepsilon_T\,,$$

où la dépendance explicite des constantes C_T *et* ε_T *par rapport à la donnée initiale est détaillée dans le Théorème 1.1.*

Nous allons maintenant présenter les points clés de notre démonstration. Comme nous l'avons vu, l'estimation (9) assure la convergence de f^ε vers son équilibre local $\rho^\varepsilon \mathcal{M}$. L'enjeu de cette preuve est donc de montrer la convergence de la distribution macroscopique ρ^ε vers la distribution limite ρ solution de (7).
Notre approche est basée sur l'identification d'un jeu de coordonnées adapté au régime parabolique permettant de convertir les effets régularisants de l'équation de Poisson en des estimations d'erreur par rapport à ε.
On rappelle tout d'abord que l'inégalité (10) utilisée dans [192] n'est pas optimale puisque $\|\rho^\varepsilon\|_{L^p}$ permet de contrôler la régularité Hölder du champ électrique

$$\|\mathbf{E}^\varepsilon\|_{\mathscr{C}^{0,\gamma}} \leq C_p \|\rho^\varepsilon\|_{L^p}, \quad \forall p > d, \tag{12}$$

pour un certain exposant γ. Il convient de tirer avantage de ce gain de régularité. Pour cela, nous allons suivre une stratégie initialement développée pour étudier le modèle de FitzHugh-Nagumo (voir Section 0.3.7) et dont le principe est expliqué en début de Section 5.3. Dans notre cadre spécifique, cela consiste à introduire une densité spatiale modifiée π^ε définie par

$$\pi^\varepsilon(t, \boldsymbol{x}) = \int_{\mathbb{R}^d} f^\varepsilon(t, \boldsymbol{x} - \varepsilon \boldsymbol{v}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{v},$$

dont l'équation est en définitive déjà proche du modèle limite (7) ! En effet, en opérant le changement de variable $\boldsymbol{x} \to \boldsymbol{x} + \varepsilon \boldsymbol{v}$ dans la première ligne de (8) et en intégrant par rapport à la variable de vitesse $\boldsymbol{v} \in \mathbb{R}^d$, on vérifie que π^ε résout l'équation suivante

$$\partial_t \pi^\varepsilon + \nabla_{\boldsymbol{x}} \cdot \left[\int_{\mathbb{R}^d} (\mathbf{E}^\varepsilon f^\varepsilon)(t, \boldsymbol{x} - \varepsilon \boldsymbol{v}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \right] - \Delta_{\boldsymbol{x}} \pi^\varepsilon = 0.$$

Ainsi, en définissant le champ électrique $\mathbf{I}^\varepsilon$ associé à π^ε par

$$\mathbf{I}^\varepsilon = -\nabla_{\boldsymbol{x}} \psi^\varepsilon, \quad -\Delta_{\boldsymbol{x}} \psi^\varepsilon = \pi^\varepsilon - \rho_i,$$

on obtient l'estimée suivante de $|||f^\varepsilon|||_p$ à partir de celle dérivée par F. Poupaud et J. Soler (11)

$$\frac{\mathrm{d}}{\mathrm{d}t} |||f^\varepsilon|||_p^p \leq C_p \left(\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L^\infty}^2 + \|\mathbf{I}^\varepsilon - \mathbf{E}\|_{L^\infty}^2 + \|\mathbf{E}\|_{L^\infty}^2 \right) |||f^\varepsilon|||_p^p.$$

On s'appuie d'une part sur la régularisation Hölder (12) pour montrer que $\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L^\infty}^2$ est d'ordre ε^γ et d'autre part sur les propriétés du problème limite (7) pour estimer $\|\mathbf{I}^\varepsilon - \mathbf{E}\|_{L^\infty}$ et $\|\mathbf{E}\|_{L^\infty}$. On obtient ainsi une inégalité différentielle de la forme suivante

$$\frac{\mathrm{d}}{\mathrm{d}t} |||f^\varepsilon|||_p^p \leq C \left(\varepsilon^\gamma |||f^\varepsilon|||_p^2 + 1 \right) |||f^\varepsilon|||_p^p,$$

permettant de contrôler $|||f^\varepsilon|||_p$ sur des temps arbitrairement longs lorsque $\varepsilon \to 0$. En s'appuyant sur l'estimation de $|||f^\varepsilon|||_p$ ainsi que sur la régularisation Hölder (12), on propage de l'équicontinuité, ce qui permet d'obtenir des estimations d'erreur.

2.4 Étude numérique : champ appliqué (Chapitre 2)

L'objectif principal est ici de construire de nouveaux schémas numériques précis et robustes par rapport aux paramètres physiques et numériques. Puis, d'en faire l'analyse quantitative de stabilité. Nous proposons une première étude consacrée à une équation linéaire avec un champ appliqué. Par la suite, nous généraliserons certains de nos résultats au cas non linéaire. Le point de départ est donc l'équation de Vlasov-Fokker-Planck, obtenue à partir de (3) en supprimant le couplage avec l'équation de Poisson,

$$\partial_t f \,+\, \frac{1}{\varepsilon}\left(v\,\partial_x f \,+\, E_\infty\,\partial_v f\right) \,=\, \frac{1}{\tau(\varepsilon)}\,\partial_v\left(v\,f \,+\, T_0\,\partial_v f\right), \tag{13}$$

où les variables $(t, x, v) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}$, tandis que le champ E_∞ dérive du potentiel ϕ_∞, c'est-à-dire $E_\infty = -\partial_x \phi_\infty$. Comme expliqué précédemment, la limite parabolique correspondant au cas $\tau(\varepsilon) = \tau_0\,\varepsilon^2$ est donnée par

$$f(t,x,v) \underset{\varepsilon\to 0}{\longrightarrow} \rho_{\tau_0}(t,x)\,\mathcal{M}(v)\,,$$

où la distribution limite ρ_{τ_0} résout l'équation suivante

$$\partial_t \rho_{\tau_0} \,+\, \tau_0\,\partial_x\left(E_\infty\,\rho_{\tau_0} \,-\, T_0\,\partial_x\,\rho_{\tau_0}\right) = \,0\,, \tag{14}$$

tandis que le comportement en temps long correspondant au cas $\tau(\varepsilon) = \varepsilon$ est donné par

$$f(t,x,v) \underset{\frac{t}{\varepsilon}\to\infty}{\longrightarrow} f_\infty(x,v) \,:=\, \rho_\infty(x)\,\mathcal{M}(v)\,,$$

où

$$\mathcal{M}(v) \,=\, \frac{1}{\sqrt{2\pi\,T_0}}\,\exp\left(-\frac{|v|^2}{2\,T_0}\right),$$

tandis que ρ_∞ est donné par

$$\rho_\infty \,=\, c_0\,\exp\left(-\frac{\phi_\infty}{T_0}\right).$$

Comme nous l'avons souligné en Section 0.2.2, la difficulté de l'analyse mathématique de (13) réside dans l'interprétation des mécanismes d'irréversibilité microscopiques à l'échelle macroscopique. Les méthodes d'hypocoercivité introduites par C. Villani [211] et développées par de nombreux auteurs, notamment F. Hérau [134, 135], J. Dolbeault, C. Mouhot et C. Schmeiser [97], permettent de définir un cadre systématique dans lequel il est possible de résoudre ce problème. Ces méthodes sont notamment inspirées des idées développées par L. Desvillettes et C. Villani dans le cadre de leur étude sur la convergence en temps long de l'équation de Boltzmann [91, 92].
Par la suite, elles ont été adaptées à l'étude en temps long du modèle de Vlasov-Poisson-Fokker-Planck. F. Hérau et L. Thomann [136] ont démontré le retour exponentiel vers

l'état d'équilibre dans le régime de faible non-linéarité. Leur résultat a ensuite été généralisé par M. Herda et M. Rodrigues [138] qui ont obtenu des bornes explicites sur la taille de la non-linéarité ainsi que des taux de retour uniformes dans différents régimes collisionnels en dimension $d = 2$. Pour conclure, L. Addala, J. Dolbeault, X. Li et M.L. Tayeb [1] démontrent la convergence exponentielle vers l'équilibre uniformément dans la limite parabolique et sans restriction sur la dimension pour le système linéarisé. On mentionne aussi l'étude de H. J. Hwang et J. Jang [143], qui démontre la stabilité exponentielle de la Maxwellienne homogène dans l'espace entier $\mathbb{R}^3$ sans confinement externe.

Notre étude vise à adapter les méthodes d'hypocoercivité au cadre discret. Nous proposerons une méthode numérique dont nous prouverons qu'elle *préserve l'asymptotique*, c'est-à-dire qu'elle satisfait les mêmes propriétés que le modèle continu, uniformément par rapport aux paramètres d'échelle et de discrétisation. Notre approche s'inscrit dans le cadre des schémas *préservant l'asymptotique*, introduits au début des années 2000, notamment par S. Jin [146] et A. Klar [154]. Il s'agit ici de mener une analyse de stabilité précise, uniforme non seulement par rapport au paramètre d'échelle, mais aussi par rapport aux paramètres numériques et sans contrainte de type CFL.

Nous présenterons nos résultats de manière délibérément informelle pour ne garder que les idées principales de notre analyse. On notera respectivement f_h^n, ρ_{h,τ_0}^n et $f_{h,\infty}$ les approximations numériques des solutions de (13) et (14) ainsi que de l'équilibre f_∞ au temps $n\,\Delta t$, où $n \in \mathbb{N}$ désigne le temps discret tandis que $\Delta t > 0$ (resp. $h > 0$) désigne le paramètre de discrétisation de la variable temporelle t (resp. des variables (x, v)). Nous noterons $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^1}$, $\|\cdot\|_{H^{-1}}$ certaines normes de Sobolev L^2 à poids, nécessaires pour énoncer nos résultats. La construction de ces approximations ainsi que la définition des normes est détaillée au Chapitre 2, Section 2.2.

Notre premier résultat décrit le comportement en temps long de l'approximation f_h^n : nous démontrons son retour exponentiel vers l'état d'équilibre au taux $\tau(\varepsilon)/\varepsilon^2$ de manière uniforme par rapport aux paramètres de discrétisation Δt et h. Ce résultat permet de traiter numériquement le comportement en temps long de (13) dans tous les régimes collisionnels d'intérêt

$$\varepsilon^2 \lesssim \tau(\varepsilon) \lesssim \varepsilon\,.$$

Notons que le taux de retour obtenu est identique à celui du modèle continu. De plus, nous obtenons la convergence en norme H^1, ce qui permet de propager la régularité discrète des solutions numériques. Le résultat complet est détaillé dans le Théorème 2.7, dont nous énonçons ici une version simplifiée

Théorème 3 ([25])**.** *Sous l'hypothèse suivante sur* $\tau(\varepsilon)$

$$\sup_{\varepsilon>0} \frac{\tau(\varepsilon)}{\varepsilon} \;<\; +\infty\,,$$

nos approximations numériques $(f_h^n)_{n\in\mathbb{N}}$ *et* $f_{\infty,h}$ *vérifient pour tout* $n \in \mathbb{N}$

$$\|f_h^n - f_{\infty,h}\|_{L^2} \leq C \left\|f_h^0 - f_{\infty,h}\right\|_{L^2} \left(1 + \frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,\Delta t\right)^{-\frac{n}{2}} ,$$

ainsi que

$$\|f_h^n - f_{\infty,h}\|_{H^1} \leq C \left\|f_h^0 - f_{\infty,h}\right\|_{H^1} \left(1 + \frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,\Delta t\right)^{-\frac{n}{2}} ,$$

où les constantes C *et* κ *sont indépendantes de* $\varepsilon > 0$ *ainsi que des paramètres de discrétisation* $\Delta t, h > 0$.

Il est important de noter qu'ici le pas de temps Δt n'est contraint par aucune condition CFL ! En première étape, nous identifierons un cadre discret préservant les propriétés clés du modèle continu, telles que la conservation de la masse et l'estimée de stabilité L^2 suivante

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}\times\mathbb{R}} |f - f_\infty|^2 f_\infty^{-1} \mathrm{d}x\,\mathrm{d}v + \frac{T_0}{\tau(\varepsilon)}\int_{\mathbb{T}\times\mathbb{R}} \left|\partial_v\left(\frac{f}{f_\infty}\right)\right|^2 f_\infty\,\mathrm{d}x\,\mathrm{d}v = 0\,, \qquad (15)$$

vérifiée par les solution de (13).
La difficulté principale réside dans l'obtention de l'estimation (15), qui nécessite de faire des intégrations par partie difficilement reproductibles dans le cadre discret en raison du poids f_∞^{-1} qui apparaît dans (15). Pour surmonter cette difficulté, on effectue tout d'abord une décomposition spectrale en vitesse de la solution f de (13) dans la base des fonctions d'Hermite, qui constituent un système orthonormal de $L^2\left(\mathcal{M}^{-1}\right)$. On reformule ainsi (13) en un système hyperbolique sur les coefficients d'Hermite $C_k(t,x)$ de f, pour $k \in \mathbb{N}$. On procède alors à la renormalisation suivante des coefficients : $(D_k)_{k\in\mathbb{N}} = \left(C_k/\sqrt{\rho_\infty}\right)_{k\in\mathbb{N}}$. Sur les coefficients $(D_k)_{k\in\mathbb{N}}$, l'estimation (15) se traduit naturellement dans un cadre L^2 sans poids.
Le schéma numérique est obtenu en tronquant le système hyperbolique sur les coefficients $(D_k)_{k\in\mathbb{N}}$ ce qui permet d'assurer une précision spectrale par rapport à la variable de vitesse, et en utilisant une discrétisation de type volume fini pour la variable d'espace. Concernant la discrétisation en temps, on utilise une méthode d'Euler implicite pour simplifier l'analyse, même s'il est possible de généraliser notre étude à des méthodes implicites d'ordre élevé. Du point de vue computationnel, l'implicitation en temps est relativement peu coûteuse car le système à résoudre est autonome du fait que ϕ_∞ ne dépend pas du temps. Dans la pratique, on effectue une décomposition *LU* (méthode super *LU* [158]) de la matrice du système que l'on réutilisera par la suite à chaque pas de temps.
Le reste de l'analyse consiste à adapter au cadre discret les méthodes d'hypocoercivité, classiques dans le cadre continu. Ces méthodes se déploient généralement en deux étapes [211, 97, 135]. En premier lieu, identification d'une *entropie modifiée* [211, 97, 135] équivalente à la norme L^2 dissipée selon (15). En second lieu, preuve que cette modification permet de retrouver la dissipation par rapport à la variable d'espace x, qui n'est pas assurée par (15). Ces deux étapes reposent sur la propriété de *coercivité macroscopique* [97],

vérifiée à condition que l'espace $L^2\left(\rho_\infty^{-1}\right)$ soit muni d'une inégalité de Poincaré. Une des difficultés consiste donc à démontrer une telle inégalité dans le cadre discret.
Avant d'énoncer le résultat suivant, soulignons que la décomposition d'Hermite est largement utilisée dans le cadre de la théorie cinétique. On mentionne l'étude du modèle de Vlasov-Poisson [17, 18, 88, 89, 90] mais aussi celle des plasmas fortement magnétisés [66].

Notre second résultat porte sur les régimes macroscopiques du modèle (13) correspondant à sa limite lorsque $\varepsilon \to 0$. Le Théorème 2.8 permet de traiter tous les régimes collisionnels d'intérêt $\varepsilon^2 \lesssim \tau(\varepsilon) \lesssim \varepsilon$. Par souci de simplification, on ne traitera ici que régime parabolique correspondant à

$$\frac{\tau(\varepsilon)}{\varepsilon^2} = \tau_0 \in (0, +\infty)\,. \tag{16}$$

Notre résultat permet de vérifier quantitativement que notre solution numérique a le même comportement que le modèle continu dans la limite $\varepsilon \to 0$. En outre, nos estimations décrivent simultanément les limites $\varepsilon \to 0$ et $t \to +\infty$. Le diagramme présenté en fin de Section 0.2.2 est ainsi respecté par notre schéma numérique. Comme précédemment, nos résultats sont uniformes par rapport à tous les paramètres du problème : discrétisation Δt et h, paramètre d'échelle ε, temps discret $n \in \mathbb{N}$. L'énoncé suivant est une version simplifiée du Théorème 2.8

Théorème 4 ([25])**.** *Sous la condition* (16) *sur* $\tau(\varepsilon)$*, nos approximations* $(f_h^n)_{n\in\mathbb{N}}$ *et* $\left(\rho_{\tau_0,h}^n\right)_{n\in\mathbb{N}}$ *des solutions* f *et* ρ_{τ_0} *de* (13) *et* (14) *vérifient pour tout* $n \in \mathbb{N}$

$$\left\|f_h^n - \rho_{\tau_0,h}^n \mathcal{M}\right\|_{H^{-1}} \leq C\left(\left\|\int_{\mathbb{R}} f_h^0 \mathrm{d}v - \rho_{\tau_0,h}^0\right\|_{H^{-1}} + \varepsilon\left\|f_h^0 - \rho_{\tau_0,h}^0\mathcal{M}\right\|_{H^1}\right)(1+\tau_0\kappa\Delta t)^{-\frac{n}{2}}\,,$$

$$C\left\|f_h^0 - \int_{\mathbb{R}} f_h^0 \mathrm{d}v \mathcal{M}\right\|_{L^2}\left(1 + \frac{\Delta t}{2\,\varepsilon^2}\right)^{-\frac{n}{2}}\,,$$

où les constantes C *et* κ *sont indépendantes de* $\varepsilon > 0$ *ainsi que des paramètres de discrétisation* $\Delta t, h > 0$.

Encore une fois, soulignons qu'ici le pas de temps n'est pas soumis à une condition de type CFL.

Pour démontrer ce résultat, on introduit une entropie modifiée comme pour la preuve du Théorème 3. Cependant, une difficulté majeure réside ici dans le fait que l'entropie modifiée adaptée à ce régime comprend des dérivées de la solution. Cette contrainte justifie la deuxième estimation du Théorème 3, dans laquelle on montre un résultat de convergence en norme H^1.

Pour conclure notre démarche, nous mettons en place des expériences numériques variées permettant non seulement d'illustrer nos résultats et de tester leur optimalité mais aussi d'observer des phénomènes complexes qui vont au-delà de notre analyse. Ces tests numériques sont détaillés en Section 2.4 du Chapitre 2. On y observe la transition entre régime macroscopique et temps long prévue par notre analyse (voir Section 2.4.2), ainsi que l'apparition de phénomènes oscillatoires lorsque le champ appliqué n'est pas uniforme (voir Section 2.4.1).

2.5 Étude numérique : champ auto-consistant (Chapitre 3)

Il s'agit maintenant de proposer un nouveau cadre, inspiré du précédent, pour traiter le système non linéaire de Vlasov-Poisson-Fokker-Planck (3). On construira un schéma numérique basé sur un "splitting" astucieux. Il s'agira d'abord de résoudre le système de Vlasov-Poisson-Fokker-Planck linéarisé autour de son équilibre, ce qui permettra de capturer correctement son asymptotique. En deuxième étape on prendra en compte les termes non linéaires. Grâce à ce schéma de discrétisation, on mènera des expériences numériques visant à illustrer plusieurs phénomènes complexes. En particulier, on s'intéressera à l'écho plasma, ayant récemment attiré l'attention de la communauté mathématique [10, 9, 11, 128, 68], et dont nous étudierons la stabilité dans des régimes collisionnels variés.

Dans une partie plus théorique, on démontrera un résultat en temps long pour le système de Vlasov-Poisson-Fokker-Planck linéarisé en s'inspirant des techniques développées dans la section précédente. Cette étude pose les bases pour traiter le modèle non linéaire complet.

Dans cette section, nous considérons le régime temps long $\tau(\varepsilon) = \tau_0\,\varepsilon$ et nous traitons le cas de la dimension $d = 1$. Ainsi, en posant $\psi = \phi - \phi_\infty$, où ϕ_∞ est défini par (5), le système de Vlasov-Poisson-Fokker-Planck (3) sur f^ε s'écrit (en omettant les indices ε)

$$\begin{cases} \varepsilon\,\partial_t f \,+\, v\,\partial_x f \,-\, \partial_x\phi_\infty\,\partial_v f \,-\, \partial_x\psi\,\partial_v f \,=\, \dfrac{1}{\tau_0}\partial_v\left(v\,f \,+\, T_0\,\partial_v f\right)\,, \\ -\partial_x^2\psi \,=\, \rho - \rho_\infty\,, \quad \rho(t,x) \,=\, \displaystyle\int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v\,, \end{cases} \tag{17}$$

où $(t,x,v) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}$, complété par la condition de moyenne nulle sur ψ

$$\int_{\mathbb{T}} \psi(t,x)\,\mathrm{d}x \,=\, 0\,.$$

On applique alors une méthode de "splitting" en deux étapes. Durant la première, on résout le schéma numérique pour l'équation linéarisée autour de l'équilibre global,

$$\begin{cases} \varepsilon\,\partial_t f \,+\, v\,\partial_x f \,-\, \partial_x\phi_\infty\,\partial_v f \,-\, \partial_x\psi\,\partial_v f_\infty \,=\, \dfrac{1}{\tau_0}\partial_v\left(v\,f \,+\, T_0\,\partial_v f\right)\,, \\ -\partial_x^2\psi \,=\, \rho - \rho_\infty\,, \quad \rho(t,x) \,=\, \displaystyle\int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v\,, \end{cases} \tag{18}$$

où $(t,x,v) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}$, complété par la condition de moyenne nulle sur ψ

$$\int_{\mathbb{T}} \psi(t,x)\,\mathrm{d}x \,=\, 0\,.$$

La deuxième étape du splitting consiste alors en la résolution implicite en temps du terme non linéaire restant dans l'équation (17) sur f

$$\varepsilon\, \partial_t f \;-\; \partial_x \psi\, \partial_v (f - f_\infty) \;=\; 0\,,$$

durant laquelle ψ ne varie pas au court du temps puisque la densité ρ reste inchangée. Le coût computationnel de cette méthode est discuté dans la Section 3.1 ; on précise cependant que l'implicitation de la dernière étape est négligeable en termes de coût car le système à résoudre est trivialement inversible du fait que le potentiel ψ n'est pas mis à jour.

Notre premier résultat concerne le comportement en temps long du modèle (17) linéarisé autour de l'équilibre f_∞ défini par (5). Comme plus haut, on définit f_h^n, E_h^n et $f_{h,\infty}$ les approximations numériques respectives de la distribution f, du champ $-\partial_x\psi$ solutions de (18) au temps $n\,\Delta t$, ainsi que de l'équilibre f_∞, où $n \in \mathbb{N}$ désigne le temps discret tandis que $\Delta t > 0$ (resp. $h > 0$) décrit le paramètre de discrétisation de la variable temporelle t (resp. des variables (x, v)). Nous notons $\|\cdot\|_{L^2}$ la norme discrète L^2 avec le poids f_∞^{-1}. La construction de ces approximations comme celle de la norme sont détaillées au Chapitre 3, Section 3.3.
Notre résultat décrit le comportement en temps long de l'approximation f_h^n, nous démontrons son retour exponentiel vers l'état d'équilibre au taux $1/\varepsilon$ de manière uniforme par rapport aux paramètres de discrétisation Δt et h. Le résultat complet est détaillé dans le Théorème 3.2, dont nous énonçons ici une version simplifiée

Théorème 5. *Les approximations numériques $(f_h^n)_{n\in\mathbb{N}}$ et $f_{\infty,h}$ vérifient pour tout $n \in \mathbb{N}$*

$$\mathcal{E}^n \;\leq\; 3\,\mathcal{E}^0 \left(1 + \frac{\kappa}{\varepsilon}\,\Delta t\right)^{-n}\,,$$

où la constante κ est indépendante de $\varepsilon > 0$ ainsi que des paramètres de discrétisation $\Delta t, h > 0$ tandis que l'énergie linéarisée est définie par

$$\mathcal{E}^n \;=\; \frac{1}{2}\left(\|f_h^n - f_{\infty,h}\|_{L^2}^2 + \frac{1}{T_0}\;\|E_h^n\|_{L^2(\mathbb{T})}^2\right)\,,$$

où $L^2\,(\mathbb{T})$ désigne la norme L^2 discrète sur le tore.

Pour démontrer ce résultat, l'approche est exactement la même que celle proposée dans la preuve du Théorème 3 de la section précédente. Cependant, comme on peut le voir en comparant le système (18) à l'équation de Vlasov-Fokker-Planck (13), il faut maintenant prendre en compte la discrétisation du champ électrique $-\partial_x\psi$. Sa contribution génère désormais une nouvelle estimation d'énergie exprimée par

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}\times\mathbb{R}} |f - f_\infty|^2\, f_\infty^{-1} \mathrm{d}x\,\mathrm{d}v \;+\; \frac{1}{T_0}\,\|\,\partial_x\psi\,\|_{L^2(\mathbb{T})}^2\right) \;=\; -\frac{T_0}{\tau_0\varepsilon}\int_{\mathbb{T}\times\mathbb{R}}\left|\partial_v\left(\frac{f}{f_\infty}\right)\right|^2 f_\infty\,\mathrm{d}x\,\mathrm{d}v\,.$$

Comme précédemment, le point clé consiste à discrétiser (18) de manière à préserver cette dernière estimation. La discussion complète à ce sujet est détaillée dans la Section 3.2.2 du Chapitre 3 ; il est possible de la résumer lorsque l'équilibre ρ_∞ est homogène en x puisque dans ce cas, conserver l'estimation d'énergie revient essentiellement à assurer que E_h^n vérifie une équation d'Ampère discrète.

La seconde partie de ce travail est dédiée à la simulation numérique du modèle non linéaire complet (17). Nos simulations sont présentées dans la Section 3.4 du Chapitre 3. Nous attirons en particulier l'attention du lecteur sur la plus notable, dédiée aux échos plasmas. Notre objectif est d'illustrer numériquement des travaux théoriques récents consacrés à ces phénomènes [10, 9, 11, 128, 68]. Dans un plasma, l'écho est un phénomène de résonance conduisant à une croissance soudaine et intense de l'énergie potentielle du système. Ce phénomène contrevient à l'amortissement Landau qui prédit la décroissance exponentielle de l'intensité du champ électrique pour des données proches de l'équilibre. Ce type de situation se produit également lorsque l'on considère des plasmas magnétisés [67]. Nos expériences mettent en lumière la complexité des échos plasmas. D'une part, la comparaison des solutions aux problèmes linéarisé et non linéaire fait apparaître que l'écho est un processus non linéaire. D'autre part, on capture la répétition des échos au cours du temps contrairement aux études numériques précédentes [164, 110]. Pour finir, on étudie la suppression des échos par les collisions, ce qui permet de faire le lien avec les travaux récents sur le sujet [10, 68].

3 Neurosciences : panorama et contributions

Dans la section précédente, nous avons utilisé les outils de la physique statistique développés par Lord Kelvin, J.C. Maxwell et L. Boltzmann pour étudier un modèle classique en théorie cinétique. Dans cette section, nous verrons qu'il est possible d'appliquer ce formalisme à des systèmes issus des neurosciences.

Cette partie est donc dédiée à la modélisation et à l'analyse quantitative de réseaux de neurones biologiques. Notre approche relève des neurosciences théoriques, domaine devenu central en neurosciences du fait qu'il permet d'expliquer et de prédire comment une population de neurones se coordonne pour accomplir des tâches. Nous verrons comment, à partir de la description des neurones à l'échelle microscopique, il est possible d'obtenir des modèles macroscopiques décrivant la dynamique d'une large assemblée de neurones en interactions.

3.1 Phénoménologie de la dynamique d'un neurone

Comme évoqué plus haut, Lord Kelvin, J.C. Maxwell et L. Boltzmann ont utilisé la loi de Newton pour décrire un gaz à l'échelle microscopique. De manière analogue, nous utiliserons le modèle de Hodgkin et Huxley qui régit la dynamique de chaque neurone,

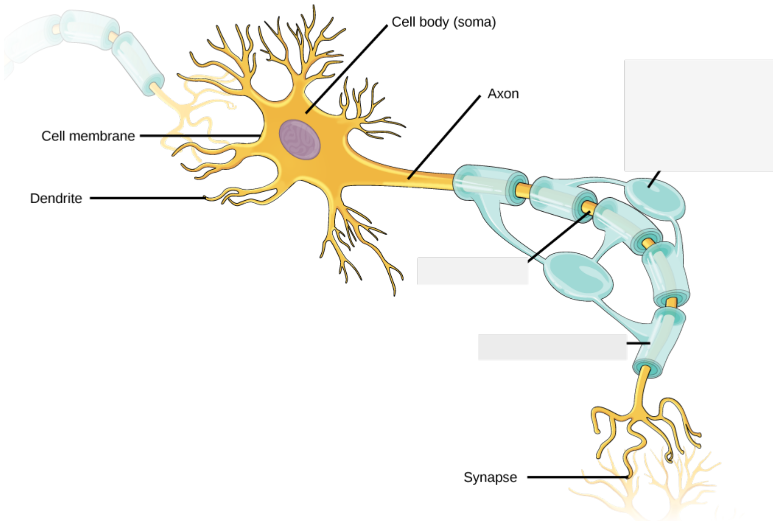


FIGURE 1 – Représentation de l'anatomie d'un neurone

dans l'optique de construire un modèle statistique décrivant un réseau.
Nous discuterons de l'élaboration du modèle de Hodgkin et Huxley en commençant par la phénoménologie du comportement des neurones. Pour cela, nous nous appuierons sur les ouvrages [38, 84, 119, 149, 207] dans lesquels le lecteur pourra trouver une introduction complète.

Morphologie d'un neurone.

D'un point de vue biologique, les neurones sont des cellules qui se singularisent par leur capacité à émettre et recevoir de l'information *via* des signaux électro-chimiques. La morphologie même des neurones témoigne de cette aptitude particulière : outre son corps cellulaire, appelé soma et au sein duquel est stockée l'ADN, un neurone se distingue par deux composantes remarquables, représentées sur la Figure 1 :

(i) une structure branchante entrante, appelée *arbre dendritique*, permettant au neurone de recevoir les signaux émis par les autres neurones ;

(ii) un structure branchante sortante appelée *axone*, permettant au neurone d'émettre des signaux.

Bien que dotés de composantes communes, les neurones ont des morphologies très variées, comme illustré par la Figure 2, sur laquelle sont représentées trois types de neurones différents. Ainsi, l'arbre dendritique des neurones est d'une complexité variable : pour donner un ordre d'idée, il peut établir plus de 10^5 connexions avec les axones d'autres neurones, comme c'est le cas des cellules de Purkinje, représentées sur la Figure 2. De plus, les axones peuvent être très longs par rapport à la taille de la cellule : par exemple

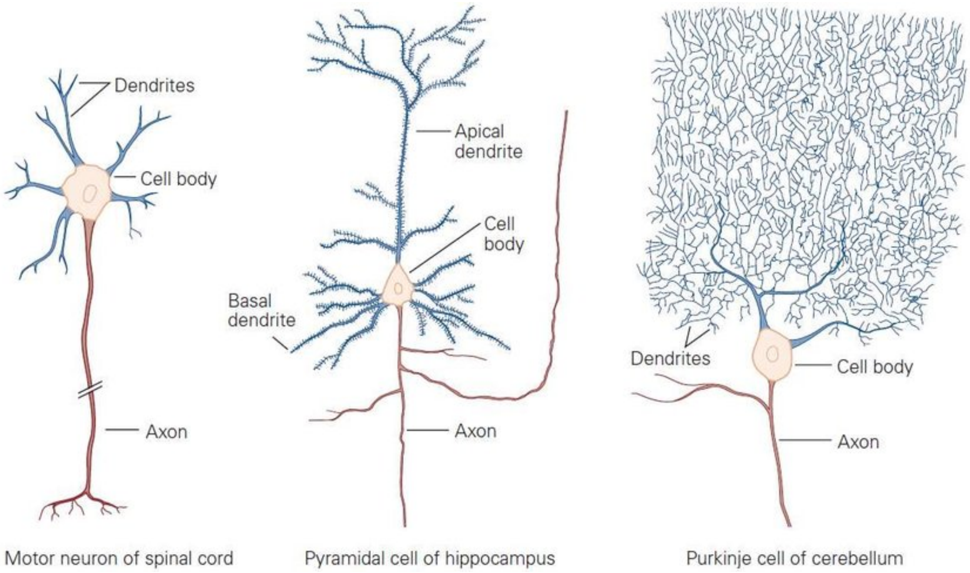


FIGURE 2 – Représentation de trois familles de neurones [149]

les axones présents dans le cerveau d'une souris mesurent en moyenne 40 mm tandis que le diamètre typique d'un soma varie entre 10 et 50 μm.

Potentiel de membrane et activité électrique.

L'activité d'un neurone peut être mesurée en fonction de la tension induite par la différence de potentiel électrique entre l'intérieur du neurone et son environnement. Cette quantité est appelée *potentiel de membrane* du neurone.
On distingue deux mécanismes permettant d'agir sur le potentiel de membrane :

1. les *canaux à ions*, protéines présentes sur la membrane d'un neurone, ont la capacité de reconnaître et de sélectionner une espèce particulière parmi les ions sodium ($Na+$), potassium ($K+$), calcium ($Ca2+$), et chlorure ($Cl-$), ainsi que de s'ouvrir en réponse à un signal, qu'il soit électrique, mécanique ou chimique. Leur ouverture permet le passage de l'espèce d'ion sélectionnée au travers de la membrane cellulaire, entraînant ainsi une variation du potentiel membranaire ;

2. les *synapses* assurent la jonction entre terminaison axonale et arbre dendritique de deux neurones. Elles transmettent le signal électrique d'un neurone à l'autre de manière directe dans le cas de synapses "électriques" ou de manière indirecte dans le cas de synapses "chimiques", *via* des molécules appelées neurotransmetteurs.

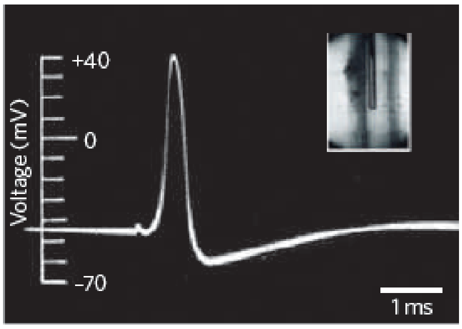


FIGURE 3 – Première mesure intracellulaire du potentiel d'action [139]

Phénoménologie du potentiel de membrane.

Lorsqu'un neurone est au repos, son potentiel de membrane est généralement maintenu entre $-60\,mV$ et $-70\,mV$ grâce à des protéines appelées "pompes à ions" : on dit que le neurone est polarisé. Cependant, suite à une entrée d'ions chargés positivement, le potentiel peut augmenter jusqu'à atteindre des valeurs positives : on dit alors que le neurone est dépolarisé. Lorsque le potentiel de membrane atteint une valeur seuil, il subit un pic intense, de l'ordre de $100\,mV$, et d'une durée de l'ordre de la milli-seconde : c'est le *potentiel d'action*. Ce potentiel d'action fut mesuré pour la première fois par à la fin des années 1930 par Hodgkin et Huxley [139]. Pour cela, ils appliquèrent un courant extérieur à l'axone d'un calamar géant (d'environ $0,5\,mm$ de diamètre et contrôlant une partie du système de projection d'eau du calamar) et mesurèrent sa réponse. Ces résultats sont reportés sur la Figure 3.

3.2 Modélisation mathématique de la dynamique d'un neurone

En s'appuyant sur leurs résultats expérimentaux, Hodgkin et Huxley construisirent un modèle capable de reproduire la dynamique du potentiel de membrane et prenant en compte les mécanismes présentés plus haut (échanges ioniques, polarisation/dépolarisation, potentiel d'action...). Ils publièrent leur résultat [140] dans les années 1950 et se virent décerner le prix Nobel en 1963 pour leurs travaux pionniers. Dans cette section, nous présenterons leur modèle puis nous mènerons une étude numérique permettant d'illustrer ses caractéristiques principales.

Derivation du modèle de Hodgkin et Huxley.

Nous prendrons comme point de départ l'équation de conservation de la charge, à laquelle est soumis le potentiel de membrane $v(t)$ d'un neurone

$$C\,\frac{\mathrm{d}}{\mathrm{d}t}\,v(t) \;=\; -\,I_{con} \,+\, I_{ext}\,,$$

où C est la capacité électrique de la cellule par unité de surface, I_{con} représente le courant traversant sa membrane par unité de surface, et I_{ext} mesure le courant appliqué lors de l'expérience divisé par la surface totale de la cellule. Du point de vue mathématique, cette discussion a donc pour but d'obtenir un système fermé décrivant l'évolution de la quantité $v(t)$. Pour cela, on détaillera l'expression du terme I_{con} de manière à rendre compte de l'effet des échanges entre le neurone et le milieu extra-cellulaire. Comme évoqué plus haut, ces échanges sont contrôlés par les canaux ioniques. On peut donc décomposer I_{con} de la manière suivante

$$I_{con} = \sum_s g_s \left(v(t) - V_s\right) \, ,$$

où l'indice s parcourt l'ensemble des espèces d'ions sélectionnées par les canaux. Dans le modèle de Hodgkin et Huxley, on considère les canaux associés aux ions sodium et potassium ainsi qu'une espèce fictive notée l dont nous expliquerons le rôle par la suite

$$s \in \{K+, Na+, l\} \; .$$

Les potentiels d'inversion V_s correspondent aux valeurs pour lesquelles les forces électriques et de diffusion induites par les gradients de concentrations de l'espèce s entre l'extérieur et l'intérieur du neurone se compensent. Ces potentiels sont déterminés par la loi de Nernst (voir [84, 149] pour une discussion précise). Leurs valeurs, calculées par Hodgkin et Huxley [140], sont $V_K \sim 12\, mV$, $V_{Na} \sim -115\, mV$; on notera que ces deux espèces jouent des rôles opposés (polarisateur dans le cas de l'ion potassium $K+$, dépolarisateur pour l'ion sodium $Na+$).
Les coefficients g_s décrivent la conductance totale des canaux sélectionnant l'espèce ionique s renormalisée par la surface totale de la cellule. De manière intuitive, ils décrivent si les canaux associés à l'espèce s sont ouverts ou fermés. Le travail pionnier de Hodgkin et Huxley fut de déterminer la loi de dépendance reliant les conductances g_s au potentiel de membrane $v(t)$. Avant d'exposer leurs arguments, nous soulignons qu'individuellement, les canaux à ions n'ont pas un comportement purement déterministe. Cependant, ces fluctuations stochastiques peuvent être négligées lorsque l'on considère un nombre suffisamment important de canaux à ions indépendants. Dans cette approximation, la conductance g_s des canaux sélectionnant l'espèce s est donnée par

$$g_s = \bar{g}_s \, P_s \, ,$$

où $\bar{g}_s$ décrit la conductance surfacique d'un canal à ions s ouvert tandis que P_s mesure la proportion de canaux s ouverts. On considère alors que la probabilité d'ouverture de chaque canal est conditionnée par trois évènements indépendants de probabilité $\{m, n, h\} \in [0, 1]$. Hodgkin et Huxley mesurèrent la loi de dépendance liant P_s à m, n, h et aboutirent aux expressions suivantes

$$P_K = n^4 \, , \quad P_{Na} = m^3 h \, .$$

A partir de ces lois, ils proposèrent le modèle suivant

$$\begin{cases} C\,\dfrac{\mathrm{d}}{\mathrm{d}t}\,v + \bar{g}_K\,n^4\,(v - V_K) \,+\, \bar{g}_{Na}\,m^3 h\,(v - V_{Na}) \,+\, \bar{g}_l\,(v - V_l) = \,I_{ext}\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\,n \,=\, \alpha_n(v)\,(1-n) \,-\, \beta_n(v)\,n\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\,m \,=\, \alpha_m(v)\,(1-m) \,-\, \beta_m(v)\,m\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\,h \,=\, \alpha_h(v)\,(1-h) \,-\, \beta_h(v)\,h\,. \end{cases} \tag{19}$$

La conductance g_l est supposée indépendante du potentiel de membrane : les ions fictifs de cette espèce prennent en compte certains effets polarisateurs, tel que l'activité des pompes à ions qui maintiennent les gradients de concentration et ramènent le neurone au repos. Pour finir, les probabilités $X \in \{m, n, h\}$ suivent des lois exponentielles paramétrées par les coefficients (α_X, β_X), eux-mêmes déterminés de manière empirique (voir [84, 140]). On donnera les expressions originales des coefficients (α_X, β_X), devenues classiques en neurosciences [114, 132, 174, 200, 207]

$$\begin{cases} \alpha_n(v) \,=\, \dfrac{(v+10)\,/100}{\exp\left(\left(v+10\right)/10\right)-1}\,, \text{ et} \quad \beta_n(v) \,=\, \exp\left(v/80\right)/8\,, \\ \alpha_m(v) \,=\, \dfrac{(v+25)\,/10}{\exp\left(\left(v+25\right)/10\right)-1}\,, \text{ et} \quad \beta_m(v) \,=\, 4\,\exp\left(v/18\right), \\ \alpha_h(v) \,=\, 7\,\exp\left(v/20\right)/100\,, \text{ et} \quad \beta_h(v) \,=\, \dfrac{1}{1\,+\,\exp\left(\left(v+30\right)/10\right)}\,. \end{cases} \tag{20}$$

Précisons que les expressions des coefficients α_X, β_X dépendent de la température. Ainsi, les expressions données dans (20) sont valables lorsque la température est de $6,3°C$.

Comportement qualitatif du modèle de Hodgkin et Huxley

Nous allons maintenant illustrer numériquement les caractéristiques principales du modèle de Hodgkin et Huxley (19)-(20).
Nos simulations permettent d'observer les comportements du modèle lorsque l'on fait varier le courant appliqué I_{ext}. Nous conserverons les autres paramètres du modèle constants dans nos simulations.

Avant de présenter nos résultats numériques, nous allons détailler les paramètres utilisés dans nos simulations. Nous utiliserons les valeurs originales des paramètres dans la première ligne de (19)

$$C \,=\, 1\,,\; V_{Na} \,=\, -115\,,\; V_K \,=\, 12\,,\; \bar{g}_{Na} \,=\, 120\,,\; \bar{g}_K \,=\, 36\,,\; \bar{g}_l \,=\, 0.3\,,$$

où la capacité est donnée en micro-faraday par centimètre carré, les tensions en millivolts et les coefficients $\bar{g}_X$ en milli-siemens par centimètre carré. La valeur de V_l est calculée à posteriori de manière à assurer que l'état $v(t) = 0$ soit un état équilibre pour (19). Pour cela, on ré-écrit les trois dernières équations de (19) comme suit

$$\tau_X(v)\,\frac{\mathrm{d}}{\mathrm{d}t}X(t) \;=\; -\left(X(t) \;-\; X_\infty(v)\right)\,, \tag{21}$$

où

$$X_\infty(v) \;:=\; \frac{\alpha_X(v)}{\alpha_X(v)\;+\;\beta_X(v)}\,,\;\; et \quad \tau_X(v) \;:=\; \frac{1}{\alpha_X(v)\;+\;\beta_X(v)}\,,$$

et on fixe V_l par l'expression suivante

$$V_l \;=\; -\frac{1}{\bar{g}_l}\left(\bar{g}_K\, n_\infty^4(0)\, V_K \;+\; \bar{g}_{Na}\, m_\infty^3(0)\, h_\infty(0)\, V_{Na}\right)\,.$$

Avec les paramètres indiqués ci-dessus, on trouve $V_l \;\sim\; -10.5\,mV$, une valeur proche de celle proposée par Hodgkin et Huxley [140]. Nous prendrons pour donnée initiale l'équilibre $(V_0, n_0, m_0, h_0) \;:=\; (0, n_\infty(0), m_\infty(0), h_\infty(0))$.

Nous allons maintenant présenter nos résultats.
Pour commencer, le modèle permet de retrouver l'effet de seuil observé expérimentalement. Lorsque le courant I_{ext} injecté est faible, le potentiel de membrane reste proche de son équilibre ; cependant lorsque I_{ext} atteint une valeur seuil, un potentiel d'action apparaît. Ceci est illustré Figure 4 qui représente l'évolution temporelle du potentiel de membrane pour un courant appliqué constant variant entre 0 et $2.3\,\mu A/cm^2$. L'étude numérique et théorique de cet effet de seuil fut initiée dans l'article fondateur de Hodgkin et Huxley et poursuivie au cours de la décennie suivante [74, 75, 115, 202, 207].
Une autre caractéristique du modèle s'observe lorsque l'on augmente le courant appliqué. À partir d'un nouveau seuil, on observe l'apparition de solutions périodiques, appelées "potentiels d'action répétés" dans la littérature. Nous avons illustré ce phénomène sur la Figure 5 qui représente l'évolution temporelle du potentiel de membrane pour un courant appliqué constant variant entre 5.9 et $6.3\,\mu A/cm^2$. Cet autre effet de seuil, qui n'avait pas été mesuré expérimentalement, a aussi fait l'objet d'un nombre important de travaux [74, 75, 115, 132, 200, 203].
Pour finir, on peut observer un dernier effet de seuil si l'on continue d'augmenter le courant appliqué. Lorsque celui-ci est très grand, les oscillations rétrécissent jusqu'à disparaître. Nous avons illustré ce phénomène sur la Figure 6 qui représente l'évolution temporelle du potentiel de membrane pour un courant appliqué constant variant entre 100 et $160\,\mu A/cm^2$. Dans ce cas, on observe numériquement l'apparition d'un nouvel équilibre qui semble stable.
Pour conclure, nous soulignons que nous avons traité ici un modèle de câble pour l'axon du calamar géant dans le sens où nous avons considéré que le potentiel de membrane

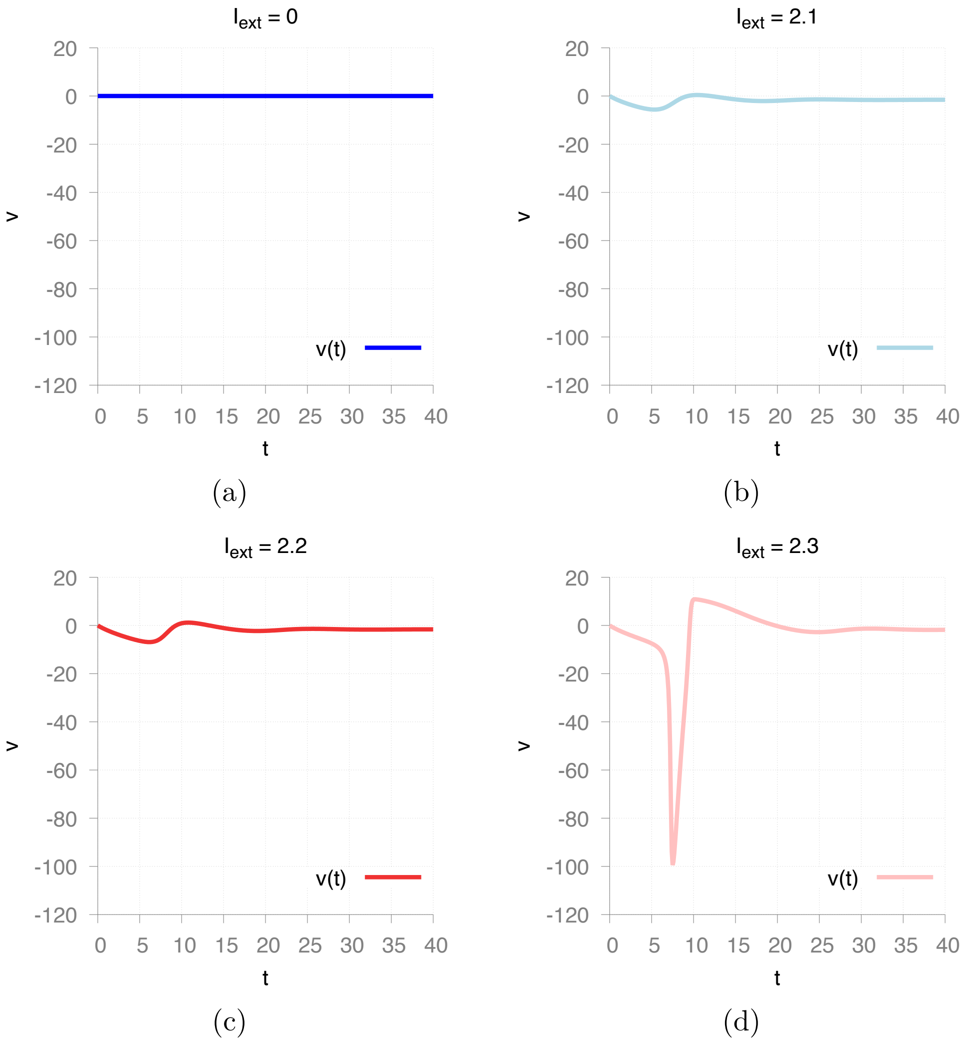


FIGURE 4 – **Simulation du modèle de Hodgkin-Huxley.** Variation temporelle du potentiel de membrane en réponse à un courant (exprimé en $\mu A/cm^2$) appliqué de (a) 0, (b) 2.1, (c) 2.2 et (d) 2.3.

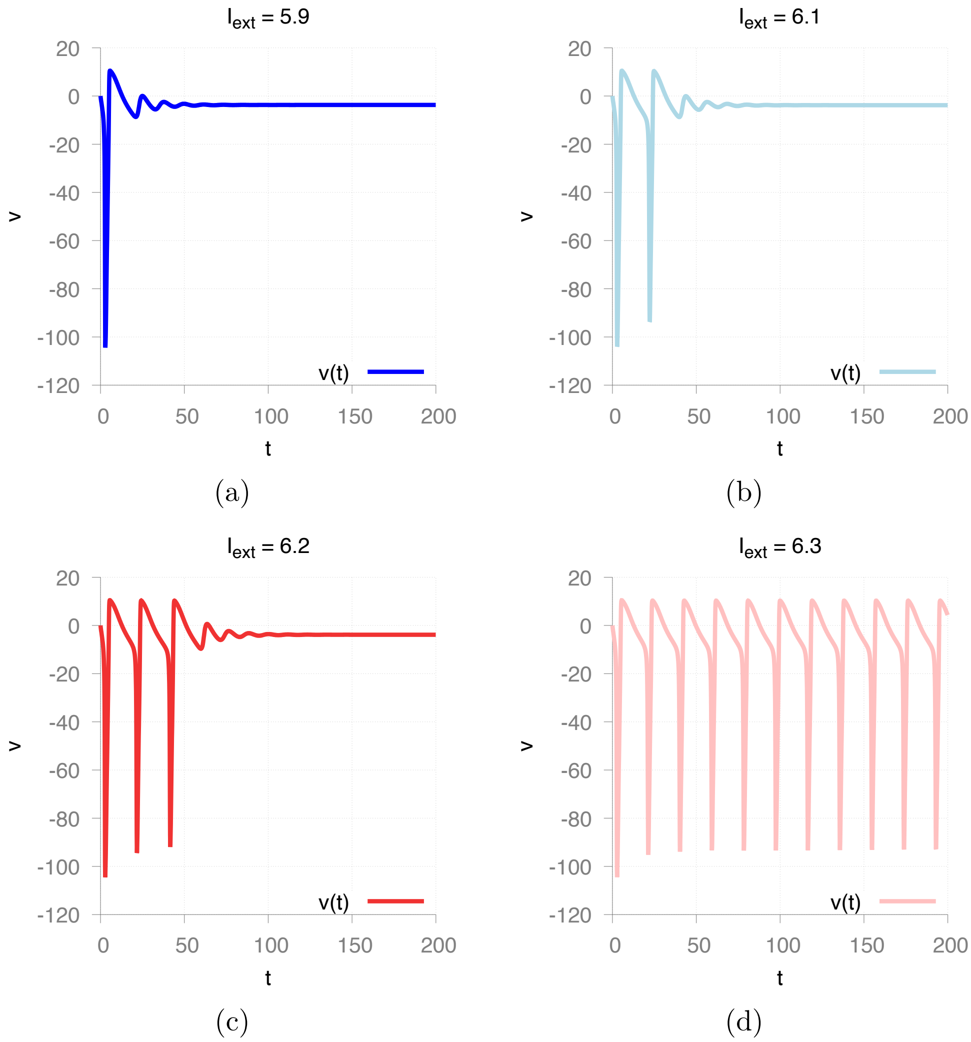


FIGURE 5 – **Simulation du modèle de Hodgkin-Huxley.** Variation temporelle du potentiel de membrane en réponse à un courant (exprimé en $\mu A/cm^2$) appliqué de (a) 5.9, (b) 6.1, (c) 6.2 et (d) 6.3.

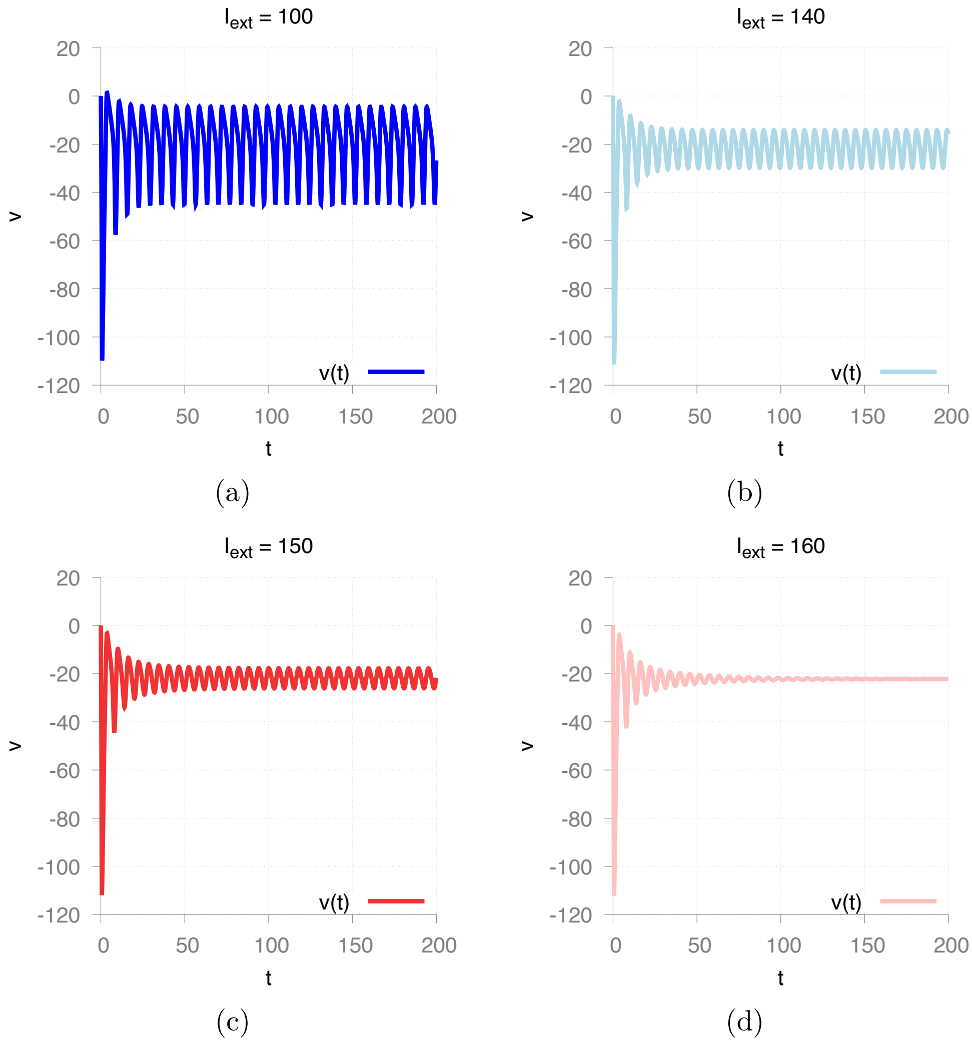


FIGURE 6 – **Simulation du modèle de Hodgkin-Huxley.** Variation temporelle du potentiel de membrane en réponse à un courant (exprimé en $\mu A/cm^2$) appliqué de (a) 100, (b) 140, (c) 150 et (d) 160.

est constant le long de l'axone. Plusieurs travaux se sont penchés sur la propagation du potentiel d'action le long de l'axone lorsque cette hypothèse simplificatrice est levée [38, 140, 207].

3.3 Modèles à potentiel d'action forcé

Le modèle de Hodgkin et Huxley fait référence en neurosciences car il propose une vision simple des mécanismes de la dynamique du potentiel de membrane. Cependant, l'analyse du système (19)-(20) est complexe en raison de sa taille mais aussi de son caractère très non linéaire. Ainsi, ce modèle n'est pas toujours adapté, en particulier lorsque l'on souhaite étudier un grand nombre de neurones en interaction. C'est pourquoi de nombreux modèles plus ou moins simplifiés mais conservant les caractéristiques qualitatives principales de la dynamique du potentiel de membrane ont été proposés dans la littérature. Dans cette section, nous présenterons une famille de tels modèles, appelés à "potentiel d'action forcé", et nous dresserons un panorama de la littérature existante à leur sujet.

Principe général du potentiel d'action forcé

Malgré sa complexité, le modèle de Hodgkin et Huxley (19) a un comportement stéréotypé. En effet, l'allure du pic lors d'un potentiel d'action est relativement indépendante des paramètres de l'expérience, que ce soit en termes de durée ($\sim\ 10\,ms$), d'intensité ($\sim\ 100\,mV$), de potentiel seuil ($V_F \sim -10\,mV$) ou de potentiel au repos ($V_R \sim 0\,mV$), comme illustré par les Figures 4 et 5. Cela suggère que l'on peut résumer l'activité d'un neurone i à la donnée unique des temps auxquels les potentiels d'action se produisent : lorsque le potentiel de membrane est inférieur au potentiel seuil V_F, c'est-à-dire sous la condition $v^i(t) < V_F$, sa dynamique est décrite par une équation différentielle d'ordre 1 tandis que lorsque le neurone atteint le potentiel seuil V_F au temps τ, c'est-à-dire sous la condition $v^i(\tau^-) = V_F$, on néglige la durée du potentiel d'action et le neurone est instantanément réinitialisé à sa valeur au repos $v^i(\tau) = V_R$. On aboutit ainsi au système générique suivant, prescrivant l'évolution du potentiel de membrane

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\, v^i(t) \;=\; b^i(t, v^i(t))\,, \quad \text{lorsque} \quad t \neq \tau_k^i\,, \\[2ex] v^i(\tau_k^i) \;=\; V_R\,, \quad \text{lorsque} \quad k \geq 1\,, \end{cases} \tag{22}$$

où la suite $\left(\tau_k^i\right)_{k\in\mathbb{N}}$ des temps de décharge est définie de la manière suivante : $\tau_0^i \;=\; 0$ et

$$\tau_{k+1}^i \;=\; \inf\left\{\quad t \in \mathbb{R} \quad \text{tels que} \quad t > \tau_k^i \quad \text{et} \quad \lim_{\substack{s \to t \\ <}} v^i(s) \;=\; V_F \right\}.$$

Ce type de modèle est très populaire en neurosciences puisqu'il permet d'écarter la dynamique complexe du potentiel d'action pour se concentrer uniquement sur les interactions entre neurones. Le reste de cette section est dédié à la présentation de trois modèles construits à partir du potentiel d'action forcé.

Neurones intègre et tire

Dans cette section, nous introduirons le modèle qui est probablement le plus populaire en neurosciences du fait de sa simplicité mais aussi de la grande variétés des phénomènes capturés. On considère une assemblée de m neurones (numérotés de 1 à m) décrits par la donnée unique de leur potentiel de membrane dont la dynamique est prescrite par l'équation (22), avec

$$b^i(t, v^i(t)) \;=\; -\,v^i(t) \;+\; \frac{\alpha}{m}\sum_{j=1}^{m}\sum_{l\in\mathbb{N}^*}\delta_0\left(t-\tau_l^j\right) \;+\; \sqrt{2}\,\eta_i(t)\,, \tag{23}$$

pour $i \in \{1, \cdots, m\}$, où η_i désigne un bruit blanc et δ_0 la masse de Dirac centrée en 0. On prend en compte les interactions entre neurones par la somme dans l'équation précédente : à chaque décharge, le potentiel du neurone i subit un saut dont l'intensité est pondérée par le coefficient α décrivant la connectivité du réseau ainsi que son caractère inhibiteur ($\alpha < 0$) ou excitateur ($\alpha > 0$). Nous proposons ici une présentation sommaire de quelques résultats connus sur ce modèle mais le lecteur intéressé pourra trouver une introduction plus détaillée dans [198, 204]. Le système (22)-(23) ainsi que sa limite de champ moyen ont été introduits dans [40, 41], notamment par N. Brunel. Dans la limite de champ moyen $m \to +\infty$, le modèle microscopique peut être décrit de manière équivalente par l'équation stochastique suivante

$$v_t \;=\; v_0 \;-\; \int_0^t v_t\,\mathrm{d}t \;+\; \alpha\mathbb{E}\left[M_t\right] \;-\; \left(V_F - V_R\right) M_t \;+\; \sqrt{2}\,\mathrm{d}B_t\,, \tag{24}$$

où B_t désigne le mouvement Brownien standard et M_t compte le nombre de fois où v_t a atteint le potentiel seuil V_F avant t ou bien par la loi $p(t, v)$ de la variable aléatoire v_t représentant la densité de neurones ayant un potentiel de membrane $v \leq V_F$ au temps $t \geq 0$ et qui résout l'équation aux dérivées partielles suivante

$$\partial_t p \;+\; \partial_v\left[\left(-v \;+\; \alpha\, N(t)\right) p\right] \;-\; \partial_v^2 p \;=\; N(t)\,\delta(v - V_R)\,, \tag{25}$$

complétée par des conditions de Dirichlet au bord

$$p(t, -\infty) \;=\; p(t, V_F) \;=\; 0\,. \tag{26}$$

Le couplage non linéaire dû aux interactions est pris en compte par le terme $N(t)$ qui décrit l'activité du réseau, c'est-à-dire la densité de décharges au temps t, il est défini par

$$N(t) \;=\; -\,\partial_v p(t, V_F)\,,$$

et vérifie $N(t) \;=\; \partial_t\mathbb{E}\left[M_t\right]$. Le lien entre les formulations stochastiques et déterministes a été prouvé rigoureusement dans un cadre linéaire par J.-G. Liu, Z. Wang, Y. Zhang, Z. Zhou [160].

L'analyse de ce modèle à été initiée par M. Càceres, J. Carrillo et B. Perthame [44]. Ils

démontrent notamment que les solutions de (25)-(26) peuvent développer des singularités en temps fini lorsque le réseaux est excitateur ($\alpha > 0$). Leur méthode de démonstration est similaire à celle utilisée dans le cadre de l'étude des équations de Keller-Segel : elle consiste en une estimation de moments et permet de montrer que, quelle que soit la valeur de $\alpha > 0$, lorsque la condition initiale est suffisamment concentrée autour du potentiel seuil, la solution faible associée ne peut être globale. Par ailleurs, ils étudient le problème stationnaire et montrent que dans le cas inhibiteur ($\alpha < 0$) et de faible connectivité ($0 < \alpha \ll 1$), il existe un unique état stationnaire, tandis qu'il n'en existe aucun dans le régime de grande connectivité ($1 \ll \alpha$). Dans certains régimes intermédiaires, ils montrent qu'il existe au moins 2 états d'équilibre.
Le premier résultat d'existence globale pour (25)-(26) a été obtenu par J.-A. Carrillo *et al.* [55] dans le cas de réseaux inhibiteurs ($\alpha < 0$). De plus, les auteurs démontrent que dans le cas excitateur ($\alpha > 0$), le temps d'existence T^* de la solution est égal au temps d'explosion de l'activité $N(t)$ du réseau

$$T^* = \sup \{ t > 0 \mid \quad \forall s \leq t \ : \ N(s) < +\infty \} \ .$$

Ce dernier résultat est important car il assure que l'explosion en temps fini est la conséquence de la synchronisation spontanée des neurones du réseaux, qui conduit à la formation d'une singularité autour du potentiel seuil V_F. L'argument développé consiste à reformuler l'équation en un problème de Stephan, établissant ainsi un lien avec des modèles appliqués à la finance.
J. Carrillo *et al* [56] ont obtenu le premier résultat quantitatif de retour vers l'équilibre. Ils prouvent une relaxation exponentielle dans le régime de faible connectivité $|\alpha| \ll 1$ et obtiennent aussi des estimées uniformes en temps sur la distribution p et l'activité N dans le cas inhibiteur.
L'étude du modèle stochastique (24) a permis de mieux comprendre la formation de singularité. En effet, F. Delarue *et al.* [87] réussissent à montrer *via* des méthodes probabilistes l'existence globale de solutions de (24) dans le cas excitateur, sous des conditions jointes entre donnée initiale et connectivité du réseau. Les mêmes auteurs [86] introduisent une nouvelle notion de solution de (24) pouvant être continuée après explosion, lorsque les temps de synchronisation n'ont pas de point d'accumulation. Ces travaux permettent aussi de justifier rigoureusement la limite de champ moyen $m \to +\infty$.
L'existence globale de solutions a aussi été obtenue dans le cas excitateur par des méthodes déterministes en introduisant un délai synaptique [46] ainsi que pour le modèle initial sans délai [201], sous des conditions jointes entre la concentration initiale au voisinage du potentiel seuil et la connectivité. Dans ce dernier article, les auteurs prouvent aussi que pour des connectivités assez grandes, la création de singularité se produit de manière systématique, quelle que soit la donnée initiale.
Pour conclure, des solutions périodiques ont été observées numériquement, ainsi que tous les phénomènes abordés précédemment [81, 82, 83].

Modèle potentiel-conductance

Nous allons maintenant introduire le modèle potentiel-conductance, obtenu à partir du modèle intègre et tire en y incorporant une variable de conductance. Cette modification est motivée par des considérations biologiques [209].
On considère une assemblée de m neurones en interaction dont les potentiels de membrane suivent le modèle de décharges forcées (22) : le drift b^i associé au i-ème neurone prend la forme particulière

$$b\left(t, v^i(t)\right) \;=\; -g_L\, v^i(t) \;-\; g^i(t)(v^i(t) - V_E)\,, \tag{27}$$

et l'on suppose sans perte de généralité $V_R = 0$ dans (22). la quantité V_E décrit le potentiel d'inversion du neurone, on supposera $0 < V_F < V_E$, ce qui assure $v_i(t) \in [0, V_F]$ puisque g_L et $g_i(t)$ sont des quantités positives. Le coefficient g_L décrit la conductance associée au potentiel de repos tandis que $g_i(t)$ décrit la conductance entre le neurone i et les autres neurones du réseau. Lorsqu'un neurone décharge, la conductance $g^i(t)$ du i-ème neurone subit un saut dont l'amplitude est dictée par l'équation suivante

$$\sigma\,\frac{\mathrm{d}}{\mathrm{d}t}\,g^i(t) \;=\; -g^i(t) \;+\; f\sum_{l\in\mathbb{N}^*}\delta_0\left(t-\tau_l^i\right) \;+\; \frac{S}{m}\sum_{j\neq i}\sum_{l\in\mathbb{N}^*}\delta_0\left(t-\tau_l^j\right)\,, \tag{28}$$

où $\sigma > 0$ est le temps caractéristique de relaxation de la conductance et δ_0 désigne la masse de Dirac centrée en 0. Chaque somme dans (28) décrit un type de connexion synaptique : la première peut être interprétée comme la capacité du neurone à s'auto-exciter tandis que la seconde décrit l'influence du réseau. Elles sont pondérées par les coefficients $f, S > 0$ décrivant l'intensité des connexions. On aboutit ainsi au système d'équations couplées (22)-(27)-(28) pour $i \in \{1, \cdots, m\}$ fournissant une description du réseau à l'échelle microscopique. Comme présenté en Section 0.1, il est intéressant à de nombreux points de vue d'adopter une description du réseau à l'échelle mésoscopique lorsque le nombre de neurones est grand. Dans le cadre de ce changement d'échelle, correspondant aux limites jointes $(f, m) \to (0, +\infty)$ et dont la justification formelle ainsi que numérique fait l'objet des études [47, 48, 197], le réseau de neurone est décrit par la densité $p(t, v, g)$ de neurones ayant un potentiel de membrane $v \in [V_R, V_F]$ et une conductance $g \in \mathbb{R}^+$ au temps $t \in \mathbb{R}^+$. La dynamique de la fonction de distribution est alors prescrite par l'équation suivante

$$\partial_t\, p \;-\; \partial_v\left[\left(g_L\, v + g\,(v - V_E)\right) p\right] \;-\; \frac{1}{2\sigma}\partial_g\left[\left(g - g_{in}(t)\right) p + a(t)\partial_g p\,\right] \;=\; 0\,, \tag{29}$$

complétée par les conditions de flux nul aux bords suivantes

$$\begin{cases} (g - g_{in}(t))\, p(t,v,g) + a(t)\partial_g p(t,v,g) \;=\; 0\,, & \text{lorsque} \quad g \in \{0\,,\, +\infty\}\,, \\[2ex] J\left(V_F, g\right) p(t, V_F, g) \;=\; J\left(V_R, g\right) p(t, V_R, g)\,, & \text{lorsque} \quad J\left(V_F, g\right) > 0\,, \\[2ex] p(t, V_F, g) \;=\; p(t, V_R, g) \;=\; 0\,, & \text{lorsque} \quad J\left(V_F, g\right) \leq 0\,, \end{cases} \tag{30}$$

où J désigne le flux associé à la variable de potentiel

$$J\,(v,g)\;=\;-\,g_L\,v\;-\;g\,(v-V_E)\;.$$

La dernière équation du système de conditions au bord (30) permet d'empêcher tout flux entrant qui ne soit pas causé par la décharge de neurones. Les couplages non linéaires induits par les interactions entre neurones dans (29) sont donnés par

$$g_{in}(t)\;=\;f\,\nu(t)+S\,\mathcal{N}(t)\,,\qquad a(t)\;=\;f^2\,\nu(t)+\frac{S^2}{n}\,\mathcal{N}(t)\,,$$

où $n>0$ est une constante de normalisation, $\nu(t)$ est une fonction donnée tandis que $\mathcal{N}(t)$ désigne le taux de décharge total du réseau

$$\mathcal{N}(t)\;=\;\int_{\mathbb{R}^+} J(V_F,g)\,p(t,V_F,g)\,\mathrm{d}g\,.$$

L'analyse mathématique de (29)-(30) a fait l'objet de nombreuses études. La résolution du problème stationnaire a été initiée par B. Perthame et D. Salort [184]. Ils démontrent l'existence et l'unicité d'une solution stationnaire dans le régime de faible connectivité ($S\ll 1$) ainsi qu'un résultat de non-existence dans le régime connectivité-bruit intense ($1\ll S,f$). D'autre part, les auteurs proposent des estimées de moments et de régularité pour la solution au problème de Cauchy.
D. Xu'an, B. Perthame, D. Salort et Z. Zhou [99] proposent la première analyse en temps long du problème. Ils démontrent la convergence vers l'état stationnaire pour l'équation linéaire associée à (29)-(30) ainsi qu'une amélioration des résultats sur la régularité du problème stationnaire obtenus dans [184]. Nous mentionnons aussi deux autres travaux récents qui traitent la question du comportement en temps long pour des problèmes liés [54, 100].
D'autre part certaines études numériques ont mis en évidence la diversité des comportements possibles, allant de la relaxation vers l'état d'équilibre jusqu'à l'observation de solutions périodiques [45, 197].
Pour finir, plusieurs travaux se sont penchés sur les limites asymptotiques de ce modèle. C'est le cas de [185] dans lequel B. Perthame et D. Salort démontrent la convergence vers un modèle de type intègre et tire dans la limite de conductance rapide $\sigma\to 0$ ainsi que de [153] dans lequel J. Kim, B. Perthame et D. Salort prouvent la convergence vers une équation de Fokker-Planck non linéaire dans la limite de potentiel rapide.

Neurones structurés par âge

Le dernier modèle que nous présenterons propose une vision transversale par rapport aux modèles présentés précédemment : plutôt que de décrire un neurone par son potentiel, on s'intéresse au temps écoulé depuis sa dernière décharge. Ainsi, on représente le réseau par la densité $n(s,t)$ de neurones au temps $t\geq 0$ dont le temps écoulé depuis le dernier potentiel d'action est $s \geq 0$. La dynamique de la distribution de probabilité $n(s,t)$ est prescrite par l'équation suivante, justifiée dans [187, 188]

$$\begin{cases} \partial_t n(s,t) \,+\, \partial_s n(s,t) \,+\, p(s,X(t))n(s,t) \,=\, 0 \,, \\ n(0,t) \,=\, \displaystyle\int_0^{+\infty} p(s,X(t))n(s,t)\,\mathrm{d}s\,. \end{cases} \tag{31}$$

La seconde ligne de (31) assure la conservation de la masse et décrit le fait qu'un neurone est réinitialisé à son état de repos après chaque décharge. L'équation (31) est non linéaire car le taux de décharge $p(s,X(t))$, à l'âge s, dépend de l'activité totale du réseau $X(t)$, déterminée implicitement par l'équation suivante

$$X(t) \,=\, J\int_0^t \alpha(u)\int_0^{+\infty} p(s,X(t-u))n(s,t-u)\,\mathrm{d}s\,\mathrm{d}u\,.$$

En notant $N(t)$ le nombre total de décharges au temps t,

$$N(t) \,=\, \int_0^{+\infty} p(s,X(t))n(s,t)\,\mathrm{d}s\,,$$

on obtient la formulation suivante de l'activité du réseau

$$X(t) \,=\, J\int_0^t \alpha(u)N(t-u)\,\mathrm{d}u\,.$$

Le coefficient $J \geq 0$ mesure la connectivité du réseau tandis que la fonction à valeurs positives α décrit le délaie entre l'émission d'une décharge par un neurone et la réception du signal par le reste du réseau : le cas où α est donné par une masse de Dirac correspond donc à une transmission instantanée. On suppose la normalisation suivante

$$\int_{\mathbb{R}^+} \alpha(u)\,\mathrm{d}u \,=\, 1\,.$$

Le taux de décharge $p(s,x)$ permet notamment de décrire la période réfractaire suivant un potentiel potentiel d'action, au cours de laquelle le neurone est à l'état de repos : pendant une certaine durée suivant une décharge, le neurone est comme déconnecté du réseau. Ainsi, une forme canonique pour p est donnée par

$$p(s,x) \,=\, \mathbb{1}_{s<s^\star(x)}\,,$$

où $s^\star(x)$ est la durée de la période réfractaire pour une activité x.

L'analyse mathématique du modèle (31) a été initiée par K. Pakdaman, B. Perthame et D. Salort [181]. Ils démontrent son caractère bien posé ainsi que des bornes uniformes sur sa solution $n(s,t)$ et l'activité $X(t)$. De plus, les auteurs montrent l'existence d'équilibres stationnaires ainsi que leur unicité dans les régimes de connectivité faible ($J \ll 1$) et forte ($J \gg 1$). Les auteurs démontrent aussi la stabilité asymptotique de ces équilibres

lorsqu'ils sont uniques et obtiennent des taux de retour exponentiels dans le cas sans délai ($\alpha = \delta_0$), mettant ainsi en évidence la capacité du modèle à décrire des effets de désynchronisation. Pour finir, les auteurs mettent numériquement en évidence l'existence de solutions périodiques dans les régimes de connectivités intermédiaires, illustrant ainsi que le modèle est aussi capable de prendre en compte des phénomènes de synchronisation.
L'analyse du modèle a été poursuivie dans [182] où sont précisés les résultats obtenus dans [181] dans le cas de réseaux sans délais tandis qu'est donnée la preuve constructive de l'existence des solutions périodiques observées précédemment. Les auteurs étudient numériquement la stabilité de ses solutions et conjecturent notamment que certaines d'entre elles sont instables.
Jusqu'à maintenant, nous n'avons évoqué que le pendant déterministe du modèle (31). J. Chevallier *et al.* [71] ont initié l'étude du modèle par des méthodes probabilistes en montrant que celui-ci peut être interprété de manière stochastique à l'aide de processus à sauts dits de Hawkes. Il fut aussi montré dans [70, 194] que (31) peut être obtenu comme la limite de champ moyen d'un système microscopique dans lequel chaque neurone est décrit par un processus de Hawkes.
Pour revenir à l'étude déterministe de ce modèle, nous soulignons que dans le régime de faible connectivité ($J \ll 1$) la relaxation exponentielle vers les états d'équilibre a été démontrée par S. Mischler et Q. Weng dans [170] sous des hypothèses générales, autorisant notamment des réseaux avec délais et levant l'hypothèse de [181] consistant à considérer des taux de décharge p en escalier. Les résultats de cette étude sont étendus par S. Mischler *et al.* [169], qui traitent non seulement le cas de faible mais aussi de forte connectivité ($J \gg 1$) et relaxent encore les hypothèses de régularité sur le taux de décharge p.
Pour conclure sur l'analyse de ce modèle, le cas de réseaux inhibiteurs a été traité dans [43]. Les auteurs étudient aussi de manière analytique et numérique de nouvelles solutions périodiques de (31).
Le modèle (31) a été généralisé dans différentes directions. Par exemple, M.-J. Kang, B. Perthame et D. Salort [151] proposent une version permettant de traiter le cas de réseaux dont le taux de décharge est inhomogène et démontrent la relaxation exponentielle vers l'état d'équilibre dans le régime de faible connectivité. Dans le même esprit, N. Torres et D. Salort [210] proposent un réseau spatialement structuré qui suit une loi d'apprentissage. Ils analysent le comportement en temps long du modèle dans le régime de faible connectivité et mènent une étude numérique du modèle qui permet notamment d'observer la formation de motifs pour la fonction d'apprentissage.

3.4 Le modèle de FitzHugh-Nagumo

Cette section vise à introduire le modèle de FitzHugh-Nagumo sur lequel nous travaillerons durant le reste de notre étude. Le modèle de FitzHugh-Nagumo fait l'intermédiaire entre celui de Hodgkin et Huxley et ceux à potentiel d'action forcé : bien que moins complexe comparé au modèle de Hodgkin et Huxley, il permet tout de même de retrouver le potentiel d'action sans recourir à un forçage.

Au début des années 1960, R. FitzHugh [114] suivi de J. Nagumo, S. Arimoto et S. Yoshizawa [174] ont observé des similarités entre le comportement du modèle de Hodgkin et Huxley et celui de l'oscillateur de Van der Pool, un modèle notamment utilisé pour simuler les pulsations cardiaques. À partir de ces observations, ils ont proposé un modèle simplifié permettant de préserver les caractéristiques principales du modèle de Hodgkin et Huxley. Ce modèle ne compte que deux composantes et son caractère non linéaire est considérablement réduit. Nous ré-écrivons ici le modèle originale (aux notations près, dans un souci d'unité avec le reste du manuscrit)

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} v \,=\, N(v) \,+\, w \,-\, I_{ext}\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} w \,=\, A(v,w)\,, \end{cases} \tag{32}$$

où $N(v)$ prend la forme typique suivante

$$N(v) \,=\, v \,-\, v^3\,,$$

et $A(v,w)$ est donné par

$$A(v,w) \,=\, a\,v \,-\, b\,w \,+\, c\,,$$

où les coefficients a, c sont deux réels et b est un réel positif. Dans la suite de ce paragraphe nous exposerons les arguments présentés dans [114, 174] pour justifier ce modèle et nous illustrerons numériquement les similarités entre (19)-(20) et (32).

Pour commencer, R. FitzHugh [113] distingue deux échelles temporelles distinctes dans la dynamique des solutions de (19). En effet, il observe que les variables $(v(t), m(t))$ sont plus rapides que les variables $(n(t), h(t))$. Cette différence est illustrée numériquement sur la Figure 7 qui représente le temps caractéristique de relaxation du potentiel de membrane $\tau_v := C/g_{Na}$ ainsi que les temps caractéristiques $\tau_X(v)$, $X \in \{h, m, n\}$, définis par (21). Ces temps sont calculés sur l'intervalle des potentiels observés au cours de nos simulations : $v \in [-120, 20]$.

R. FitzHugh propose donc de ne garder qu'une variable de chaque type en effectuant des projections linéaires des couples (v, m) et (n, h) sur des droites. Il aboutit au changement de variable suivant $(u, w) = (v - 36\,m\,,\,(n - h)/2)$ mais nous considèrerons ici le couple (v, n) par souci de simplicité.

Il propose alors la modification (32) du modèle de Van der Pool permettant de retrouver toutes les caractéristiques du modèle Hodgkin-Huxley. Il montre notamment que les trajectoires du couple $(v(t), n(t))$ solution de (19) et celles de $(v(t), w(t))$ solution de (32) dans l'espace des phases ont un comportement qualitatif similaire. Sur la Figure 8 nous proposons une reproduction de ces comparaisons dans laquelle nous observons l'effet seuil à partir duquel apparaît le potentiel d'action.

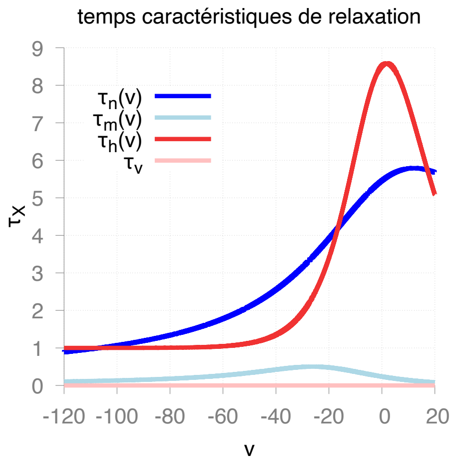


FIGURE 7 – courbes associées aux temps caractéristiques de relaxation de m, n, h définis par (21)

Pour conclure cette partie, nous mentionnons que d'autres modèles ont été dérivés en suivant des arguments similaires à ceux utilisés par FitzHugh et Nagumo. Le lecteur intéressé pourra par exemple consulter [119].

3.5 Limite macroscopique pour le modèle de FitzHugh-Nagumo

Dans cette partie, on introduira un modèle décrivant un réseau de neurones de FitzHugh-Nagumo et, sur la base de considérations biologiques, nous nous pencherons sur une situation dans laquelle le réseau exhibe une structure spatiale particulière. De manière analogue aux problématiques présentées dans la Partie 0.1, notre étude consiste à décrire quantitativement comment cette structuration spatiale induit une dynamique macroscopique dans le comportement du réseau. Dans cette partie, nous mènerons une analyse formelle permettant d'exposer les principaux enjeux.

Réseau neuronal de FitzHugh-Nagumo spatialement étendu

À l'instant $t \geq 0$, chaque neurone sera décrit par son potentiel de membrane noté $v_t \in \mathbb{R}$ couplé à une variable d'adaptation $w_t \in \mathbb{R}$. L'évolution du couple (v_t, w_t) sera prescrite par le modèle de FitzHugh-Nagumo (32) auquel nous ajouterons un bruit stochastique brownien B_t de manière à prendre en compte les fluctuations aléatoires du potentiel de membrane. Ainsi, le couple potentiel de membrane - variable d'adaptation satisfait l'équation différentielle stochastique suivante

$$\begin{cases} \mathrm{d}v_t = (N(v_t) - w_t - I_{ext})\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t\,, \\ \mathrm{d}w_t = A(v_t, w_t)\,\mathrm{d}t\,, \end{cases}$$

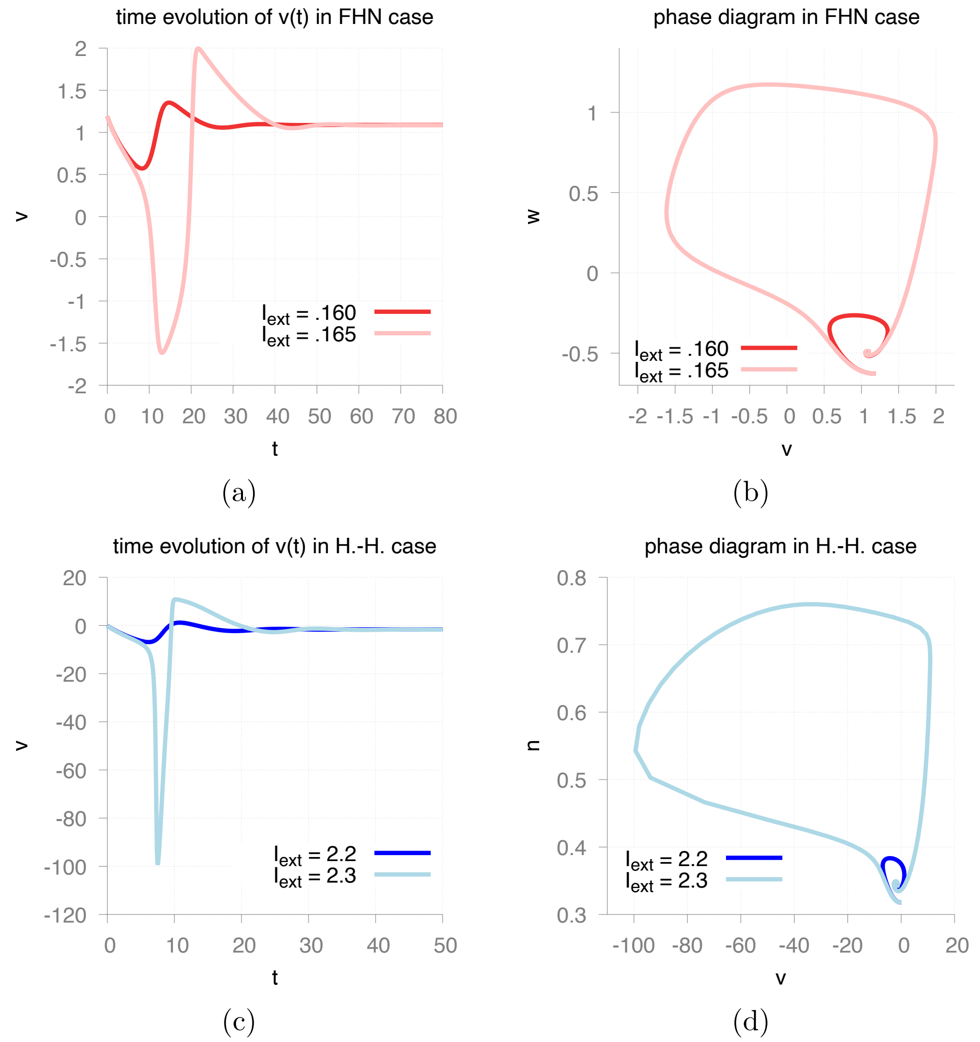


FIGURE 8 – **Comparaison des modèles de Hodgkin-Huxley (HH) et FitzHugh-Nagumo (FHN).** Variation du potentiel de membrane en réponse à différents courants appliqués pour (*a*) FHN, (*c*) HH ; Diagramme de phase pour différents courants appliqués (*b*) FHN $(v(t), w(t))$, (*b*) HH $(v(t), n(t))$. les paramètres choisis pour FHN sont $(a, b, c, v_0, w_0) = (-0.111, 0.088, 0.077, 1.19, -0.62)$.

où le coefficient A est défini par (32) tandis que $N \in \mathscr{C}^2$ vérifie la condition de confinement super-linéaire suivante : il existe un exposant $p \geq 2$ tel que

$$\begin{cases} \limsup_{|v| \to +\infty} \dfrac{N(v)}{v} \;=\; -\infty\,, & (33a) \\[2ex] \sup_{|v| \geq 1} \dfrac{|N(v)|}{|v|^p} \;<\; +\infty\,. & (33b) \end{cases}$$

Remarque. *Tout au long de notre analyse, nous veillerons attentivement à ce que nos hypothèses sur N soient consistantes avec le modèle de FitzHugh-Nagumo* (32). *Cette contrainte sera satisfaite car, comme nous le verrons, tous nos résultats sont valables pour la classe de drifts suivante*

$$P(v) \;=\; Q(v) \;-\; Cv|v|^{p-1}\,,$$

pour toute constante $C > 0$ tandis que Q est une fonction polynomiale de degré au plus p.

Le terme I_{ext} représente le courant extérieur appliqué au neurone. Dans la Section 0.3.2, ce terme prenait en compte le courant appliqué de manière artificielle lors d'une expérience mais dans notre cas, il décrira les signaux électriques émis par les autres neurones. Plus précisément, nous considérons que les neurones sont connectés entre eux *via* des synapses électriques dont le fonctionnement est régi par la loi d'Ohm. Ainsi, si l'on considère un réseau constitué de m neurones, le courant reçu par le i-ème est donné par l'expression suivante

$$I_{ext} \;=\; \frac{1}{m}\sum_{j=1}^{m} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)\,(v_t^i - v_t^j)\,.$$

Le coefficient $\Phi(\boldsymbol{x}_i, \boldsymbol{x}_j) \in \mathbb{R}$ décrit la conductance entre les neurones i et j. Celle-ci dépend de manière non symétrique de leur position respective $(\boldsymbol{x}_i, \boldsymbol{x}_j) \in K^2$, où K désigne un sous ensemble compact de $\mathbb{R}^d$. Nous aboutissons donc au système suivant, décrivant la dynamique microscopique d'un réseau constitué de m neurones

$$\begin{cases} \mathrm{d}v_t^i \;=\; \left(N(v_t^i) \;-\; w_t^i \;-\; \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)\,(v_t^i - v_t^j)\right)\,\mathrm{d}t \;+\; \sqrt{2}\,\mathrm{d}B_t^i\,, \\[2ex] \mathrm{d}w_t^i \;=\; A\left(v_t^i, w_t^i\right)\,\mathrm{d}t\,, \end{cases}$$

pour tout $i \in \{1, \cdots, m\}$. Lorsque le nombre de neurones composant le réseau est grand, une description statistique du réseau est plus adaptée car moins coûteuse et plus facile à analyser, comme expliqué en Partie 0.1. Cela correspond à considérer la loi de probabilité $f(t, \boldsymbol{x}, v, w)$ décrivant la densité de neurones au temps $t \geq 0$, à la position $\boldsymbol{x} \in K$ ayant un potentiel de membrane $v \in \mathbb{R}$ et une variable d'adaptation $w \in \mathbb{R}$. Dans la limite de champ moyen $m \to +\infty$, la dynamique de f est prescrite par l'équation suivante

$$\partial_t f \,+\, \nabla_{\boldsymbol{u}} \cdot \left[\begin{pmatrix} N(v) \,-\, w \,-\, \mathcal{K}_\Phi[f] \\ A(v,w) \end{pmatrix} f \right] \,-\, \partial_v^2 f \,=\, 0\,,$$

où l'on a utilisé la notation $\boldsymbol{u} := (v,w) \in \mathbb{R}^2$. Cette équation est non linéaire à cause du terme $\mathcal{K}_\Phi[f]$ prenant en compte les interactions spatiales et défini par

$$\mathcal{K}_\Phi[f](t,\boldsymbol{x},v) \,=\, \int_{K\times\mathbb{R}^2} \Phi\left(\boldsymbol{x},\boldsymbol{x}'\right)\left(v-v'\right) f(t,\,\boldsymbol{x}',v',w')\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}v'\,\mathrm{d}w'\,.$$

L'analyse mathématique de la limite de champ moyen $m \to +\infty$ est maintenant classique : elle a été justifiée dans cas de neurones de FitzHugh-Nagumo [77, 168] ainsi que dans le cas de neurones plus généraux, comprenant le modèle de Hodkin-Huxley [3, 30, 161]. Nous mentionnons aussi [144, 195] dans lesquels le lecteur trouvera une analyse similaire pour des systèmes présentant des structures spatiales non-symétriques.

Régime des interactions locales fortes

À la fin des années 1950, une série de travaux permit de mieux comprendre la structuration spatiale des différents cortex sensoriels. Plusieurs expériences furent menées et aboutirent à la conclusion que les cortex visuels [142] et sensoriels [172, 173] sont organisés en sous-régions appelées colonnes corticales et composées de neurones dont les réponses aux stimulations sont similaires. Depuis, de nombreux scientifiques ont tenté d'identifier clairement les propriétés permettant de définir les caractéristiques des colonnes corticales. C'est par exemple le cas de [162], dans lequel on pourra retrouver une étude des propriétés fonctionnelles des colonnes corticales dans le cortex visuel du macaque. Il apparaît notamment dans cette étude qu'au sein d'une même colonne corticale, les neurones sont fortement connectés, comme on pourra l'observer sur la Figure 9, reprise de [162], qui représente les connexions entre neurones mises en évidence par l'ajout d'un traceur. On observe sur cette figure que la répartition des connexions n'est pas homogène, elles sont plus nombreuses au sein des colonnes corticales.

Dans ce qui suit, nous prendrons en compte cette organisation spatiale et nous analyserons ses conséquences sur la dynamique globale du réseau. Cela passera par l'étude d'un régime asymptotique pour le modèle de champ moyen introduit précédemment. En effet, nous considèrerons une situation dans laquelle les interactions courte portée sont dominantes. Ainsi, nous décomposons le noyau d'interaction de la manière suivante

$$\Phi(\boldsymbol{x},\boldsymbol{x}') \,=\, \Psi(\boldsymbol{x},\boldsymbol{x}') \,+\, \frac{1}{\varepsilon}\,\delta_0\left(\boldsymbol{x}-\boldsymbol{x}'\right)\,, \tag{34}$$

où la masse de Dirac δ_0 prend en compte les interactions locales, dont l'intensité est décrite par le paramètre $\varepsilon > 0$: nous nous intéresserons à la limite $\varepsilon \to 0$. Quant à lui, le noyau

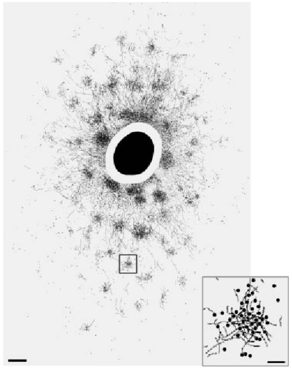


FIGURE 9 – Reconstruction des connexions au sein d'une région du cortex visuel du macaque après injection d'un traceur (cercle noir). les échelles sont de $500\,\mu m$ et $100\,\mu m$

$(\Psi : \boldsymbol{x} \longmapsto \Psi(\boldsymbol{x}\,,\,\cdot)) \in \mathscr{C}^0\left(K, L^1(K)\right)$ prend en compte les interactions longue portée. Nous supposerons qu'il existe $r \in]1, +\infty]$ vérifiant

$$\sup_{\boldsymbol{x}' \in K} \int_K \left|\Psi(\boldsymbol{x}, \boldsymbol{x}')\right| \mathrm{d}\boldsymbol{x} < +\infty, \qquad \sup_{\boldsymbol{x} \in K} \int_K \left|\Psi(\boldsymbol{x}, \boldsymbol{x}')\right|^r \mathrm{d}\boldsymbol{x}' < +\infty, \tag{35}$$

et nous noterons $r' \geq 1$ l'exposant conjugué de r défini par $1/r + 1/r' = 1$.

Remarque. *Du point de vue de la modélisation, Ψ recèle la complexité de la structure spatiale du réseau ; il est donc important de prendre les hypothèses les moins restrictives possibles. Nos hypothèses permettent notamment de traiter des interactions non-symétriques [39], polynomiales [73, 131, 161, 176] ainsi qu'à support compact [161, 177, 178].*

Certains travaux sont déjà dédiés à l'analyse mathématique et numérique de ce régime [39, 78, 79, 80, 196], nous mentionnons aussi [168], dans lequel S. Mischler *et al.* traitent la limite des interactions locales faibles dans un cadre spatialement homogène. Nous commencerons donc par expliquer comment nos travaux s'inscrivent dans la littérature existante en présentant les résultats disponibles. Pour cela nous allons mener une analyse formelle du régime $\varepsilon \to 0$. En remplaçant Φ par l'*ansatz* (34), l'équation de champ moyen se ré-écrit de la manière suivante

$$\partial_t \mu^\varepsilon + \nabla_{\boldsymbol{u}} \cdot \left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Psi[\rho_0^\varepsilon\, \mu^\varepsilon] \\ A(v, w) \end{pmatrix} \mu^\varepsilon\right] - \partial_v^2 \mu^\varepsilon = \frac{1}{\varepsilon}\, \rho_0^\varepsilon\, \partial_v \left[(v - \mathcal{V}^\varepsilon)\, \mu^\varepsilon\right], \tag{36}$$

avec $\boldsymbol{u} := (v, w) \in \mathbb{R}^2$. Le membre de droite de (36) est induit par les interactions locales et met en jeu les quantités macroscopiques associées au réseau : d'une part la distribution spatiale de neurones

$$\rho_0^\varepsilon(\boldsymbol{x}) = \int_{\mathbb{R}^2} f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,,$$

ainsi que la valeur moyenne du potentiel de membrane et de la variable d'adaptation à la position $\boldsymbol{x} \in K$, données par le couple $\mathcal{U}^\varepsilon(t,\boldsymbol{x}) = (\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)(t,\boldsymbol{x})$ et définies par

$$\begin{cases} \rho_0^\varepsilon(\boldsymbol{x})\, \mathcal{V}^\varepsilon(t,\boldsymbol{x}) & = \displaystyle\int_{\mathbb{R}^2} v\, f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,, \\ \rho_0^\varepsilon(\boldsymbol{x})\, \mathcal{W}^\varepsilon(t,\boldsymbol{x}) & = \displaystyle\int_{\mathbb{R}^2} w\, f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,. \end{cases} \tag{37}$$

Comme on peut le voir en intégrant (36) par rapport à v et w, la distribution ρ_0^ε est indépendante du temps. C'est pour cette raison que nous travaillerons avec une version renormalisée μ^ε de f^ε définie par

$$\rho_0^\varepsilon\, \mu^\varepsilon = f^\varepsilon\,,$$

ayant l'avantage d'être de masse constante sur K puisque μ^ε est une fonction positive vérifiant

$$\int_{\mathbb{R}^2} \mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\, \mathrm{d}\boldsymbol{u} = 1, \quad \forall\, (t,\,\boldsymbol{x}) \in \mathbb{R}^+ \times K\,.$$

Afin de déterminer formellement la limite de μ^ε lorsque $\varepsilon \to 0$, on considère le terme singulier en ε dans l'équation (36) : pour assurer qu'aucun terme n'explose, μ^ε doit vérifier

$$(v - \mathcal{V}^\varepsilon(t,\boldsymbol{x}))\, \mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \underset{\varepsilon\to 0}{\sim} 0\,.$$

Étant donné que l'équation (36) conserve la masse totale, on déduit de cette relation que μ^ε se concentre en une masse de Dirac autour du potentiel de membrane moyen, c'est-à-dire

$$\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \underset{\varepsilon\to 0}{\approx} \delta_{\mathcal{V}^\varepsilon(t,\boldsymbol{x})}(v) \otimes \bar{\mu}^\varepsilon(t,\boldsymbol{x},w)\,,$$

où $\mathcal{V}^\varepsilon$ est défini par (37) et $\bar{\mu}^\varepsilon$ désigne la distribution marginale de μ^ε par rapport à w

$$\bar{\mu}^\varepsilon(t,\boldsymbol{x},w) = \int_{\mathbb{R}} \mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\, \mathrm{d}v\,.$$

Ainsi, dans le régime où les interactions locales dominent, μ^ε se réduit en une distribution mono-potentiel dans le sens où le couple $(\mathcal{V}^\varepsilon, \bar{\mu}^\varepsilon)$ suffit pour décrire la dynamique du réseau. Pour conclure cette analyse formelle, nous devons déterminer la limite du couple $(\mathcal{V}^\varepsilon, \bar{\mu}^\varepsilon)$. Dans cette optique, nous multiplions (36) par v (resp. 1) et en intégrons par rapport à $\boldsymbol{u} \in \mathbb{R}^2$ (resp. $v \in \mathbb{R}$)

$$\begin{cases} \partial_t \mathcal{V}^\varepsilon = N(\mathcal{V}^\varepsilon) - \mathcal{W}^\varepsilon - \mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon] + \mathcal{E}(\mu^\varepsilon)\,, \\ \partial_t \bar{\mu}^\varepsilon + \partial_w \left(a \displaystyle\int_{\mathbb{R}} v\, \mu^\varepsilon \mathrm{d}v - b\, w\, \bar{\mu}^\varepsilon + c\, \bar{\mu}^\varepsilon \right) = 0\,, \end{cases} \tag{38}$$

où $\mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon]$ est l'opérateur non-local induit par les interactions longue portée ; il est défini par

$$\mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon] = \mathcal{V}^\varepsilon\, \Psi *_r \rho_0^\varepsilon - \Psi *_r (\rho_0^\varepsilon \mathcal{V}^\varepsilon)\,,$$

tandis que $\mathcal{E}(\mu^\varepsilon)$ est un terme d'erreur s'annulant lorsque $\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) = \delta_{\mathcal{V}^\varepsilon}(v) \otimes \bar{\mu}^\varepsilon$. Ainsi, dans la limite $\varepsilon \to 0$, on obtient un système fermé sur les quantités macroscopiques : le couple $(\mathcal{V}^\varepsilon, \bar{\mu}^\varepsilon)$ converge vers $(\mathcal{V}, \bar{\mu})$ dont la dynamique est prescrite par l'équation suivante

$$\left\{ \begin{array}{l} \partial_t \mathcal{V} \,=\, N(\mathcal{V}) \,-\, \mathcal{W} \,-\, \mathcal{L}_{\rho_0}[\mathcal{V}]\,, \\[2mm] \partial_t \bar{\mu} \,+\, \partial_w \left(A\left(\mathcal{V}, w\right) \bar{\mu} \right) = 0\,, \end{array} \right. \tag{39}$$

où $\mathcal{W}$ vérifie

$$\mathcal{W}(t,\boldsymbol{x}) \,=\, \int_{\mathbb{R}} w \, \bar{\mu}(t,\boldsymbol{x},w) \, \mathrm{d}w\,,$$

et où ρ_0 est défini par $\rho_0 \,=\, \lim_{\varepsilon\to 0} \rho_0^\varepsilon$. Nous soulignons qu'en multipliant la seconde ligne de (39) par w et en intégrant sur $\mathbb{R}$, on retrouve le système suivant de réaction-diffusion

$$\left\{ \begin{array}{l} \partial_t \mathcal{V} \,=\, N(\mathcal{V}) \,-\, \mathcal{W} \,-\, \mathcal{L}_{\rho_0}[\mathcal{V}]\,, \\[2mm] \partial_t \mathcal{W} \,=\, A\left(\mathcal{V}, \mathcal{W}\right)\,, \end{array} \right.$$

qui a été intensivement étudié car il permet d'observer la propagation de front d'ondes [53, 59, 60, 133, 147]. Dans le régime des interactions locales fortes, la distribution μ^ε vérifie donc

$$\mu^\varepsilon\left(t,\boldsymbol{x},\boldsymbol{u}\right) \underset{\varepsilon\to 0}{\longrightarrow} \delta_{\mathcal{V}(t,\boldsymbol{x})}(v) \otimes \bar{\mu}(t,\boldsymbol{x},w)\,. \tag{40}$$

Cette convergence a été étudiée dans un cadre déterministe [78, 79].

Profil de concentration

Notre contribution sur ce sujet a consisté à raffiner le résultat de convergence (40) en analysant le profil de concentration de la distribution μ^ε autour du potentiel de membrane moyen $\mathcal{V}$. Par profil de concentration, nous désignons une fonction $g \,:\, \mathbb{R}\times K \to \mathbb{R}$ telle que

$$\mu^\varepsilon\left(t,\boldsymbol{x},v,w\right) \underset{\varepsilon\to 0}{\sim} \frac{1}{\theta^\varepsilon}\, g\left(t,\boldsymbol{x},\frac{v-\mathcal{V}(t,\boldsymbol{x})}{\theta^\varepsilon}\right) \otimes \bar{\mu}(t,\boldsymbol{x},w)\,,$$

pour un certain taux de concentration $\theta^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$ à déterminer. Le lecteur pourra se référer à la Figure 10 pour une illustration visuelle. Cette approche est classique en théorie cinétique où elle intervient dans le cadre de l'analyse des milieux poreux [58], pour l'analyse de modèles de type Keller-Segel [20], ainsi que dans l'étude de l'équation de Boltzmann en milieu granulaire [171, 199]. Si le résultat de convergence (40) décrit le développement de μ^ε à l'ordre 0 dans la limite $\varepsilon \to 0$, la convergence du profil de concentration peut être interprétée comme le développement de μ^ε à l'ordre 1. Une étude du profil de concentration permet donc d'améliorer la convergence (40) à de nombreux égards. D'une part le changement d'échelle désingularise la limite puisque l'on s'attend à ce que le profil g soit plus régulier qu'une masse de Dirac, ce qui permet d'obtenir des résultats de convergence

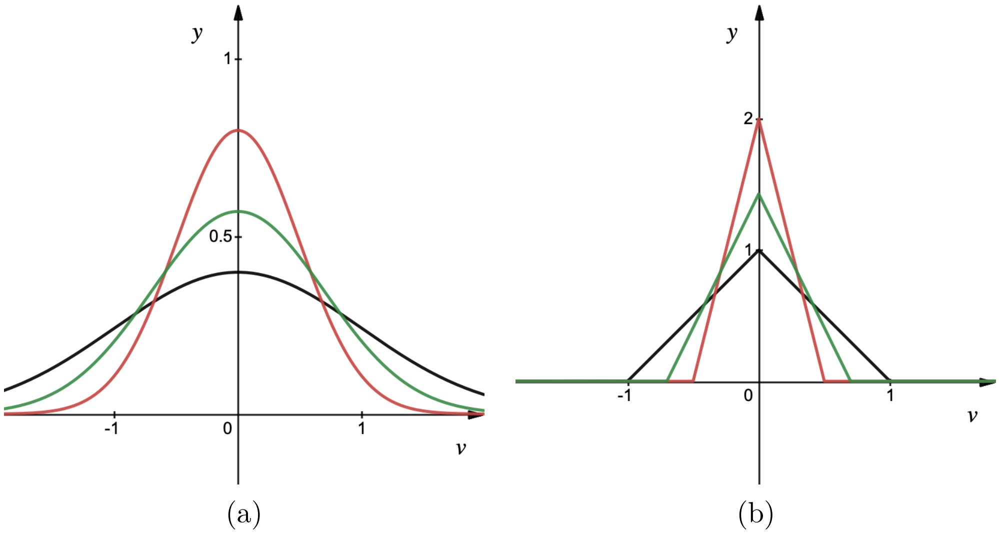


FIGURE 10 – courbe représentative de $y = \dfrac{1}{\theta^\varepsilon} g\left(\dfrac{v}{\theta^\varepsilon}\right)$, pour les valeurs $\theta^\varepsilon = 1; 0.7; 0.5$ avec profil g (a) gaussien ; (b) triangulaire.

forte. D'autre part, la convergence du profil fournit des taux optimaux à l'ordre 0, c'est-à-dire permet de montrer que (40) est en fait un équivalent asymptotique. Comme on le verra, ces considérations fournissent des indications plus précises quant au comportement asymptotique des quantités macroscopiques : on pourra par exemple utiliser nos résultats pour montrer que $\mathcal{V}^\varepsilon$ est $\varepsilon^{3/2}$-proche de la correction suivante au système limite (39)

$$\partial_t \mathcal{V} = N(\mathcal{V}) - \mathcal{W} - \mathcal{L}_{\rho_0}[\mathcal{V}] + \frac{\varepsilon}{2}\rho_0 N''(\mathcal{V}) \, .$$

Comme nous l'avons fait lors du Chapitre 1 pour le système de Vlasov-Poisson-Fokker-Planck, notre objectif est ici de quantifier le mieux possible la convergence de μ^ε vers son profil d'explosion. Pour cela, nous donnerons des résultats de convergence correspondant à des topologies différentes en utilisant d'une part une approche inspirée de la théorie cinétique basée sur des estimations d'énergie et de la propagation de régularité et d'autre part une approche de type Hamilton-Jacobi.

3.6 Convergence au sens de Wasserstein (Chapitre 4)

Dans cette partie, nous étudierons la convergence faible de la distribution μ^ε vers son profil asymptotique. À cet égard, nous introduisons l'espace de probabilité $\mathscr{P}_2(\mathbb{R}^2)$, qui servira de cadre fonctionnel à notre étude

$$\mathscr{P}_2\left(\mathbb{R}^2\right) = \left\{\mu \in \mathscr{P}(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} |\boldsymbol{u}|^2 \mathrm{d}\mu(\boldsymbol{u}) < +\infty\right\},$$

que nous munissons de la distance de Wasserstein d'ordre 2, notée W_2. Dans cette section, on notera $\mu^\varepsilon_{t,\boldsymbol{x}}$ la densité de probabilité définie par $\mu^\varepsilon_{t,\boldsymbol{x}} = \mu^\varepsilon(t, \boldsymbol{x}, \cdot)$.

Avant d'énoncer notre résultat principal, nous soulignons que l'équation (36) est bien posée. En effet, sous certaines conditions de régularité et de moment non-uniformes par rapport à ε, il est possible de montrer qu'elle admet une unique solution faible globale

$$\mu^\varepsilon \in \mathscr{C}^0\left(\mathbb{R}^+ \times K\,,\, L^1\left(\mathbb{R}^2\right)\right)\,,$$

vérifiant pour tout $T \geq 0$

$$\sup_{(t,\boldsymbol{x})\in[0,T]\times K} \int_{\mathbb{R}^2} e^{|\boldsymbol{u}|^2/2}\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u} < +\infty\,.$$

Nous renvoyons le lecteur au Théorème 4.3 pour un énoncé complet de ce résultat. De même, le système limite (39) est bien posé d'après le Théorème 4.6.
Dans ce qui suit, nous prendrons les hypothèses **uniformes en** ε suivantes : nous supposons qu'il existe trois constantes $m_*, m_p, \overline{m}_p > 0$ telles que pour tout $\varepsilon > 0$

$$m_* \leq \rho_0^\varepsilon \leq 1/m_*\,, \quad \rho_0^\varepsilon \in \mathscr{C}^0\left(K\right)\,, \tag{41}$$

ainsi que

$$\sup_{\boldsymbol{x}\in K} \int_{\mathbb{R}^2} |\boldsymbol{u}|^{2p}\mu_0^\varepsilon(\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \,\leq\, m_p\,, \tag{42}$$

et

$$\int_{K\times\mathbb{R}^2} |\boldsymbol{u}|^{2pr'}\,\mu_0^\varepsilon(\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \,\leq\, \overline{m}_p\,, \tag{43}$$

où p et r' sont donnés par (33b) et (35).

Nous pouvons maintenant énoncer notre résultat principal. Celui-ci assure que l'explosion de μ^ε en une masse de Dirac se produit selon le profil Gaussien suivant

$$\mathcal{M}_{\rho_0,\mathcal{V}}(v) = \sqrt{\frac{\rho_0}{2\pi}}\exp\left(-\frac{\rho_0\,|v-\mathcal{V}|^2}{2}\right)\,,$$

avec un taux de concentration $\sqrt{\varepsilon}$. De plus, la distribution marginale par rapport à la variable d'adaptation w est complètement caractérisée puisque l'on prouve la convergence de $\bar{\mu}^\varepsilon$ vers $\bar{\mu}$ donnée par (39). Notre résultat est global en temps, uniforme en $\boldsymbol{x} \in K$ et il fournit des taux de convergence optimaux en ε.

Théorème 6 ([24]). *Sous les hypothèses* (33a)-(33b) *sur* N, (35) *sur* Ψ, (41)-(43) *sur la donnée initiale* μ_0^ε, *on considère les solutions* μ^ε *et* $(\mathcal{V},\,\bar{\mu})$ *aux problèmes* (36) *et* (39) *respectivement. Sous la condition de compatibilité suivante*

$$\|\,\mathcal{U}_0 - \mathcal{U}_0^\varepsilon\,\|_{L^\infty(K)} + \|\,\rho_0 - \rho_0^\varepsilon\,\|_{L^\infty(K)} + \sup_{\boldsymbol{x}\in K} W_2\left(\bar{\mu}_{0,\boldsymbol{x}}^\varepsilon\left(\mathcal{W}^\varepsilon + \cdot\right), \bar{\mu}_{0,\boldsymbol{x}}\left(\mathcal{W} + \cdot\right)\right) \underset{\varepsilon\to 0}{=} O\left(\varepsilon\right)\,,$$

il existe $(C,\varepsilon_0) \in \left(\mathbb{R}_*^+\right)^2$ *tel que pour tout* $\varepsilon \leq \varepsilon_0$ *ainsi que pour tout* $(t,\boldsymbol{x}) \in \mathbb{R}^+ \times K$ *l'estimation suivante soit vérifiée*

$$W_2\left(\mu_{t,\boldsymbol{x}}^\varepsilon\,,\,\mathcal{M}_{\frac{1}{\varepsilon}\rho_0(\boldsymbol{x}),\mathcal{V}(t,\boldsymbol{x})} \otimes \bar{\mu}_{t,\boldsymbol{x}}\right) \,\leq\, C\,\left(\min\left(e^{C\,t}\,\varepsilon\,,\,1\right) \,+\, e^{-\rho_0^\varepsilon(\boldsymbol{x})\,t/\varepsilon}\right)\,.$$

Commençons par interpréter ce résultat. Premièrement, les hypothèses (42)-(43) n'imposent pas à la solution μ^ε d'être initialement concentrée par rapport à v, ce qui permet de considérer des données initiales générales. C'est pour cette raison qu'apparaît dans nos estimations le terme $e^{-\rho_0^\varepsilon(\boldsymbol{x})\,t/\varepsilon}$, d'ordre 1 à $t=0$ et exponentiellement décroissant lorsque $t>0$.
D'autre part, ce résultat permet d'obtenir des estimations d'erreur quantifiant la convergences des quantités macroscopiques $(\mathcal{V}^\varepsilon,\mathcal{W}^\varepsilon)$ définies par (37) vers $(\mathcal{V},\mathcal{W})$ définies par (39)

Corollaire ([24])**.** *Sous les hypothèses du Théorème 6, il existe* $(C,\varepsilon_0)\in\left(\mathbb{R}_*^+\right)^2$ *tel que pour tout* $\varepsilon\le\varepsilon_0$ *on ait*

$$\sup_{\boldsymbol{x}\in K}\left(\left|\mathcal{U}^\varepsilon\left(t,\boldsymbol{x}\right)\,-\,\mathcal{U}\left(t,\boldsymbol{x}\right)\right|\,+\,W_2\left(\bar{\mu}^\varepsilon_{t,\boldsymbol{x}},\bar{\mu}_{t,\boldsymbol{x}}\right)\right)\,\le\,C\,\left(e^{-m_*\,t/\varepsilon}\,+\,\min\left(e^{C\,t}\varepsilon,\,1\right)\right)\,,$$

pour tout $t\ \ge\ 0$*, où* m_* *est donné par* (41)*.*

Pour conclure, nous déduisons du Théorème 6, qui quantifie la convergence de μ^ε vers un profil d'explosion, un résultat optimal quantifiant la convergence de μ^ε vers sa limite $\delta_{\mathcal{V}}\otimes\bar{\mu}$. Plus précisément, on montre que la distance entre μ^ε et sa limite $\delta_{\mathcal{V}}\otimes\bar{\mu}$ est exactement d'ordre $\sqrt{\varepsilon}$. Cela justifie notre approche pour deux raisons. Premièrement, la convergence vers un profil assurée par le Théorème 6 décrit μ^ε à l'ordre ε tandis que la limite $\delta_{\mathcal{V}}\otimes\bar{\mu}$ est au mieux $\sqrt{\varepsilon}$-proche de μ^ε. Deuxièmement, mener notre analyse dans $\mathscr{P}_2\left(\mathbb{R}^2\right)$ rend possible l'interprétation de notre résultat à l'ordre 0, ce qui n'aurait pas été le cas si nous avions choisi un espace normé ne contenant pas les masses de Dirac. Ainsi, nous obtenons l'estimée suivante

Corollaire ([24])**.** *Sous les hypothèses du Théorème 6 et en considérant deux temps* t_0 *et* T *tels que* $0<t_0\,<\,T$*, il existe* $(C,\varepsilon_0)\,\in\,\left(\mathbb{R}_*^+\right)^2$ *tel que pour tout* $\varepsilon\le\varepsilon_0$ *on ait*

$$C^{-1}\sqrt{\varepsilon}\,\le\,W_2\left(\mu^\varepsilon_{t,\boldsymbol{x}}\,,\,\delta_{\mathcal{V}(t,\boldsymbol{x})}\otimes\bar{\mu}_{t,\boldsymbol{x}}\right)\,\le\,C\sqrt{\varepsilon}\,,\quad\forall(t,\boldsymbol{x})\in[\,t_0\,,\,T\,]\times K\,.$$

Nous allons maintenant expliquer les principales étapes nécessaires pour démontrer le Théorème 6. Le premier point important consiste à estimer les moments de la solution μ^ε uniformément par rapport à ε (voir Proposition 4.10 et 4.12 du Chapitre 4). Nous soulignons que les bornes obtenues sont uniformes par rapport au temps, grâce au propriétés de confinement de N, ainsi que par rapport à la variable spatiale $\boldsymbol{x}$ grâce aux propriétés d'uniforme intégrabilité prises sur Ψ. Le second point crucial est de considérer le changement d'échelle ν^ε suivant

$$\mu^\varepsilon(t,\,\boldsymbol{x},\boldsymbol{u})\,=\,\frac{1}{\varepsilon^{\frac{1}{2}}}\,\nu^\varepsilon\left(t,\,\boldsymbol{x},\,\frac{v-\mathcal{V}^\varepsilon}{\varepsilon^{\frac{1}{2}}},\,w-\mathcal{W}^\varepsilon\right)\,,$$

ainsi que son équivalent sur le profil asymptotique limite annoncé dans le Théorème 6

$$\nu(t,\boldsymbol{x},\boldsymbol{u})\,=\,\mathcal{M}_{\rho_0(\boldsymbol{x})}(v)\otimes\bar{\mu}(t,\boldsymbol{x},\mathcal{W}+w)\,.$$

L'avantage d'étudier ν^ε réside dans le fait que contrairement à μ^ε, sa limite ν n'est pas singulière. Pour estimer la convergence de ν^ε au sens de Wasserstein, on met en place une méthode de couplage : on considère l'équation vérifiée par les couplages π^ε entre ν^ε et ν et on étudie l'évolution des quantités suivantes le long des trajectoires de l'équation couplée

$$\left\{ \begin{aligned} \mathcal{A}(t) &:= \int_{\mathbb{R}^4} |v - v'|^2 \, \mathrm{d}\pi^\varepsilon(\boldsymbol{u}, \boldsymbol{u}') \,, \\ \mathcal{B}(t) &:= \int_{\mathbb{R}^4} |w - w'|^2 \, \mathrm{d}\pi^\varepsilon(\boldsymbol{u}, \boldsymbol{u}') \,. \end{aligned} \right.$$

Pour estimer ces deux quantités, on s'appuiera sur nos premières estimées des moments de μ^ε. Cette étape de la preuve correspond à l'estimation du terme $\mathcal{D}_1$ dans la Section 4.4 du Chapitre 4. Les méthodes de couplages sont classiques pour ce type de problème : on pourra par exemple se référer à [117] pour avoir un panorama des situations dans lesquelles il est possible de l'appliquer.

3.7 Convergence forte (Chapitre 5)

Dans ce paragraphe, nous améliorerons l'approche présentée en Section 0.3.6 pour obtenir un résultat de convergence forte de la distribution f^ε vers le profil asymptotique gaussien.
Les méthodes développées dans cette section ont démontré une certaine robustesse puisque nous avons été capable d'étendre leur champ d'application à l'étude de la limite parabolique du modèle de Vlasov-Poisson-Fokker-Planck présentée en Section 0.2.3.
Nous avons proposé deux méthodes, l'une permet d'obtenir un résultat de convergence en norme L^1 par rapport aux variables v et w tandis que l'autre produit un résultat de convergence dans des espaces L^2 à poids inverses gaussiens. Dans un souci de clarté, nous n'exposerons que le premier résultat aboutissant à une convergence L^1. Le second résultat, qui s'appuie sur des estimées de propagation de régularité, se trouve dans la Section 5.2.2.
Nous utiliserons la norme suivante

$$L^\infty_{\boldsymbol{x}} L^1_{\boldsymbol{u}} := L^\infty \left(K \,,\, L^1 \left(\mathbb{R}^2 \right) \right) .$$

De plus, nous désignerons par H l'entropie de Boltzmann, définie pour toute fonction $\mu : \mathbb{R}^2 \to \mathbb{R}^+$ par

$$H\left[\,\mu\,\right] = \int_{\mathbb{R}^2} \mu \, \ln(\mu) \, \mathrm{d}\boldsymbol{u} \,.$$

Pour finir, on notera τ_{w_0} la translation de longueur $w_0 \in \mathbb{R}$ par rapport à la variable w

$$\tau_{w_0} \, \mu\,(t, \boldsymbol{x}, v, w) = \mu\,(t, \boldsymbol{x}, v, w + w_0) \,.$$

Nous pouvons maintenant énoncer notre résultat principal. Comme le Théorème 6, celui-ci assure d'une part la convergence vers un profil de concentration Gaussien et d'autre part, caractérise la distribution marginale limite par rapport à la variable d'adaptation

w. Cependant, ce résultat améliore le Théorème 6 remarquablement car on obtient ici la **convergence forte** de la distribution μ^ε. De plus, on quantifie la convergence en démontrant des estimations d'erreur en ε uniformes par rapport à la variable d'espace $\boldsymbol{x} \in K$.

Théorème 7 ([21]). *Sous les hypothèses* (33a)-(33b) *sur* N, (35) *sur* Ψ, (41)-(43) *sur la donnée initiale, on considère l'unique suite de solutions* $(\mu^\varepsilon)_{\varepsilon>0}$ *de* (36) *et l'unique solution* $(\mathcal{V}, \bar{\mu})$ *de* (39) *dont la solution initiale vérifie*

$$\bar{\mu}_0 \in L^\infty\left(K\,,\, W^{2\,,1}(\mathbb{R})\right)\,, \quad \textit{et} \quad \sup_{\boldsymbol{x}\in K} \int_{\mathbb{R}} |w\,\partial_w\,\bar{\mu}_0(\boldsymbol{x}, w)|\,\mathrm{d}w < +\infty\,.$$

De plus, on suppose les conditions suivantes sur μ_0^ε

$$\sup_{\substack{(w_0,\gamma)\in\mathbb{R}^2\\ \varepsilon>0}} \left(\|H\,[\mu_0^\varepsilon]\|_{L^\infty(K)} + \frac{1}{|w_0|}\,\|\mu_0^\varepsilon - \tau_{w_0}\,\mu_0^\varepsilon\|_{L^\infty_{\boldsymbol{x}}L^1_{\boldsymbol{u}}} + \frac{1}{|\gamma|}\,\|\mu_0^\varepsilon - \tau_{\gamma v}\,\mu_0^\varepsilon\|_{L^\infty_{\boldsymbol{x}}L^1_{\boldsymbol{u}}} \right) < +\infty, \tag{44}$$

ainsi que la condition de compatibilité

$$\|\,\mathcal{U}_0\, - \,\mathcal{U}_0^\varepsilon\,\|_{L^\infty(K)} \,+\, \|\,\rho_0 - \rho_0^\varepsilon\,\|_{L^\infty(K)} \,+\, \|\,\bar{\mu}_0^\varepsilon\, - \,\bar{\mu}_0\|^{1/2}_{L^\infty_{\boldsymbol{x}}L^1_w} \underset{\varepsilon\to 0}{=} O\left(\sqrt{\varepsilon}\right)\,.$$

Il existe $(C\,,\,\varepsilon_0) \in \left(\mathbb{R}_+^*\right)^2$ *tel que pour tout* ε *inférieur à* ε_0*, on ait*

$$\left\|\,\mu^\varepsilon\, - \,\mathcal{M}_{\rho_0\,|\theta^\varepsilon|^{-2}}\,(\cdot\, - \,\mathcal{V})\otimes\bar{\mu}\,\right\|_{L^\infty(K,L^1([0,t],\mathbb{R}^2))} \,\leq\, C\,e^{C\,t}\,\sqrt{\varepsilon}\,, \quad \forall\,t\,\in\,\mathbb{R}^+\,,$$

où le taux de concentration θ^ε *est donné par*

$$\theta^\varepsilon(t,\boldsymbol{x})^2 \,=\, \varepsilon\,(1\, - \,\exp\left(-(\,2\,\rho_0^\varepsilon(\boldsymbol{x})\,t\,)\,/\,\varepsilon\right)) \,+\, \exp\left(-(\,2\,\rho_0^\varepsilon(\boldsymbol{x})\,t\,)\,/\,\varepsilon\right). \tag{45}$$

Comme pour le Théorème 6, il n'est pas demandé à la donnée initiale d'être bien préparée. Cependant, considérer des données générales est plus compliqué ici que dans le cadre de la convergence faible. En effet, comme dans la preuve du Théorème 6, le changement d'échelle

$$\mu^\varepsilon(t,\,\boldsymbol{x},\boldsymbol{u}) \,=\, \frac{1}{\varepsilon^{\frac{1}{2}}}\,\nu^\varepsilon\left(t,\,\boldsymbol{x},\,\frac{v-\mathcal{V}^\varepsilon}{\varepsilon^{\frac{1}{2}}},\,w-\mathcal{W}^\varepsilon\right)\,,$$

joue un rôle important dans notre analyse. Hélas, il n'est pas possible de l'utiliser tel quel dans notre situation car on a

$$H\,[\,\mu_0^\varepsilon\,] \,=\, H\,[\,\nu_0^\varepsilon\,] \,-\, \frac{1}{2}\,\ln(\varepsilon) \,\geq\, -\frac{1}{2}\,\ln(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} +\infty\,,$$

ce qui reviendrait à considérer des données initiales bien préparées. C'est pour cette raison que nous introduisons le taux de concentration θ^ε donné par (45) : on effectue maintenant le changement d'échelle

$$\mu^\varepsilon(t,\,\boldsymbol{x},\boldsymbol{u}) \,=\, \frac{1}{\theta^\varepsilon}\,\nu^\varepsilon\left(t,\,\boldsymbol{x},\,\frac{v-\mathcal{V}^\varepsilon}{\theta^\varepsilon},\,w-\mathcal{W}^\varepsilon\right)\,, \tag{46}$$

ce qui assure $H\,[\,\mu_0^\varepsilon\,] \;=\; H\,[\,\nu_0^\varepsilon\,]$ puisque $\theta^\varepsilon(t=0) \;=\; 1$. On retrouve alors le bon taux de concentration puisque $\theta^\varepsilon \underset{\varepsilon\to 0}{\sim} \sqrt{\varepsilon}$ pour tout $t > 0$.
Pour démontrer ce théorème, le point principal consiste à obtenir la convergence de ν^ε défini par (46). Pour cela, on commence par appliquer un argument d'entropie relative classique permettant de montrer que ν^ε converge vers l'équilibre local suivant

$$\nu^\varepsilon \underset{\varepsilon\to 0}{\sim} \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon\,,$$

où $\bar{\nu}^\varepsilon$ est la marginale de ν^ε par rapport à w (voir Proposition 5.9).
La preuve devient plus originale lorsque l'on en vient à l'analyser la convergence de $\bar{\nu}^\varepsilon$. La difficulté provient du fait que l'équation sur $\bar{\nu}^\varepsilon$ ne fait pas apparaître explicitement son problème limite. Pour la surmonter, on emploie la méthode déjà mise en œuvre dans la Section 0.2.3 pour étudier la limite parabolique du modèle Vlasov-Poisson-Fokker-Planck. Cette stratégie est expliquée à l'aide d'un exemple simplifié au début de la Section 5.3 ; dans notre contexte particulier, elle consiste à introduire le changement d'échelle suivant

$$\bar{g}^\varepsilon\,(t\,,\,\boldsymbol{x}\,,\,w) \;=\; \int_{\mathbb{R}} \nu^\varepsilon\,(t\,,\,\boldsymbol{x}\,,\,v\,,\,w - \gamma^\varepsilon\,v)\,\mathrm{d}v\,,$$

où γ^ε est choisi de manière à ce que l'équation vérifiée par $\bar{g}^\varepsilon$ fasse apparaître explicitement l'équation vérifiée par la limite de $\bar{\nu}^\varepsilon$. Cela permet d'obtenir un résultat de convergence sur cette nouvelle quantité (voir Proposition 5.10). Pour conclure, on propage la régularité par rapport à w assurée par l'hypothèse (44) (voir Proposition 5.7), de manière à déduire un résultat de convergence sur $\bar{\nu}^\varepsilon$ à partir de celui obtenu sur le changement d'échelle $\bar{g}^\varepsilon$ (voir Proposition 5.11).

3.8 Approche Hamilton-Jacobi et convergence uniforme (Chapitre 6)

Dans cette section, on suit une approche différente de celles proposées précédemment, permettant d'obtenir des résultats de convergence uniformes par rapport à v et w. Au lieu de travailler sur la distribution f^ε, on étudie la convergence de sa transformée de Hopf-Cole, définie par

$$\phi^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) := \varepsilon \ln\left(\sqrt{\frac{2\pi\varepsilon}{\rho_0}} f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\right), \qquad \forall\,(t,\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2\,. \tag{47}$$

Cette approche à été intensément utilisée, notamment pour décrire la concentration génomique dans des modèles de sélection-mutation [93, 8, 6, 166, 37, 156, 50], la propagation de fronts [49, 51, 36]. De plus, elle a donné lieu à des discussions complexes autour de l'unicité pour certaines équations de Hamilton Jacobi avec contraintes [167, 52]. C'est en effet un outil puissant pour décrire la concentration d'une distribution puisqu'en inversant (6.9), on obtient

$$f^\varepsilon = \sqrt{\frac{\rho_0}{2\pi\varepsilon}}\,\exp\left(\frac{\phi^\varepsilon}{\varepsilon}\right),$$

ce qui signifie que les points de concentration de f^ε sont caractérisés par l'ensemble de niveau $\{\phi^\varepsilon = 0\}$. De plus, le comportement de ϕ^ε lorsque $v \to +\infty$ fournit une description précise sur la queue de distribution asymptotique de f^ε par rapport à v.
Dans notre situation, nous savons que le profil de concentration est Gaussien. On s'attend donc à la convergence suivante

$$\phi^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \underset{\varepsilon\to 0}{\longrightarrow} -\frac{1}{2}\,\rho_0(\boldsymbol{x})\,|v - \mathcal{V}(t, \boldsymbol{x})|^2 \,,$$

où $\mathcal{V}$ et ρ_0 sont donnés par (39). Cette convergence est l'objet de notre résultat principal, dans lequel nous prouvons qu'elle se produit uniformément par rapport à toutes les variables $(t, \boldsymbol{x}, \boldsymbol{u})$ dans la limite $\varepsilon \to 0$. Chose peu commune pour ce type d'approche, nous obtenons le taux de convergence (formellement) optimal $O(\varepsilon)$. Notre résultat est vérifié sous les hypothèses supplémentaires suivantes sur le drift N

$$\limsup_{|v|\to+\infty} \frac{N(v)}{\operatorname{sgn}(v)|v|^p} < 0, \qquad \sup_{|v|\geq 1}\left|\frac{N(v)}{|v|^p}\right| < +\infty, \tag{48}$$

où $p \geq 2$, ainsi que

$$\sup_{|v|\geq 1}\left(|N''(v)| + |N'(v)|\right)|v|^{-p'} < +\infty, \tag{49}$$

où $p' \geq 0$. On donne ici une version allégée du résultat de convergence, dont on pourra trouver l'énoncé complet dans le Théorème 6.6

Théorème 8 ([23])**.** *Sous les hypothèses* (48)-(49) *sur* N*,* (35) *sur* Ψ*,* (41)-(43) *sur la donnée initiale et en supposant qu'il existe une constante* $C > 0$ *indépendante de* ε *telle que la condition de compatibilité suivante soit vérifiée*

$$\|\mathcal{U}_0 - \mathcal{U}_0^\varepsilon\|_{L^\infty(K)} + \|\rho_0 - \rho_0^\varepsilon\|_{L^\infty(K)} \leq C\varepsilon,$$

ainsi que sous l'hypothèse de "petitesse" suivante

$$\begin{cases} \left|\phi_0^\varepsilon + \dfrac{1}{2}\rho_0\,|v - \mathcal{V}|^2 - \varepsilon n\right| \leq \varepsilon C\left(1 + |\boldsymbol{u}|^2\right), \quad \forall(\boldsymbol{x}, \boldsymbol{u}) \in K \times \mathbb{R}^2, & \text{(50a)} \\[2ex] \displaystyle\int_{\mathbb{R}^2}\left(|v - \mathcal{V}_0^\varepsilon|^2 + |v - \mathcal{V}_0^\varepsilon|^{p'+1}\right) f_0^\varepsilon(\boldsymbol{x}, \boldsymbol{u})\mathrm{d}\boldsymbol{u} \leq C\varepsilon, \quad \forall \boldsymbol{x} \in K\,, & \text{(50b)} \end{cases}$$

où n *désigne une primitive de* N *:* $n'(v) = N(v)$*, la suite* $(\phi^\varepsilon)_{\varepsilon>0}$ *définie par* (47) *converge localement uniformément sur* $\mathbb{R}^+ \times K \times \mathbb{R}^2$ *vers* $-\rho_0\,|v - \mathcal{V}|^2/2$ *à taux* ε*. Plus précisément, il existe deux constantes* C *et* ε_0 *telles que pour tout* $\varepsilon \leq \varepsilon_0$*, l'estimée suivante soit vérifiée*

$$\left|\phi^\varepsilon + \frac{1}{2}\rho_0\,|v - \mathcal{V}|^2 - \varepsilon n\right|(t, \boldsymbol{x}, \boldsymbol{u}) \leq \varepsilon C e^{Ct}\left(1 + |\boldsymbol{u}|^2\right), \quad \forall(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K, \textit{p.p. en } \boldsymbol{u} \in \mathbb{R}^2,$$

où $\mathcal{V}$ *et* ρ *sont donnés par* (39)*.*

Avant d'aller plus loin, faisons quelques commentaires sur notre résultat. Nous insistons sur le fait qu'il fournit une estimée de convergence uniforme par rapport à toutes les variables, ce qui représente une grande amélioration par rapport aux Théorèmes 6 et 7. De plus, notre approche s'inscrit dans la lignée de [196] dans lequel fut proposé d'utiliser la transformée de Hopf-Cole pour étudier ce modèle. Cependant, la preuve présentée dans [196] repose sur un argument de compacité, ce qui a deux conséquences : premièrement, la limite $-\frac{1}{2}\rho_0 \left|v-\mathcal{V}\right|^2$ n'est pas identifiée et deuxièmement, le résultat de convergence n'est pas quantitatif. Pour finir, l'étude proposée dans [196] se place dans un cadre spatialement homogène.
D'autre part, nous soulignons que contrairement aux Théorèmes 6 et 7, ce résultat s'applique à des données initiales bien préparées. En effet, les hypothèses (50a)-(50b) s'interprètent sur f_0^ε de la manière suivante

$$f_0^\varepsilon\left(\boldsymbol{x},\boldsymbol{u}\right) \underset{\varepsilon\to 0}{=} \sqrt{\frac{\rho_0(\boldsymbol{x})}{2\pi\varepsilon}}\exp\left(-\frac{\rho_0(\boldsymbol{x})}{2\varepsilon}\left|v-\mathcal{V}_0\left(\boldsymbol{x}\right)\right|^2+O(1)\right),$$

ce qui signifie que f^ε est initialement concentrée. Cette restriction est fréquente dans les études utilisant la transformation de Hopf-Cole : c'est par exemple le cas dans [93, 6, 37, 166, 167, 50, 196]. Pour finir, il est possible d'interpréter notre résultat de convergence sur la distribution du réseau puisqu'il assure que f^ε converge uniformément vers 0 sur les sous ensembles compacts de

$$\mathbb{R}^+\times K\times\mathbb{R}^2\setminus\left\{v\neq\mathcal{V}\left(t,\boldsymbol{x}\right)\right\}$$

Nous allons maintenant expliquer la stratégie pour démontrer le Théorème 8. La difficulté principale est due au fait que le drift N n'est pas Lipschitz. Cela rend difficile l'application des méthodes usuelles consistant à propager de la régularité uniformément par rapport à ε en appliquant la méthode de Bernstein (voir [5] pour une introduction générale ainsi que [6, 196] pour des applications à des cas concrets). Plutôt que d'estimer les dérivées de ϕ^ε, nous étudions donc le second terme de son développement asymptotique. Cette stratégie est motivée par l'observation suivante : le drift non linéaire N induit des fluctuations de ϕ^ε à l'ordre ε puisque l'équation sur ϕ^ε s'écrit

$$\frac{1}{\varepsilon}\,\partial_v\left(-\frac{1}{2}\rho_0^\varepsilon\left|v-\mathcal{V}^\varepsilon\right|^2+\varepsilon n(v)-\phi^\varepsilon\right)\partial_v\phi^\varepsilon+\ldots=0\,,$$

où la correction $n(v)$ est donnée par $n'(v)=N(v)$ tandis que "$\ldots$" désigne les termes d'ordre inférieur par rapport à v et w. Ainsi nous définissons ϕ_1^ε le second terme dans le développement asymptotique de ϕ^ε par

$$\phi^\varepsilon=-\frac{1}{2}\rho_0^\varepsilon\left|v-\mathcal{V}^\varepsilon\right|^2+\varepsilon\phi_1^\varepsilon\,.$$

Nous devons alors montrer que ϕ_1^ε est uniformément borné par rapport à ε. Pour ce faire, nous construisons des sur- et sous-solutions de l'équation vérifiée par ϕ_1^ε qui incorporent le drift N. Enfin, nous utilisons un principe de comparaison pour d'encadrer ϕ_1^ε par les sur- et sous-solutions construites, concluant ainsi notre argument.

Première partie

Plasma physics

Chapitre 1

Diffusive limit of the Vlasov-Poisson-Fokker-Planck model : quantitative and strong convergence results

This work tackles the diffusive limit for the Vlasov-Poisson-Fokker-Planck model. We derive *a priori* estimates which hold without restriction on the phase-space dimension and propose a strong convergence result in a L^2 space. Furthermore, we strengthen previous results by obtaining an explicit convergence rate arbitrarily close to the (formal) optimal rate, provided that the initial data lies in some L^p space with p large enough. Our result holds on bounded time intervals whose size grow to infinity in the asymptotic limit with explicit lower bound. The analysis relies on identifying the right set of phase-space coordinates to study the regime of interest. In this set of coordinates the limiting model arises explicitly.

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Contents

1.1 Introduction

1.1.1 Physical model and motivation

In this article, we study a plasma composed of moving electrons and fixed ions. We denote by $f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{v}) \geq 0$ the density of electron at time $t \in \mathbb{R}^+$, position $\boldsymbol{x} \in \mathbb{K}^d$ with $\mathbb{K} \in \{\mathbb{T}, \mathbb{R}\}$ and velocity $\boldsymbol{v} \in \mathbb{R}^d$ whereas the density of ions is given by $\rho_i(\boldsymbol{x}) \geq 0$ with $\rho_i \in L^1\left(\mathbb{K}^d\right)$. We focus on the Vlasov-Poisson-Fokker-Planck (VPFP) model, which describes a situation where the particles interact through electrostatic effects and where electrons are subjected to collisions with the ion background. Considering the regime in which the electron/ion mass ratio and the mean free path of electrons have the same magnitude, the VPFP model reads

$$\begin{cases} \partial_t f^\varepsilon + \dfrac{1}{\varepsilon}\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f^\varepsilon + \dfrac{1}{\varepsilon}\mathbf{E}^\varepsilon\cdot\nabla_{\boldsymbol{v}} f^\varepsilon = \dfrac{1}{\varepsilon^2}\nabla_{\boldsymbol{v}}\cdot\left[\boldsymbol{v} f^\varepsilon + \nabla_{\boldsymbol{v}} f^\varepsilon\right], \\[2ex] \mathbf{E}^\varepsilon = -\nabla_{\boldsymbol{x}}\phi^\varepsilon, \quad -\Delta_{\boldsymbol{x}}\phi^\varepsilon = \rho^\varepsilon - \rho_i, \quad \rho^\varepsilon = \displaystyle\int_{\mathbb{R}^d} f^\varepsilon\,\mathrm{d}\boldsymbol{v}, \\[2ex] f^\varepsilon(0,\boldsymbol{x},\boldsymbol{v}) = f_0^\varepsilon(\boldsymbol{x},\boldsymbol{v}), \end{cases} \tag{1.1}$$

where the self consistent electric field $\mathbf{E}^\varepsilon$ is induced by Coulombian interactions between charges whereas the Fokker-Planck operator on the right-hand side of the first line in (1.1) accounts for collisions with the ion background. A detailed description of the re-scaling process in order to derive (1.1) may be found in [109, 192]. Since mass is conserved along the trajectories of (1.1), we normalize f^ε as follows

$$\int_{\mathbb{K}^d\times\mathbb{R}^d} f_0^\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v} = 1\,.$$

When $\mathbb{K} = \mathbb{T}$, we also impose the compatibility assumption

$$\int_{\mathbb{K}^d\times\mathbb{R}^d} f_0^\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v} = \int_{\mathbb{K}^d}\rho_i\,\mathrm{d}\boldsymbol{x}\,,$$

which is then satisfied for all positive time $t \geq 0$. In this article, we focus on the asymptotic analysis of (1.1) in the diffusive regime corresponding to the limit $\varepsilon \ll 1$.

1.1.2 Formal derivation

In this section, we carry out a formal analysis of the diffusive regime. Our purpose is to derive the limit of f^ε as $\varepsilon \to 0$ and to explain the key idea of this article in order to rigorously justify this limit. Considering the leading order term in equation (1.1), we deduce that as ε vanishes, f^ε converges to the following local equilibrium of the Fokker-Planck operator

$$f^\varepsilon \underset{\varepsilon\to 0}{\sim} \rho^\varepsilon \otimes \mathcal{M}\,,$$

where $\mathcal{M}$ stands for the standard Maxwellian distribution over $\mathbb{R}^d$

$$\mathcal{M}(\boldsymbol{v}) \,=\, (2\pi)^{-\,d/2}\, \exp\left(-\,\frac{1}{2}\,|\boldsymbol{v}|^2\right), \tag{1.2}$$

and where the spatial distribution ρ^ε given in (1.1) solves the following equation, obtained after integrating the first line in (1.1) with respect to $\boldsymbol{v}$

$$\partial_t\,\rho^\varepsilon \,+\, \frac{1}{\varepsilon}\,\nabla_{\boldsymbol{x}}\cdot\int_{\mathbb{R}^d} \boldsymbol{v}\, f^\varepsilon\,\mathrm{d}\boldsymbol{v} \,=\, 0\,.$$

It is unfortunately trickier to compute the limit of ρ^ε. The difficulty stems from the free transport operator in the first line of (1.1) which induces a stiff dependence with respect to f^ε in the equation on ρ^ε. The key idea is therefore to cancel this stiff dependence by considering a modified spatial distribution π^ε. To this aim, we introduce a re-scaled version g^ε of f^ε

$$(\tau_{\varepsilon\boldsymbol{v}}\, g^\varepsilon)\,(t,\boldsymbol{x},\boldsymbol{v}) \,=\, f^\varepsilon(t,\boldsymbol{x},\boldsymbol{v})\,,\quad \forall\,(t,\boldsymbol{x},\boldsymbol{v})\in\mathbb{R}^+\times\mathbb{K}^d\times\mathbb{R}^d\,,$$

where $\tau_{\boldsymbol{x}_0}\, g$ denotes the function obtained by translating any function g in the direction $\boldsymbol{x}_0 \in \mathbb{R}^d$, that is

$$\tau_{\boldsymbol{x}_0}\, g(t,\boldsymbol{x},\boldsymbol{v}) \,:=\, g(t,\boldsymbol{x}+\boldsymbol{x}_0,\boldsymbol{v})\,,\quad \forall\,(t,\boldsymbol{x},\boldsymbol{v})\in\mathbb{R}^+\times\mathbb{K}^d\times\mathbb{R}^d\,.$$

Defining the new variable $\boldsymbol{y} := \boldsymbol{x} + \varepsilon\,\boldsymbol{v}$ and operating the change of variable $\boldsymbol{x}\to\boldsymbol{y}$ in the first line of (1.1), it turns out that g^ε solves the following equation

$$\partial_t g^\varepsilon \,+\, \left(\nabla_{\boldsymbol{y}} + \frac{1}{\varepsilon}\nabla_{\boldsymbol{v}}\right)\cdot\left[\tau_{\varepsilon\boldsymbol{v}}^{-1}\,(\,\mathbf{E}^\varepsilon\, f^\varepsilon)\right] \,-\, \Delta_{\boldsymbol{y}}\, g^\varepsilon \,=\, \frac{1}{\varepsilon^2}\,\nabla_{\boldsymbol{v}}\cdot[\,\boldsymbol{v}\, g^\varepsilon + \nabla_{\boldsymbol{v}}\, g^\varepsilon\,] \,+\, \frac{2}{\varepsilon}\,\nabla_{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{y}}\, g^\varepsilon,$$

where one may notice that the free transport operator has been canceled in the re-scaling process. Therefore, the marginal π^ε of g^ε defined as

$$\pi^\varepsilon\,(t,\boldsymbol{y}) \,=\, \int_{\mathbb{R}^d} g^\varepsilon\,(t,\boldsymbol{y},\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \,=\, \int_{\mathbb{R}^d} \tau_{\varepsilon\boldsymbol{v}}^{-1} f^\varepsilon\,(t,\boldsymbol{y},\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}\,,\quad \forall\,(t,\boldsymbol{y})\in\mathbb{R}^+\times\mathbb{K}^d\,,$$

solves the following equation, obtained after integrating the equation on g^ε with respect to $\boldsymbol{v}$

$$\partial_t\,\pi^\varepsilon \,+\, \nabla_{\boldsymbol{y}}\cdot\left[\int_{\mathbb{R}^d}\tau_{\varepsilon\boldsymbol{v}}^{-1}\,(\,\mathbf{E}^\varepsilon\, f^\varepsilon)\,(t,\boldsymbol{y},\boldsymbol{v})\;\mathrm{d}\boldsymbol{v}\right] \,-\, \Delta_{\boldsymbol{y}}\,\pi^\varepsilon \,=\, 0\,. \tag{1.3}$$

In the limit $\varepsilon \to 0$, we have $\rho^\varepsilon \sim \pi^\varepsilon$ and thus $\mathbf{E}^\varepsilon \sim \mathbf{I}^\varepsilon$, where

$$\mathbf{I}^\varepsilon = -\nabla_{\boldsymbol{y}} \psi^\varepsilon , \quad -\Delta_{\boldsymbol{y}} \psi^\varepsilon = \pi^\varepsilon - \rho_i . \tag{1.4}$$

Furthermore, the translation operator $\tau_{\varepsilon \boldsymbol{v}}^{-1}$ in (1.3) cancels as $\varepsilon \to 0$. Therefore, the dependence with respect to f^ε is removed from the nonlinear term in (1.3), that is

$$\int_{\mathbb{R}^d} \tau_{\varepsilon \boldsymbol{v}}^{-1} \left(\mathbf{E}^\varepsilon f^\varepsilon \right) (t, \boldsymbol{y}, \boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \sim \mathbf{I}^\varepsilon \pi^\varepsilon .$$

Therefore, we deduce

$$\pi^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \rho ,$$

where ρ solves the following drift-diffusion-Poisson equation

$$\begin{cases} \partial_t \rho + \nabla_{\boldsymbol{x}} \cdot [\mathbf{E} \rho] - \Delta_{\boldsymbol{x}} \rho = 0 , \\ \\ -\Delta_{\boldsymbol{x}} \phi = \rho - \rho_i , \quad \mathbf{E} = -\nabla_{\boldsymbol{x}} \phi , \end{cases} \tag{1.5}$$

Since we have $\pi^\varepsilon \sim \rho^\varepsilon$, this leads to the following formal result

$$f^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \rho \otimes \mathcal{M} .$$

As it turns out, the set of coordinates $(\boldsymbol{y}, \boldsymbol{v})$, with $\boldsymbol{y} := \boldsymbol{x} + \varepsilon \boldsymbol{v}$ is very convenient to study the diffusive regime : it removes from the equation on ρ^ε the stiff dependence with respect to f^ε due to the transport operator. Instead, we end up with equation (1.3) which is close to the limiting model (1.5). Therefore, the modified spatial distribution π^ε will play a key role in our analysis.

The present work consists in rigorously justifying this formal derivation. It is the continuation of a long process in order to justify the diffusive limit of the VPFP model.
The story starts in [192], in which F. Poupaud and J. Soler demonstrate strong convergence of f^ε on short time intervals without restriction on the dimension. The present article is somehow close to [192] since it uses the same functional framework. Their key idea is to estimate the norm $|||f^\varepsilon|||_p$ (defined in Section 1.2 below), which in turn provides a L^∞-control over the field $\mathbf{E}^\varepsilon$. F. Poupaud and J. Soler manage to control $|||f^\varepsilon|||_p$ on short time intervals. This estimate allow them to prove strong compactness of the sequence $(\rho_\varepsilon)_{\varepsilon>0}$ and to pass to the limit in the nonlinear term thanks to their L^∞-control over $\mathbf{E}^\varepsilon$. In the same article, F. Poupaud and J. Soler remove the short time restriction and obtain local in time weak convergence in dimension $d = 2$. In this case, the Coulomb potential has a particular structure which allows them to pass to the limit in the nonlinear term. Their strategy was then extended by T. Goudon [125] who proved similar results for Newtonian interactions when $d = 2$.
The story goes on in [103], in which N. El Ghani and N. Masmoudi prove strong and local in time convergence without restriction on the dimension d. Their method relies on averaging lemmas [122, 96] to prove strong compactness of the spatial distributions $(\rho_\varepsilon)_{\varepsilon>0}$

associated to free energy solutions $(f_\varepsilon)_{\varepsilon>0}$ to (1.1). In such a weak regularity framework, the nonlinear term in (1.1) may not be defined. Hence, authors use renormalization techniques introduced by R. J. Diperna and P.-L. Lions [95] to pass to the limit in the nonlinear term. This approach was initially designed to treat the case of a linearized Boltzmann operator [165]. Since then, it has been extended in various directions including multi-species models [212], Vlasov-Maxwell-Fokker-Planck model [102] and strongly magnetized plasma models [137].

More recently, authors adapted hypocoercivity methods [211, 97, 135] to the present situation in order to achieve global in time convergence. This is the case of [138], in which M. Herda and M. Rodrigues prove global in time strong convergence in weakly nonlinear regime (that is, under joint restrictions on the Debye length and the size of the initial data) when $d = 2$. The main difficulty to prove global in time convergence is that the Fokker-Planck operator in (1.1) acts only on the velocity variable and therefore gives no straightforward information regarding the asymptotic behavior of ρ_ε as $t \to +\infty$. Hypocoercivity methods rely on constructing of a modified relative entropy functional designed to recover dissipation with respect to the spatial variable. In [138], M. Herda and M. Rodrigues design such functional, allowing to deduce global in time convergence of the sequence $(f^\varepsilon)_{\varepsilon>0}$. Using similar methods, exponential relaxation as $t \to +\infty$ had already been proved in [136], in a weakly nonlinear setting. We also mention [1], which proves that the linearized model associated to (1.1) relaxes exponentially fast towards equilibrium as $t \to +\infty$ with uniform rates in the limit $\varepsilon \to 0$. Authors deduce the same result on the nonlinear model in the case $d = 1$.

In perturbative settings, precise results are available [213], based on a spectral analysis of (1.1), treating simultaneously the diffusion limit with explicit convergence rates in ε and the long time behavior with optimal exponential rate in t.

In this article, we propose a strong convergence result in some L^2 space and prove that it occurs at rate $O(\varepsilon^\beta)$, with $\beta = (p-d)/(p-1)$ if f_0^ε lies in some L^p space : the (formal) optimal convergence rate is reached as $p \to +\infty$. Our analysis is non-perturbative and it holds in any dimension $d \geq 1$. Our convergence estimate holds on bounded time intervals $[0, T^\varepsilon]$, for some explicitly given T^ε (see Theorem 1.1) which satisfies $T^\varepsilon \to +\infty$ as $\varepsilon \to 0$. We point out that it should be possible to adapt our analysis to Newtonian interactions as long as the macroscopic model does not develop singularities but we do not follow this path to avoid these issues.

The article is organized as follows : in Section 1.2, we give our functional setting and state our main result, in Section 1.3 we derive uniform estimates for the solution to (1.1), then we conclude with Section 1.4 in which we prove our main result.

1.2 Functional setting and main result

In the forthcoming analysis, we work with the following norm defined for all exponent $p \geq 1$

$$|||f|||_p \,=\, \left(\int_{\mathbb{K}^d\times\mathbb{R}^d} \left|\frac{f}{\mathcal{M}}\right|^p \mathcal{M}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\right)^{1/p},$$

and denote by $L^p\left(\mathcal{M}\right)$ the set of all function whose latter norm is finite. Furthermore, we denote $\|\cdot\|_{L^p}$ the usual norm over $L^p\left(\mathbb{K}^d\right)$.

Existence and uniqueness theory for (1.1) has been widely investigated and therefore it will not be our concern here. We mention [85] in which global classical solutions are constructed in dimension $d = 1, 2$, [179, 180] which extend this result to dimension $d = 3$ in both the friction and frictionless cases and notably prove regularizing effects and thus obtain infinite regularity, [33] in which existence and uniqueness of a global strong solution is obtained when $d = 3$ and with uniform bounds on the initial data and then [34] in which regularizing effects are proved for this weak solution, finally, we mention [57] which treats the case of an initial data in L^p and constructs solutions in dimensions $d = 3, 4$. Since this article does not require any constraint on the dimension, we consider a strong solution to (1.1) in dimension $d \geq 1$. Our main result reads as follows

Theorem 1.1. *Consider some exponent $p > d$ and set*

$$(\gamma\,,\,\beta) = \left(1-\frac{d}{p}\,,\,\frac{p-d}{p-1}\right).$$

Suppose that the sequence $(f_0^\varepsilon)_{\varepsilon>0}$ meets the following assumption

$$|||f_0^\varepsilon|||_p + \sup_{|\boldsymbol{x}_0|\leq 1}\left(|\boldsymbol{x}_0|^{-\beta}\,|||\tau_{\boldsymbol{x}_0}f_0^\varepsilon - f_0^\varepsilon|||_p\right) + \varepsilon^{-\beta}\left(\|\pi_0^\varepsilon - \rho_0^\varepsilon\|_{L^p} + \|\rho_0^\varepsilon - \rho_0\|_{L^p}\right) \leq\, m_p\,, \quad (1.6)$$

for some positive constant m_p independent of ε. On top of that, suppose

$$\rho_0 \in L^p\cap L^\infty\left(\mathbb{K}^d\right), \quad \textit{and} \quad \rho_i \in L^{p+1}\cap L^\infty\left(\mathbb{K}^d\right), \quad (1.7)$$

and define

$$C_{\rho_0,\rho_i} \,:=\, \max\left(\|\,\rho_0\,\|_{L^p}^2\,,\,\|\,\rho_i\,\|_{L^{p+1}}^{2-2/p^2}\,,\,\|\,\rho_0\,\|_{L^\infty}\,,\,\|\,\rho_i\,\|_{L^\infty}\right).$$

Consider strong solutions $(f^\varepsilon)_{\varepsilon>0}$ to (1.1) and ρ to (1.5) with initial data $(f_0^\varepsilon)_{\varepsilon>0}$ and ρ_0 respectively. For all time $T > 0$, there exist two positive constants C_T and ε_T such that for all $\varepsilon \leq \varepsilon_T$, it holds

$$\|f^\varepsilon \,-\, \rho\,\mathcal{M}\|_{L^2([0,T]\,,\,L^2(\mathcal{M}))} \,\leq\, C_T\,\varepsilon^\beta\,.$$

More precisely, there exists a constant C only depending on exponent p and dimension d such that for all $\varepsilon \leq 1$ it holds

$$\|f^\varepsilon \,-\, \rho\,\mathcal{M}\|_{L^2([0,t]\,,\,L^2(\mathcal{M}))} \,\leq\, \varepsilon^\beta\left(C\,m_p^{2\,p'}\,e^{C_{\rho_0,\rho_i}C\,t} \,+\, e^{C_{\rho_0,\rho_i}^{-1}C\,m_p^2\,e^{C_{\rho_0,\rho_i}C\,t}}\right),$$

where $p' = p/(p-1)$ and for all time t less than T^ε, where T^ε is defined as

$$T^\varepsilon = \frac{1}{C_{\rho_0,\rho_i}\, C} \ln\left(1 + \frac{C_{\rho_0,\rho_i}}{4 \, |||f_0^\varepsilon|||_p^2 + C\, m_p^2\, \varepsilon^{\gamma\left(1+\frac{2}{p-1}\right)}}\, \varepsilon^{-\gamma}\right),$$

which ensures $T^\varepsilon \longrightarrow +\infty$ *as* $\varepsilon \to 0$.

The main difficulty consists in estimating the nonlinear term $\mathbf{E}^\varepsilon f^\varepsilon$ in (1.1). Indeed, if we follow the same method as in [192] and take directly its $|||\cdot|||_p$-norm, we end up with the following differential inequality for $|||f^\varepsilon|||_p$, which blows up in finite time

$$\frac{\mathrm{d}}{\mathrm{d}t} |||f^\varepsilon|||_p^p \lesssim |||f^\varepsilon|||_p^{p+2}.$$

The key point is therefore to include the re-scaled marginal π^ε in our computations. More precisely, we perform the following decomposition

$$\mathbf{E}^\varepsilon = (\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon) + (\mathbf{I}^\varepsilon - \mathbf{E}) + \mathbf{E},$$

where $\mathbf{E}$ and $\mathbf{I}^\varepsilon$ are given in (1.5) and (1.4) respectively. We rely on a functional inequality to prove that $(\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon)$ is of order $\varepsilon^\gamma\, |||f^\varepsilon|||_p$ and estimate $(\mathbf{I}^\varepsilon - \mathbf{E})$ and $\mathbf{E}$ thanks to the properties of drift-diffusion equation (1.5). It enables to derive the following differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} |||f^\varepsilon|||_p^p \lesssim \varepsilon^\gamma\, |||f^\varepsilon|||_p^{p+2}.$$

From the latter inequality, we bound $|||f^\varepsilon|||_p$ on time intervals with size of order $|\ln\varepsilon|$; this provides a global in time estimate in the limit $\varepsilon \to 0$.

1.3 *A priori* estimates

The main object of this section consists in deriving *a priori* estimates for (1.1) in $L^p(\mathcal{M})$ (see (1) in Proposition 1.4). Building on this key estimate, we deduce that π^ε given by (1.3) converges towards the solution ρ to the macroscopic model (1.5) (see (2) in Proposition 1.4) and prove equicontinuity for solutions to (1.1) (see Proposition 1.6). Let us first introduce some notations and recall a functional inequality that will be used in the proofs of this section. For all function $\rho \in L^p \cap L^1\left(\mathbb{K}^d\right)$ with $p > 1$, which in the case $\mathbb{K} = \mathbb{T}$ meets the compatibility assumption

$$\int_{\mathbb{K}^d} \rho\, \mathrm{d}\boldsymbol{x} = 0,$$

there exists a unique solution $\Delta_{\boldsymbol{x}}^{-1}\rho$ in $W^{2,p}\left(\mathbb{K}^d\right)$ to the Poisson equation (see [120, Section 9.6])

$$\begin{cases} \Delta_{\boldsymbol{x}}\, \phi = \rho, \\ \displaystyle\int_{\mathbb{K}^d} \phi\, \mathrm{d}\boldsymbol{x} = 0, \quad \text{if } \mathbb{K} = \mathbb{T}. \end{cases}$$

Furthermore, thanks to Morrey's inequality, we have the following estimate

$$\left\|\nabla_{\boldsymbol{x}}\, \Delta_{\boldsymbol{x}}^{-1}\, \rho\right\|_{\mathscr{C}^{0,\gamma}} \,\leq\, m_{d,p}\, \|\rho\|_{L^p}\ , \tag{1.8}$$

for all exponent $p > d$, where the constant $m_{d,p}$ only depends on (d,p) and where $\mathscr{C}^{0,\gamma}$ stands for the set of bounded, γ-Hölder functions, with $\gamma \,=\, 1 \,-\, d/p$.
In this section, we denote by Δ_p the dissipation of L^p-norms due to the Laplace operator

$$\Delta_p\,[\,\rho\,] \,=\, (p-1)\,\int_{\mathbb{K}^d} |\,\nabla_{\boldsymbol{x}}\,\rho\,|^2\,|\,\rho\,|^{p-2}\,\mathrm{d}\boldsymbol{x}\,,$$

and by $\mathcal{D}_p$ the dissipation of $L^p\,(\mathcal{M})$-norms due to the Fokker Planck operator

$$\mathcal{D}_p\,[\,f\,] \,=\, (p-1)\int_{\mathbb{K}^d\times\mathbb{R}^d} \left|\nabla_{\boldsymbol{v}}\left(\frac{f}{\mathcal{M}}\right)\right|^2\left|\frac{f}{\mathcal{M}}\right|^{p-2}\mathcal{M}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,.$$

We start with the following intermediate result which will be essential in order to propagate $L^p\,(\mathcal{M})$-norms.

Lemma 1.2. *Consider a smooth solution f^ε to equation* (1.1)*. For all exponent $p > d$ and all positive ε, it holds*

$$\|\,\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\,\|_{L^\infty} \,\leq\, C_{d,p}\,\varepsilon^\gamma\,\|\!|f^\varepsilon|\!\|_p\,,\quad \forall\, t\in\mathbb{R}^+\,,$$

where exponent γ is given by $\gamma \,=\, 1 - d\,/\,p$, and where $C_{d,p}$ is a constant only depending the dimension d and exponent p. In the latter estimates, the electric fields $\mathbf{E}^\varepsilon$ and $\mathbf{I}^\varepsilon$ are given by (1.1) *and* (1.4).

Proof. We consider some positive ε and some $(t,\boldsymbol{x}) \in \mathbb{R}^+ \times \mathbb{K}^d$. We have

$$(\rho^\varepsilon \,-\, \pi^\varepsilon)\,(t,\boldsymbol{x}) \,=\, \int_{\mathbb{R}^d}\left(f^\varepsilon \,-\, \tau_{\varepsilon\boldsymbol{v}}^{-1} f^\varepsilon\right)(t,\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}\,.$$

Applying the operator $\nabla_{\boldsymbol{x}}\,\Delta_{\boldsymbol{x}}^{-1}$ to the latter relation and taking the supremum over all $\boldsymbol{x}$ in $\mathbb{K}^d$, we obtain

$$\|\,\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\,\|_{L^\infty} \,\leq\, \int_{\mathbb{R}^d}\left\|\,\nabla_{\boldsymbol{x}}\,\Delta_{\boldsymbol{x}}^{-1}\,\left[f^\varepsilon \,-\, \tau_{\varepsilon\boldsymbol{v}}^{-1} f^\varepsilon\right](t,\cdot,\boldsymbol{v})\,\right\|_{L^\infty}\,\mathrm{d}\boldsymbol{v}\,.$$

To estimate $\left(f^\varepsilon \,-\, \tau_{\varepsilon\boldsymbol{v}}^{-1} f^\varepsilon\right)(t,\cdot,\boldsymbol{v})$, we notice that for each $\boldsymbol{v}\in\mathbb{R}^d$, it holds

$$\left\|\,\nabla_{\boldsymbol{x}}\,\Delta_{\boldsymbol{x}}^{-1}\,\left[f^\varepsilon \,-\, \tau_{\varepsilon\boldsymbol{v}}^{-1} f^\varepsilon\right](t,\cdot,\boldsymbol{v})\,\right\|_{L^\infty} \,\leq\, |\varepsilon\,\boldsymbol{v}|^\gamma\,\left\|\,\nabla_{\boldsymbol{x}}\,\Delta_{\boldsymbol{x}}^{-1}\,f^\varepsilon(t,\cdot,\boldsymbol{v})\,\right\|_{\mathscr{C}^{0,\gamma}}\,,$$

and therefore apply Morrey's inequality (1.8) which yields

$$\|\,\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\,\|_{L^\infty} \,\leq\, m_{d,p}\,\varepsilon^\gamma\int_{\mathbb{R}^d}\|f^\varepsilon(t,\cdot,\boldsymbol{v})\|_{L^p}\,|\boldsymbol{v}|^\gamma\,\mathrm{d}\boldsymbol{v}\,.$$

Then we rewrite the latter inequality as follows

$$\| \mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon \|_{L^\infty} \leq m_{d,p}\, \varepsilon^\gamma \int_{\mathbb{R}^d} \left(\int_{\mathbb{K}^d} \left| \frac{f^\varepsilon(t,\boldsymbol{x},\boldsymbol{v})}{\mathcal{M}(\boldsymbol{v})} \right|^p \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}} |\boldsymbol{v}|^\gamma\, \mathcal{M}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v}\,.$$

Applying Hölder's inequality to the latter relation, we deduce the result

$$\| \mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon \|_{L^\infty} \leq m_{d,p}\, \varepsilon^\gamma \left(\int_{\mathbb{R}^d} |\boldsymbol{v}|^{\frac{p\,\gamma}{p-1}}\, \mathcal{M}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} \right)^{\frac{p-1}{p}} \| f^\varepsilon \|_p\,.$$

□

We now provide estimates for the macroscopic model (1.5)

Proposition 1.3. *Consider a smooth solution ρ to equation* (1.5)*. For all exponent p, lying in* $(1, +\infty)$*, it holds*

$$\| \rho(t) \|_{L^p} \leq \max\left(\| \rho_0 \|_{L^p}\,,\, \| \rho_i \|_{L^{p+1}}^{1-1/p^2} \right), \quad \forall\, t \in \mathbb{R}^+\,.$$

Furthermore, it holds

$$\| \rho(t) \|_{L^\infty} \leq \max\left(\| \rho_0 \|_{L^\infty}\,,\, \| \rho_i \|_{L^\infty} \right), \quad \forall\, t \in \mathbb{R}^+\,.$$

where ρ_i is given in (1.5)*.*

We postpone the proof of this result to Appendix A.1 since it is not the main point in our analysis.

We turn to the main result of this section, in which we provide estimates in $L^p(\mathcal{M})$ for the solution f^ε to (1.1). As a direct consequence, we derive convergence estimates for π^ε and f^ε respectively towards ρ and the local equilibrium $\rho^\varepsilon\,\mathcal{M}$.

Proposition 1.4. *Consider some exponent $p > d$ and set $\gamma = 1 - d/p$. Let $(f^\varepsilon)_{\varepsilon>0}$ be a sequence of smooth solutions to* (1.1) *whose initial conditions meet assumption* (1.6) *and ρ be a smooth solution to* (1.5) *whose initial condition meets* (1.7)*. There exists a constant C only depending on exponent p and dimension d such that for all $\varepsilon \leq 1$, and for all t less than T^ε (where T^ε is given in Theorem 1.1)*

1. *it holds*

$$\| f^\varepsilon \|_p \leq 2 \left(\| f_0^\varepsilon \|_p + \varepsilon^{\gamma\left(\frac{1}{2} + \frac{1}{p-1}\right)}\, C\, m_p \right) e^{C_{\rho_0,\rho_i} C\, t}\,,$$

   where m_p and C_{ρ_0,ρ_i} are respectively defined in (1.6) *and* (1.7)*;*

2. *it holds*

$$\| \pi^\varepsilon - \rho \|_{L^2}(t) \leq \varepsilon^\beta\, C\, m_p^{2\,p'}\, e^{C_{\rho_0,\rho_i} C\, t}\,,$$

   where $\beta = (p-d)/(p-1)$ and $p' = p/(p-1)$;

3. *it holds*

$$\|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2([0,t]\,,\,L^2(\mathcal{M}))} \leq \varepsilon\, C\, m_p^2\, e^{C_{\rho_0,\rho_i} C\, t}\,.$$

Proof. The core of this proof consists in deriving item (1) in Proposition 1.4. To do so, we consider some positive ε and for $t \in \mathbb{R}^+$ define u as follows

$$u(t) = |||f^\varepsilon(t)|||_p^2 + \varepsilon^{-\alpha}\, \|(\pi^\varepsilon - \rho)\,(t)\|_{L^p}^2\,,$$

for some positive α which needs to be determined. Our strategy consists in estimating each one of the term composing $u(t)$ separately and then to propose a combination of these two estimates which allow us to close the estimate on u. In order to simplify notations, we omit the dependence with respect to $(t, \boldsymbol{x}, \boldsymbol{v})$ when the context is clear. Furthermore, we denote by $C_{d,p}$ a generic positive constant depending only on exponent p and dimension d in this proof.

We start by estimating $|||f^\varepsilon(t)|||_p$. To do so, we multiply the first line in (1.1) by $|f^\varepsilon/\mathcal{M}|^{p-1}$ and integrate over $\mathbb{K}^d \times \mathbb{R}^d$, this yields

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}|||f^\varepsilon|||_p^p = \\ -\frac{1}{\varepsilon}\int_{\mathbb{K}^d\times\mathbb{R}^d}\left[\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f^\varepsilon + \nabla_{\boldsymbol{v}}\cdot\left(\mathbf{E}^\varepsilon f^\varepsilon - \frac{1}{\varepsilon}\,(\,\boldsymbol{v}\, f^\varepsilon + \nabla_{\boldsymbol{v}} f^\varepsilon\,)\right)\right]\left|\frac{f^\varepsilon}{\mathcal{M}}\right|^{p-1}\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,.$$

To estimate the latter integral, we first point out that the contribution of the free transport operator cancels since we have

$$\int_{\mathbb{K}^d\times\mathbb{R}^d}\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f^\varepsilon\left|\frac{f^\varepsilon}{\mathcal{M}}\right|^{p-1}\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v} = \frac{1}{p}\int_{\mathbb{K}^d\times\mathbb{R}^d}\nabla_{\boldsymbol{x}}\cdot\left(\frac{|f^\varepsilon|^p}{\mathcal{M}^{p-1}}\,\boldsymbol{v}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v} = 0\,.$$

According to the latter observation, we deduce that the time derivative of $|||f^\varepsilon|||_p$ verifies

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}|||f^\varepsilon|||_p^p + \frac{1}{\varepsilon}\int_{\mathbb{K}^d\times\mathbb{R}^d}\nabla_{\boldsymbol{v}}\cdot\left[\mathbf{E}^\varepsilon f^\varepsilon - \frac{1}{\varepsilon}\mathcal{M}\,\nabla_{\boldsymbol{v}}\left(\frac{f^\varepsilon}{\mathcal{M}}\right)\right]\left|\frac{f^\varepsilon}{\mathcal{M}}\right|^{p-1}\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v} = 0\,,$$

where we also used the relation $\boldsymbol{v}\, f^\varepsilon + \nabla_{\boldsymbol{v}} f^\varepsilon = \mathcal{M}\,\nabla_{\boldsymbol{v}}\,(f^\varepsilon/\mathcal{M})$. An integration by part with respect to $\boldsymbol{v}$ in the integral term of the latter relation yields

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}|||f^\varepsilon|||_p^p + \frac{1}{\varepsilon^2}\mathcal{D}_p[\,f^\varepsilon\,] = \frac{p-1}{\varepsilon}\int_{\mathbb{K}^d\times\mathbb{R}^d}\mathbf{E}^\varepsilon\,\frac{f^\varepsilon}{\mathcal{M}}\,\nabla_{\boldsymbol{v}}\left(\frac{f^\varepsilon}{\mathcal{M}}\right)\left|\frac{f^\varepsilon}{\mathcal{M}}\right|^{p-2}\mathcal{M}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,.$$

Taking the uniform norm of $\mathbf{E}^\varepsilon$ and applying Young's inequality in the right hand side of the latter relation, we deduce the following differential inequality

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}|||f^\varepsilon|||_p^p + \frac{1-\eta}{\varepsilon^2}\mathcal{D}_p[\,f^\varepsilon\,] \leq \frac{p-1}{4\,\eta}\,\|\mathbf{E}^\varepsilon\|_{L^\infty}^2\,|||f^\varepsilon|||_p^p\,, \tag{1.9}$$

for all positive η. Then we replace $\|\mathbf{E}^\varepsilon\|^2_{L^\infty}$ as follows in the latter estimate

$$\|\mathbf{E}^\varepsilon\|^2_{L^\infty} \leq 3\left(\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|^2_{L^\infty} + \|\mathbf{I}^\varepsilon - \mathbf{E}\|^2_{L^\infty} + \|\mathbf{E}\|^2_{L^\infty}\right),$$

and estimate the norm $\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L^\infty}$ applying Lemma 1.2, the quantities $\|\mathbf{I}^\varepsilon - \mathbf{E}\|_{L^\infty}$, $\|\mathbf{E}\|_{L^\infty}$ applying Morrey's inequality (1.8). It yields for all $\eta > 0$

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}|||f^\varepsilon|||^p_p + \frac{1-\eta}{\varepsilon^2}\mathcal{D}_p[f^\varepsilon] \leq \frac{C_{d,p}}{\eta}\left(\varepsilon^{2\gamma}|||f^\varepsilon|||^2_p + \|\pi^\varepsilon - \rho\|^2_{L^p} + \|\rho\|^2_{L^p}\right)|||f^\varepsilon|||^p_p. \tag{1.10}$$

We now estimate $\|(\pi^\varepsilon - \rho)(t)\|_{L^p}$. To do so, we multiply the difference between equation (1.3) and the first line of (1.5) by $(\pi^\varepsilon - \rho)|\pi^\varepsilon - \rho|^{p-2}$ and integrate with respect to $\boldsymbol{y}$ over $\mathbb{K}^d$, this yields

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\pi^\varepsilon - \rho\|^p_{L^p} =$$
$$-\int_{\mathbb{K}^d}\nabla_{\boldsymbol{y}}\cdot\left[\int_{\mathbb{R}^d}\tau^{-1}_{\varepsilon\boldsymbol{v}}(\mathbf{E}^\varepsilon f^\varepsilon)\,\mathrm{d}\boldsymbol{v} - \mathbf{E}\rho - \nabla_{\boldsymbol{y}}(\pi^\varepsilon - \rho)\right](\pi^\varepsilon - \rho)|\pi^\varepsilon - \rho|^{p-2}\,\mathrm{d}\boldsymbol{y},$$

then we integrate by part with respect to $\boldsymbol{y}$ in the latter integral term and obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\pi^\varepsilon - \rho\|^p_{L^p} + \Delta_p[\pi^\varepsilon - \rho] = \mathcal{A},$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = (p-1)\int_{\mathbb{K}^d}\nabla_{\boldsymbol{y}}(\pi^\varepsilon - \rho)\cdot\left(\int_{\mathbb{R}^d}\tau^{-1}_{\varepsilon\boldsymbol{v}}(\mathbf{E}^\varepsilon f^\varepsilon)\,\mathrm{d}\boldsymbol{v} - \mathbf{E}\rho\right)|\pi^\varepsilon - \rho|^{p-2}\,\mathrm{d}\boldsymbol{y}.$$

To estimate $\mathcal{A}$, we use the following decomposition

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3,$$

where $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are given by

$$\begin{cases} \mathcal{A}_1 = (p-1)\displaystyle\int_{\mathbb{K}^d}\nabla_{\boldsymbol{y}}(\pi^\varepsilon - \rho)\cdot\int_{\mathbb{R}^d}\left(\tau^{-1}_{\varepsilon\boldsymbol{v}}\mathbf{E}^\varepsilon - \mathbf{E}^\varepsilon\right)g^\varepsilon\,\mathrm{d}\boldsymbol{v}\,|\pi^\varepsilon - \rho|^{p-2}\,\mathrm{d}\boldsymbol{y}, \\ \mathcal{A}_2 = (p-1)\displaystyle\int_{\mathbb{K}^d}\nabla_{\boldsymbol{y}}(\pi^\varepsilon - \rho)\cdot(\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon)\,\pi^\varepsilon\,|\pi^\varepsilon - \rho|^{p-2}\,\mathrm{d}\boldsymbol{y}, \\ \mathcal{A}_3 = (p-1)\displaystyle\int_{\mathbb{K}^d}\nabla_{\boldsymbol{y}}(\pi^\varepsilon - \rho)\cdot(\mathbf{I}^\varepsilon\pi^\varepsilon - \mathbf{E}\rho)|\pi^\varepsilon - \rho|^{p-2}\,\mathrm{d}\boldsymbol{y}. \end{cases}$$

$\mathcal{A}_1$ and $\mathcal{A}_2$ are error terms which we estimate using Sobolev inequalities whereas we estimate $\mathcal{A}_3$ using the properties of the limiting macroscopic equation (1.5).

Let us start with $\mathcal{A}_1$, which we estimate using Young inequality

$$\mathcal{A}_1 \,\leq\, \eta\,\Delta_p\,[\,\pi^\varepsilon -\, \rho\,] \,+\, \frac{p-1}{4\,\eta}\,\mathcal{A}_{11}\,,$$

for all positive η and where $\mathcal{A}_{11}$ is given by

$$\mathcal{A}_{11} \,=\, \int_{\mathbb{K}^d}\left|\int_{\mathbb{R}^d}\left(\tau_{\varepsilon\boldsymbol{v}}^{-1}\,\mathbf{E}^\varepsilon -\, \mathbf{E}^\varepsilon\right) g^\varepsilon\,\mathrm{d}\boldsymbol{v}\right|^2 |\pi^\varepsilon -\, \rho|^{p-2}\,\mathrm{d}\boldsymbol{y}\,.$$

Thanks to Morrey inequality (1.8), we have for all $\boldsymbol{v}\in\mathbb{R}^d$

$$\left\|\tau_{\varepsilon\boldsymbol{v}}^{-1}\,\mathbf{E}^\varepsilon -\, \mathbf{E}^\varepsilon\right\|_{L^\infty} \,\leq\, C_{d,p}\,|\varepsilon\,\boldsymbol{v}|^\gamma\,\|\rho^\varepsilon\|_{L^p}\,.$$

After applying Jensen's inequality to estimate $\|\rho^\varepsilon\|_{L^p}$, this yields

$$\left\|\tau_{\varepsilon\boldsymbol{v}}^{-1}\,\mathbf{E}^\varepsilon -\, \mathbf{E}^\varepsilon\right\|_{L^\infty} \,\leq\, C_{d,p}\,|\varepsilon\,\boldsymbol{v}|^\gamma\,|||f^\varepsilon|||_p\,.$$

We substitute $\tau_{\varepsilon\boldsymbol{v}}^{-1}\,\mathbf{E}^\varepsilon -\, \mathbf{E}^\varepsilon$ with the latter estimate in the definition of $\mathcal{A}_{11}$ and deduce

$$\mathcal{A}_{11} \,\leq\, C_{d,p}\,\varepsilon^{2\gamma}\,|||f^\varepsilon|||_p^2\int_{\mathbb{K}^d}\left|\int_{\mathbb{R}^d}|\boldsymbol{v}|^\gamma\,g^\varepsilon\,\mathrm{d}\boldsymbol{v}\right|^2|\pi^\varepsilon -\, \rho|^{p-2}\,\mathrm{d}\boldsymbol{y}\,.$$

Applying Hölder's inequality, we obtain

$$\mathcal{A}_{11} \,\leq\, C_{d,p}\,\varepsilon^{2\gamma}\,|||f^\varepsilon|||_p^2\,\|\,\pi^\varepsilon -\, \rho\,\|_{L^p}^{p-2}\left(\int_{\mathbb{K}^d}\left|\int_{\mathbb{R}^d}|\boldsymbol{v}|^\gamma\,g^\varepsilon\,\mathrm{d}\boldsymbol{v}\right|^p\,\mathrm{d}\boldsymbol{y}\right)^{2/p}\,.$$

To estimate the integral in the latter inequality, we apply Hölder inequality

$$\left(\int_{\mathbb{K}^d}\left|\int_{\mathbb{R}^d}|\boldsymbol{v}|^\gamma\,g^\varepsilon\,\mathrm{d}\boldsymbol{v}\right|^p\,\mathrm{d}\boldsymbol{y}\right)^{2/p} \,\leq\, \left(\int_{\mathbb{R}^d}|\boldsymbol{v}|^{p'\gamma}\,\mathcal{M}\,\mathrm{d}\boldsymbol{v}\right)^{2/p'}|||g^\varepsilon|||_p^2\,.$$

Then we notice that $|||g^\varepsilon|||_p \,=\, |||f^\varepsilon|||_p$ and deduce

$$\mathcal{A}_{11} \,\leq\, C_{d,p}\,\varepsilon^{2\gamma}\,|||f^\varepsilon|||_p^4\,\|\,\pi^\varepsilon -\, \rho\,\|_{L^p}^{p-2}\,.$$

In the end, it yields the following bound for $\mathcal{A}_1$

$$\mathcal{A}_1 \,\leq\, \eta\,\Delta_p\,[\,\pi^\varepsilon -\, \rho\,] \,+\, \frac{C_{d,p}}{\eta}\,\varepsilon^{2\gamma}\,|||f^\varepsilon|||_p^4\,\|\,\pi^\varepsilon -\, \rho\,\|_{L^p}^{p-2}\,.$$

To estimate $\mathcal{A}_2$, we follow the same method as before excepted that we apply Lemma 1.2 to bound $\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon$, it yields

$$\mathcal{A}_2 \,\leq\, \eta\,\Delta_p\,[\,\pi^\varepsilon -\, \rho\,] \,+\, \frac{C_{d,p}}{\eta}\,\varepsilon^{2\gamma}\,|||f^\varepsilon|||_p^4\,\|\,\pi^\varepsilon -\, \rho\,\|_{L^p}^{p-2}\,.$$

We turn to the last term $\mathcal{A}_3$, which decomposes as follows

$$\mathcal{A}_3 \,=\, \mathcal{A}_{31} \,+\, \mathcal{A}_{32} \,+\, \mathcal{A}_{33}\,,$$

where the latter terms are given by

$$\begin{cases} \mathcal{A}_{31} \,=\, (p-1)\displaystyle\int_{\mathbb{K}^d} \nabla_{\boldsymbol{y}}\left(\pi^\varepsilon-\,\rho\right)\cdot\left(\mathbf{I}^\varepsilon-\,\mathbf{E}\right)\left(\pi^\varepsilon-\,\rho\right)|\pi^\varepsilon-\,\rho|^{p-2}\,\mathrm{d}\boldsymbol{y}\,,\\ \mathcal{A}_{32} \,=\, (p-1)\displaystyle\int_{\mathbb{K}^d} \nabla_{\boldsymbol{y}}\left(\pi^\varepsilon-\,\rho\right)\cdot\left(\mathbf{I}^\varepsilon-\,\mathbf{E}\right)\rho\,|\pi^\varepsilon-\,\rho|^{p-2}\,\mathrm{d}\boldsymbol{y}\,,\\ \mathcal{A}_{33} \,=\, (p-1)\displaystyle\int_{\mathbb{K}^d} \nabla_{\boldsymbol{y}}\left(\pi^\varepsilon-\,\rho\right)\cdot\mathbf{E}\,\left(\pi^\varepsilon-\,\rho\right)|\pi^\varepsilon-\,\rho|^{p-2}\,\mathrm{d}\boldsymbol{y}\,. \end{cases}$$

We start with $\mathcal{A}_{31}$, which rewrites as follows after an integration by part

$$\mathcal{A}_{31} \,=\, -\,\frac{p-1}{p}\int_{\mathbb{K}^d} |\pi^\varepsilon-\,\rho|^p\;\nabla_{\boldsymbol{x}}\cdot\left(\mathbf{I}^\varepsilon\,-\,\mathbf{E}\right)\mathrm{d}\boldsymbol{y}\,.$$

Therefore, replacing $\mathbf{I}^\varepsilon$ and $\mathbf{E}$ according to equations (1.4) and (1.5), we deduce the following relation

$$\mathcal{A}_{31} \,=\, -\,\frac{p-1}{p}\int_{\mathbb{K}^d} |\pi^\varepsilon-\,\rho|^p\left(\pi^\varepsilon-\,\rho\right)\mathrm{d}\boldsymbol{y}\,.$$

Since π^ε has positive values and taking the absolute value of ρ, we obtain

$$\mathcal{A}_{31} \,\leq\, \frac{p-1}{p}\,\|\rho\|_{L^\infty}\,\|\,\pi^\varepsilon-\,\rho\,\|^p_{L^p}\,\,.$$

Remark 1.5. *It is the only time that we use the L^∞-norm of ρ in our analysis.*

To estimate $\mathcal{A}_{32}$ and $\mathcal{A}_{33}$, we use the same techniques as the ones already used to estimate $\mathcal{A}_1$ and $\mathcal{A}_2$. Therefore, we do not detail the computations. In the end it yields

$$\mathcal{A}_{32}\,+\,\mathcal{A}_{33}\,\leq\,\eta\,\Delta_p\left[\,\pi^\varepsilon-\,\rho\,\right]\,+\,\frac{C_{d,p}}{\eta}\,\|\rho\|^2_{L^p}\,\|\,\pi^\varepsilon-\,\rho\,\|^p_{L^p}\,\,.$$

Gathering latter computations and taking η small enough, we obtain the following estimate

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\pi^\varepsilon-\rho\,\|^p_{L^p}\leq C_{d,p}\left(\varepsilon^{2\gamma}\,|||f^\varepsilon|||^4_p\,\|\,\pi^\varepsilon-\rho\,\|^{p-2}_{L^p}+\left(\|\rho\|^2_{L^p}+\|\rho\|_{L^\infty}\right)\|\,\pi^\varepsilon-\rho\,\|^p_{L^p}\right),$$

for some constant $C_{d,p}$ only depending on d and p. Dividing by $\|\,\pi^\varepsilon\,-\,\rho\,\|^{p-2}_{L^p}$ the latter estimate, this yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\pi^\varepsilon-\,\rho\,\|^2_{L^p}\,\leq\,C_{d,p}\left(\varepsilon^{2\gamma}\,|||f^\varepsilon|||^4_p\,+\,\left(\|\rho\|^2_{L^p}\,+\,\|\rho\|_{L^\infty}\right)\|\,\pi^\varepsilon-\,\rho\,\|^2_{L^p}\right)\,. \qquad (1.11)$$

It is now possible to obtain item (1) in Proposition 1.4 : we set η to 1 in (1.10) and take the sum between estimate (1.10) divided by $\|f^\varepsilon\|_p^{p-2}$ and estimate (1.11), we deduce that u verifies the following differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}u(t) \leq$$

$$C_{d,p}\left(\left(\varepsilon^{2\gamma-\alpha}+\varepsilon^{2\gamma}\right)|||f^\varepsilon|||_p^4 + \|\pi^\varepsilon - \rho\|_{L^p}^2\,|||f^\varepsilon|||_p^2 + C_\rho\left(\varepsilon^{-\alpha}\,\|\pi^\varepsilon - \rho\|_{L^p}^2 + |||f^\varepsilon|||_p^2\right)\right),$$

where C_ρ is given by

$$C_\rho \,=\, \|\rho\|_{L^p}^2 \,+\, \|\rho\|_{L^\infty}\,.$$

Hence, taking $\alpha = \gamma$ and applying Lemma 1.3 to estimate C_ρ, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) \,\leq\, C_{d,p}\,\left(\varepsilon^\gamma\,u(t)^2 \,+\, C_{\rho_0,\rho_i}\,u(t)\right)\,,$$

where C_{ρ_0,ρ_i} is given by

$$C_{\rho_0,\rho_i} \,=\, \max\left(\|\,\rho_0\,\|_{L^p}^2, \|\,\rho_i\,\|_{L^{p+1}}^{2-2/p^2}, \|\,\rho_0\,\|_{L^\infty}\,, \|\,\rho_i\,\|_{L^\infty}\right).$$

We divide the latter estimate by $\left(\varepsilon^\gamma\,u(t)^2 \,+\, C_{\rho_0,\rho_i}\,u(t)\right)$ and notice that

$$\frac{1}{\varepsilon^\gamma\,u(t)^2 + C_{\rho_0,\rho_i}\,u(t)} \,=\, \frac{1}{C_{\rho_0,\rho_i}}\left(\frac{1}{u(t)} \,-\, \frac{\varepsilon^\gamma}{\varepsilon^\gamma\,u(t) + C_{\rho_0,\rho_i}}\right),$$

therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\frac{u(t)}{\varepsilon^\gamma\,u(t) \,+\, C_{\rho_0,\rho_i}}\right) \,\leq\, C_{\rho_0,\rho_i}\,C_{d,p}\,.$$

Integrating between 0 and t and taking the exponential of the latter estimate, it yields

$$u(t) \,\leq\, u(0)\left(1 \,-\, \varepsilon^\gamma\,\frac{u(0)}{C_{\rho_0,\rho_i}}\left(e^{C_{\rho_0,\rho_i}\,C_{d,p}\,t} \,-\, 1\right)\right)^{-1} e^{C_{\rho_0,\rho_i}C_{d,p}\,t}\,, \tag{1.12}$$

for all time t verifying

$$t \,<\, \frac{1}{C_{\rho_0,\rho_i}\,C_{d,p}}\,\ln\left(1 \,+\, \frac{C_{\rho_0,\rho_i}}{u(0)}\,\varepsilon^{-\gamma}\right).$$

To conclude this step, we estimate $u(0)$ by applying the triangular inequality

$$u(0) \,\leq\, |||f_0^\varepsilon|||_p^2 \,+\, 2\,\varepsilon^{-\gamma}\left(\|\pi_0^\varepsilon \,-\, \rho_0^\varepsilon\|_{L^p}^2 \,+\|\rho_0^\varepsilon \,-\, \rho_0\|_{L^p}^2\right). \tag{1.13}$$

Thanks to assumption (1.6), we obtain

$$u(0) \,\leq\, |||f_0^\varepsilon|||_p^2 \,+\, C_{d,p}\,m_p^2\,\varepsilon^{2\beta-\gamma}\,,$$

which yields

$$u(0) \leq |||f_0^\varepsilon|||_p^2 + C_{d,p}\, m_p^2\, \varepsilon^{\gamma\left(1+\frac{2}{p-1}\right)},$$

thanks to the relation $2\beta - \gamma = \gamma(1 + 2/(p-1))$. Replacing $u(0)$ in (1.12) thanks to the latter inequality, we deduce item (1) of Proposition 1.4, that is

$$|||f^\varepsilon|||_p \leq 2\left(|||f_0^\varepsilon|||_p + \varepsilon^{\gamma\left(\frac{1}{2}+\frac{1}{p-1}\right)}\, C_{d,p}\, m_p\right) e^{C_{\rho_0,\rho_i} C_{d,p}\, t},$$

for all time t less than T^ε, where T^ε is given by

$$T^\varepsilon = \frac{1}{C_{\rho_0,\rho_i}\, C_{d,p}} \ln\left(1 + \frac{C_{\rho_0,\rho_i}}{4\left(|||f_0^\varepsilon|||_p^2 + \varepsilon^{\gamma\left(1+\frac{2}{p-1}\right)} C_{d,p} m_p^2\right)}\, \varepsilon^{-\gamma}\right).$$

In order to prove item (2), we consider relation (1.11), replace $|||f^\varepsilon(t)|||_p$ with the estimate given by item (1) in Proposition 1.4 and apply Lemma 1.3 to bound ρ, this yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\, \pi^\varepsilon - \rho\, \|_{L^p}^2 \leq C_{d,p}\left(\varepsilon^{2\gamma}\, m_p^4\, e^{C_{\rho_0,\rho_i} C_{d,p}\, t} + C_{\rho_0,\rho_i}\, \|\, \pi^\varepsilon - \rho\, \|_{L^p}^2\right),$$

for all time t less than T^ε, where m_p and C_{ρ_0,ρ_i} are given in Theorem 1.1. Multiplying the latter estimate by $e^{-C_{\rho_0,\rho_i} C_{d,p}\, t}$, integrating between 0 and t and taking the square root on both sides of the inequality, we obtain

$$\|\, \pi^\varepsilon - \rho\, \|_{L^p} \leq e^{C_{\rho_0,\rho_i}\, C_{d,p}\, t}\left(\|\, \pi_0^\varepsilon - \rho_0\, \|_{L^p} + \varepsilon^\gamma\, C_{d,p}\, m_p^2\, t\right).$$

According to assumption (1.6), we have $\|\, \pi_0^\varepsilon - \rho_0\, \|_{L^p} \leq m_p\, \varepsilon^\beta$. Since $1 \leq m_p$, we deduce

$$\|\, \pi^\varepsilon - \rho\, \|_{L^p} \leq \varepsilon^\gamma\, C_{d,p}\, m_p^2\, e^{C_{\rho_0,\rho_i}\, C_{d,p}\, t}.$$

Item (2) in Proposition 1.4 is obtained applying the following interpolation inequality in the latter estimate

$$\|\, \pi^\varepsilon - \rho\, \|_{L^2} \leq \|\, \pi^\varepsilon - \rho\, \|_{L^1}^{\frac{p-2}{p-1}}\, \|\, \pi^\varepsilon - \rho\, \|_{L^p}^{p'},$$

where p' is given by $p' = p/(p-1)$ and noticing that $p'\,\gamma = \beta$.

To prove item (3) in Proposition 1.4, we set $p = 2$ in (1.9), apply Morrey's inequality (1.8) to estimate $\mathbf{E}^\varepsilon$ and Jensen's inequality, which ensures $|||f^\varepsilon|||_2 \leq |||f^\varepsilon|||_p$. This yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||f^\varepsilon|||_2^2 + \frac{1-\eta}{\varepsilon^2}\mathcal{D}_2[\, f^\varepsilon\,] \leq \frac{C_{d,p}}{\eta}\, |||f^\varepsilon|||_p^4.$$

Therefore, taking $\eta = 1/2$, integrating the latter estimate between 0 and t and replacing $|||f^\varepsilon|||_p$ by the estimate given in item (1) in Proposition 1.4 we obtain

$$\int_0^t \mathcal{D}_2[\, f^\varepsilon\,]\, \mathrm{d}s \leq \varepsilon^2\left(|||f_0^\varepsilon|||_p^2 + C_{d,p}\int_0^t m_p^4\, e^{C_{\rho_0,\rho_i} C_{d,p}\, s}\, \mathrm{d}s\right),$$

for all time t between 0 and T^ε. Hence, applying the Gaussian-Poincaré inequality which ensures $|||f^\varepsilon - \rho^\varepsilon \mathcal{M}|||_2^2 \leq \mathcal{D}_2[f^\varepsilon]$, we deduce

$$\int_0^t |||f^\varepsilon - \rho^\varepsilon \mathcal{M}|||_2^2 \,\mathrm{d}s \leq C_{d,p}\, \varepsilon^2\, m_p^4\, e^{C_{\rho_0,\rho_i} C_{d,p}\, t},$$

which yields the result by taking the square root of the latter estimate. □

Building on Proposition 1.4, we are able to prove equicontinuity estimates for f^ε in $L^p(\mathcal{M})$.

Proposition 1.6. *Consider some exponent p such that $p > d$ and set $\beta = (p-d)/(p-1)$. Under assumptions (1.7) on (ρ_0, ρ_i) and (1.6) on the sequence of initial conditions $(f_0^\varepsilon)_{\varepsilon>0}$, consider a sequence of solutions $(f^\varepsilon)_{\varepsilon>0}$ to (1.1). There exists a constant C only depending on exponent p and dimension d such that for all positive ε less than 1, it holds*

$$\sup_{|\boldsymbol{x}_0|\leq 1} |\boldsymbol{x}_0|^{-\beta} |||f^\varepsilon - \tau_{\boldsymbol{x}_0} f^\varepsilon|||_2(t) \leq e^{C_{\rho_0,\rho_i}^{-1} C\, m_p^2\, e^{C_{\rho_0,\rho_i} C\, t}} \sup_{|\boldsymbol{x}_0|\leq 1} |\boldsymbol{x}_0|^{-\beta} |||f_0^\varepsilon - \tau_{\boldsymbol{x}_0} f_0^\varepsilon|||_p\,.$$

for all time t less than T^ε, where T^ε, m_p and C_{ρ_0,ρ_i} are given in Theorem 1.1.

Proof. We consider $\varepsilon > 0$ and $(t, \boldsymbol{x}_0) \in \mathbb{R}^+ \times \mathbb{K}^d$ such that $|\boldsymbol{x}_0| \leq 1$. Furthermore we denote by $C_{d,p}$ a generic positive constant depending only on exponent p and dimension d in this proof.

We first compute the equation solved by $h^\varepsilon := f^\varepsilon - \tau_{\boldsymbol{x}_0} f^\varepsilon$. It is given by the difference between equation (1.1) and equation (1.1) translated by $\boldsymbol{x}_0$ with respect to the spatial variable, that is

$$\partial_t h^\varepsilon + \frac{1}{\varepsilon} \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} h^\varepsilon + \frac{1}{\varepsilon}(\mathbf{E}^\varepsilon - \tau_{\boldsymbol{x}_0}\mathbf{E}^\varepsilon)\cdot\nabla_{\boldsymbol{v}} f^\varepsilon + \frac{1}{\varepsilon}\tau_{\boldsymbol{x}_0}\mathbf{E}^\varepsilon\cdot\nabla_{\boldsymbol{v}} h^\varepsilon = \frac{1}{\varepsilon^2}\nabla_{\boldsymbol{v}}\cdot[\boldsymbol{v}\, h^\varepsilon + \nabla_{\boldsymbol{v}} h^\varepsilon].$$

To estimate the variations of $|||h^\varepsilon|||_p$, we proceed as in the estimation of $|||f^\varepsilon|||_p$ in the proof of Proposition 1.4. First, we multiply the latter equation by $(h^\varepsilon/\mathcal{M})\,|h^\varepsilon/\mathcal{M}|^{p-2}$ and integrate over $\mathbb{K}^d \times \mathbb{R}^d$. Then, we notice that the free transport operator has a zero contribution. Therefore, using the relation $\boldsymbol{v}\, h^\varepsilon + \nabla_{\boldsymbol{v}} h^\varepsilon = \mathcal{M}\,\nabla_{\boldsymbol{v}}(h^\varepsilon/\mathcal{M})$ and integrating by part with respect to $\boldsymbol{v}$, we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}|||h^\varepsilon|||_p^p + \frac{1}{\varepsilon^2}\mathcal{D}_p[h^\varepsilon] =$$

$$\frac{p-1}{\varepsilon}\int_{\mathbb{K}^d\times\mathbb{R}^d}\left|\frac{h^\varepsilon}{\mathcal{M}}\right|^{p-2}\nabla_{\boldsymbol{v}}\left(\frac{h^\varepsilon}{\mathcal{M}}\right)\cdot\left(f^\varepsilon\,(\mathbf{E}^\varepsilon - \tau_{\boldsymbol{x}_0}\mathbf{E}^\varepsilon) + h^\varepsilon\tau_{\boldsymbol{x}_0}\mathbf{E}^\varepsilon\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,.$$

We apply Young's inequality to estimate the right hand side in the latter inequality and deduce

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}|||h^\varepsilon|||_p^p \leq$$

$$\frac{p-1}{2}\int_{\mathbb{K}^d\times\mathbb{R}^d}\left|\frac{h^\varepsilon}{\mathcal{M}}\right|^{p-2}\left(\left|\frac{f^\varepsilon}{\mathcal{M}}\right|^2|\mathbf{E}^\varepsilon - \tau_{\boldsymbol{x}_0}\mathbf{E}^\varepsilon|^2 + \left|\frac{h^\varepsilon}{\mathcal{M}}\right|^2|\tau_{\boldsymbol{x}_0}\mathbf{E}^\varepsilon|^2\right)\mathcal{M}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,.$$

After taking the uniform norms of $\mathbf{E}^\varepsilon - \tau_{\boldsymbol{x}_0}\mathbf{E}^\varepsilon$ and $\mathbf{E}^\varepsilon$ and applying Hölder's inequality to estimate the cross product between f^ε and h^ε, it yields

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,|\!|\!|h^\varepsilon|\!|\!|_p^p \,\leq\, \frac{p-1}{2}\left(\|\mathbf{E}^\varepsilon - \tau_{\boldsymbol{x}_0}\mathbf{E}^\varepsilon\|_{L^\infty}^2\,|\!|\!|h^\varepsilon|\!|\!|_p^{p-2}|\!|\!|f^\varepsilon|\!|\!|_p^2 \,+\, \|\mathbf{E}^\varepsilon\|_{L^\infty}^2\,|\!|\!|h^\varepsilon|\!|\!|_p^p\right).$$

We apply Morrey's inequality (1.8) to estimate the uniform norms of $\mathbf{E}^\varepsilon - \tau_{\boldsymbol{x}_0}\mathbf{E}^\varepsilon$ and $\mathbf{E}^\varepsilon$, it yields

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,|\!|\!|h^\varepsilon|\!|\!|_p^p \,\leq\, C_{d,p}|\!|\!|h^\varepsilon|\!|\!|_p^p|\!|\!|f^\varepsilon|\!|\!|_p^2\,.$$

To conclude, we divide the latter estimate by $|\!|\!|h^\varepsilon|\!|\!|_p^{p-1}$, multiply it by $e^{-C_{d,p}\int_0^t|\!|\!|f^\varepsilon|\!|\!|_p^2\,\mathrm{d}s}$, integrate between 0 and t and replace $|\!|\!|f^\varepsilon|\!|\!|_p$ with the estimate in item (1) of Proposition 1.4, it yields

$$|\!|\!|h^\varepsilon|\!|\!|_p \,\leq\, \exp\left(C_{\rho_0,\rho_i}^{-1}\left(|\!|\!|f_0^\varepsilon|\!|\!|_p + C_{d,p}m_p\varepsilon^{\frac{\gamma}{2}}\right)^2 e^{C_{d,p}C_{\rho_0,\rho_i}t}\right)|\!|\!|h_0^\varepsilon|\!|\!|_p\,,$$

for all time t less than T^ε, where T^ε is given in Theorem 1.1. We obtain the result dividing the latter estimate by $|\boldsymbol{x}_0|^\beta$, taking the supremum over all $\boldsymbol{x}_0$ with norm less than 1 and since according to Jensen's inequality it holds $|\!|\!|h^\varepsilon|\!|\!|_2 \,\leq\, |\!|\!|h^\varepsilon|\!|\!|_p$. □

1.4 Proof of Theorem 1.1

We consider the following decomposition of the quantity that we need to estimate

$$\|f^\varepsilon \,-\, \rho\,\mathcal{M}\|_{L^2([0,t]\,,\,L^2(\mathcal{M}))} \,\leq\, \left(\mathcal{E}_1 \,+\, \mathcal{E}_2 \,+\, \mathcal{E}_3\right)(t)\,,$$

where $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are given by

$$\begin{cases} \mathcal{E}_1(t) \,=\, \|f^\varepsilon \,-\, \rho^\varepsilon\,\mathcal{M}\|_{L^2([0,t]\,,\,L^2(\mathcal{M}))}\,, \\[2ex] \mathcal{E}_2(t) \,=\, \|\rho^\varepsilon \,-\, \pi^\varepsilon\|_{L^2\left([0,t]\,,\,L^2\left(\mathbb{K}^d\right)\right)}\,, \\[2ex] \mathcal{E}_3(t) \,=\, \|\pi^\varepsilon \,-\, \rho\|_{L^2\left([0,t]\,,\,L^2\left(\mathbb{K}^d\right)\right)}\,. \end{cases}$$

We estimate $\mathcal{E}_1$ and $\mathcal{E}_3$ thanks to Proposition 1.4. Indeed, according to item (3), it holds

$$\mathcal{E}_1(t) \,\leq\, C\,\varepsilon\,m_p^2\,e^{C_{\rho_0,\rho_i}C\,t}\,,$$

and according to item (2), it holds

$$\mathcal{E}_3(t) \,\leq\, C\,\varepsilon^\beta\,m_p^{2\,p'}\,e^{C_{\rho_0,\rho_i}\,C\,t}\,,$$

for all time t less than T^ε, where T^ε is given in Theorem 1.1. We turn to the last term $\mathcal{E}_2$, which we estimate thanks to the following decomposition

$$\mathcal{E}_2(t) \,\leq\, \mathcal{E}_{21}(t) \,+\, \mathcal{E}_{22}(t)\,,$$

where $\mathcal{E}_{21}$ and $\mathcal{E}_{22}$ are defined as follows

$$\left\{\begin{aligned} \mathcal{E}_{21}(t) &= \left(\int_0^t \int_{\mathbb{K}^d} \left|\int_{\mathbb{R}^{2d}} \mathcal{M}(\tilde{\boldsymbol{v}})\left(f^\varepsilon(s,\boldsymbol{x},\boldsymbol{v}) - f^\varepsilon(s,\boldsymbol{x}-\varepsilon\tilde{\boldsymbol{v}},\boldsymbol{v})\right) \mathrm{d}\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{v}\right|^2 \mathrm{d}\boldsymbol{x}\,\mathrm{d}s\right)^{\frac{1}{2}}, \\ \mathcal{E}_{22}(t) &= \left(\int_0^t \int_{\mathbb{K}^d} \left|\int_{\mathbb{R}^d} \mathcal{M}(\boldsymbol{v})\rho^\varepsilon(s,\boldsymbol{y}-\varepsilon\boldsymbol{v}) - f^\varepsilon(s,\boldsymbol{y}-\varepsilon\boldsymbol{v},\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}\right|^2 \mathrm{d}\boldsymbol{y}\,\mathrm{d}s\right)^{\frac{1}{2}}. \end{aligned}\right.$$

To estimate $\mathcal{E}_{22}$, we divide and multiply by $\mathcal{M}(\boldsymbol{v})$ inside the integral in $\boldsymbol{v}$ in the definition of $\mathcal{E}_{22}$ and apply Jensen's inequality, which yields

$$\mathcal{E}_{22}(t) \le \left(\int_0^t \int_{\mathbb{K}^d\times\mathbb{R}^d} \left|\frac{\mathcal{M}(\boldsymbol{v})\rho^\varepsilon(s,\boldsymbol{y}-\varepsilon\boldsymbol{v}) - f^\varepsilon(s,\boldsymbol{y}-\varepsilon\boldsymbol{v},\boldsymbol{v})}{\mathcal{M}(\boldsymbol{v})}\right|^2 \mathcal{M}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}s\right)^{\frac{1}{2}},$$

then we operate the change of variable $\boldsymbol{x} = \boldsymbol{y} - \varepsilon\,\boldsymbol{v}$ in the latter relation and deduce

$$\mathcal{E}_{22}(t) \le \mathcal{E}_1(t).$$

Thanks to our estimate of $\mathcal{E}_1(t)$, we obtain

$$\mathcal{E}_{22}(t) \le C\,\varepsilon\, m_p^2\, e^{C_{\rho_0,\rho_i} C\, t},$$

Now, we estimate $\mathcal{E}_{21}$. First, we divide and multiply by $\mathcal{M}(\boldsymbol{v})$ inside the integral in the definition of $\mathcal{E}_{21}$ and apply Jensen's inequality, which yields

$$\mathcal{E}_{21}(t) \le \left(\int_0^t \int_{\mathbb{R}^d} \left\|\left| f^\varepsilon - \tau_{\varepsilon\tilde{\boldsymbol{v}}}^{-1} f^\varepsilon \right|\right\|_2^2(s)\mathcal{M}(\tilde{\boldsymbol{v}})\,\mathrm{d}\tilde{\boldsymbol{v}}\,\mathrm{d}s\right)^{\frac{1}{2}}.$$

To bound $\left\|\left| f^\varepsilon - \tau_{\varepsilon\tilde{\boldsymbol{v}}}^{-1} f^\varepsilon \right|\right\|_2$, we distinguish two cases. When, $|\varepsilon\,\tilde{\boldsymbol{v}}| > 1$ we use the triangular inequality, which ensures

$$\left\|\left| f^\varepsilon - \tau_{\varepsilon\tilde{\boldsymbol{v}}}^{-1} f^\varepsilon \right|\right\|_2(s) \le 2\,|\varepsilon\tilde{\boldsymbol{v}}|^\beta\, \||f^\varepsilon|\|_2(s),$$

whereas in the case $|\varepsilon\,\tilde{\boldsymbol{v}}| \le 1$, it holds

$$\left\|\left| f^\varepsilon - \tau_{\varepsilon\tilde{\boldsymbol{v}}}^{-1} f^\varepsilon \right|\right\|_2(s) \le |\varepsilon\tilde{\boldsymbol{v}}|^\beta \sup_{|\boldsymbol{x}_0|\le 1} |\boldsymbol{x}_0|^{-\beta}\, \||\tau_{\boldsymbol{x}_0} f^\varepsilon - f^\varepsilon|\|_2(s).$$

According to these estimates, we deduce

$$\mathcal{E}_{21}(t) \le \varepsilon^\beta \left(\int_0^t 4\,\||f^\varepsilon|\|_2^2(s) + \left(\sup_{|\boldsymbol{x}_0|\le 1} |\boldsymbol{x}_0|^{-\beta}\, \||\tau_{\boldsymbol{x}_0} f^\varepsilon - f^\varepsilon|\|_2(s)\right)^2 \mathrm{d}s\right)^{\frac{1}{2}},$$

where we used that $\int_{\mathbb{R}^d} |\tilde{\boldsymbol{v}}|^{2\beta}\mathcal{M}(\tilde{\boldsymbol{v}})\,\mathrm{d}\tilde{\boldsymbol{v}} \le 1$. Then we apply item (1) in Proposition 1.4 to estimate the norm of f^ε and Proposition 1.6 to estimate the norm of $\tau_{\boldsymbol{x}_0} f^\varepsilon - f^\varepsilon$. It yields

$$\mathcal{E}_{21}(t) \le \varepsilon^\beta \left(\int_0^t C\, m_p^2\, e^{C_{\rho_0,\rho_i} C\, s} + m_p^2 \exp\left(C_{\rho_0,\rho_i}^{-1} C\, m_p^2\, e^{C_{\rho_0,\rho_i} C\, s}\right)\mathrm{d}s\right)^{\frac{1}{2}},$$

for all time t less than T^ε, where T^ε is given in Theorem 1.1. Hence, we deduce

$$\mathcal{E}_{21}(t) \leq \varepsilon^\beta \left(C_{\rho_0,\rho_i}^{-1/2} m_p \, e^{C_{\rho_0,\rho_i} C\, t} + \exp\left(C_{\rho_0,\rho_i}^{-1} C\, m_p^2 \, e^{C_{\rho_0,\rho_i} C\, t} \right) \right),$$

where we used the following inequality to estimate the time integral of the double exponential term

$$m_p^2 \exp\left(C_{\rho_0,\rho_i}^{-1} C\, m_p^2 \, e^{C_{\rho_0,\rho_i} C\, t} \right) \leq \frac{1}{C^2} \frac{\mathrm{d}}{\mathrm{d}t} \exp\left(C_{\rho_0,\rho_i}^{-1} C\, m_p^2 \, e^{C_{\rho_0,\rho_i} C\, t} \right).$$

Gathering the estimate on $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$, we obtain the estimate in Theorem 1.1, that is

$$\| f^\varepsilon - \rho \mathcal{M} \|_{L^2([0,t]\,,\,L^2(\mathcal{M}))} \leq \varepsilon^\beta \left(C\, m_p^{2\,p'} \, e^{C_{\rho_0,\rho_i} C\, t} + \exp\left(C_{\rho_0,\rho_i}^{-1} C\, m_p^2 \, e^{C_{\rho_0,\rho_i} C\, t} \right) \right),$$

for all time t less than T^ε, where T^ε is given in Theorem 1.4.

1.5 Conclusion

We have proposed a method in order to treat the diffusive scaling for the VPFP model. Our approach provides non-perturbative strong convergence results with explicit rates. It may be regarded as an alternative to compactness methods relying on averaging lemmas widely used in this context [165, 103, 102, 212, 137] with the advantage that it provides explicit convergence rates.

An interesting and challenging continuation of this work would consist in conciliating this approach with hypocoercivity methods [136, 138, 1] which present the advantage of providing global in time convergence estimates but which fail, for now, to provide non-perturbative results. To be noted that up to our knowledge, non-perturbative results [35, 28] treating the long time behavior of (1.1) rely on compactness arguments and are thus non quantitative. Therefore, associating hypocoercivity methods with the one presented in this article might be a way to treat simultaneously the diffusive regime $\varepsilon \to 0$ and the long time behavior $t \to +\infty$ in a non-perturbative framework and with explicit rates.

Another natural question concerns the applicability of our approach to treat other asymptotic limits of the VPFP model. For example, our method might be applicable to "free-field" regimes analyzed by M. Herda and M. Rodrigues in [138]. These regimes correspond to the limit $\tau \to 0$ when ε^2 is replaced with $\tau\varepsilon$ in the right hand of the first line in (1.1) and under the assumption $\tau = o(\varepsilon)$ as $\tau \to 0$. Roughly speaking, these regimes describe situations where collisions are strong enough to cancel electrostatic effects.
Adapting our approach to the famous high-field or hyperbolic regime also constitutes a great challenge. This regime corresponds to a situation where collisions and electrostatic effects have the same magnitude, leading to unthermalized asymptotic limits. This regime

has drawn intense interest of the mathematics community [191, 61, 14, 29, 175, 126, 31]. In this case, smoothing effects due to the Fokker-Planck operator disappear in the limit. For this reason, there is no clear indication that our method would apply. However, we also mention the article [13] in which intermediate regimes are considered, where collisions slightly dominate electrostatic effects. If possible, applying our method to these intermediate regimes would constitute a first step towards treating the high-field limit.

To conclude, it would be interesting to test the robustness of our method on other collision operators such as linearized Boltzmann operators [190, 15, 165] or BGK relaxation operators [61]. In these examples, we expect less smoothing effects than in the Fokker-Planck case, leading to additional difficulties.

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Chapitre 2

On a discrete framework of hypocoercivity for kinetic equations

We propose and study a fully discrete finite volume scheme for the linear Vlasov-Fokker-Planck equation written as an hyperbolic system using Hermite polynomials in velocity. This approach naturally preserves the stationary solution and the weighted L^2 relative entropy. Then, we adapt the arguments developed in [97] based on hypocoercivity methods to get quantitative estimates on the convergence to equilibrium of the discrete solution. Finally, we prove that in the diffusive limit, the scheme is asymptotic preserving with respect to both the time variable and the scaling parameter at play.

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Contents

2.1 Introduction

This article is devoted to the numerical approximation and analysis of the linear Vlasov-Fokker-Planck equation, corresponding to the kinetic description of the Brownian motion of a large system of charged particles under the effect of a force field.

Our main motivation comes from an electrostatic plasma composed of charged particles, where the Coulomb force are taken into account. The time evolution of the electron distribution function f solves the Vlasov-Fokker-Planck system coupled with the Poisson equation giving a self-consistent potential Φ :

$$\left\{\begin{array}{l} \dfrac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f + \dfrac{q_e}{m_e}\,\mathbf{E}\cdot\nabla_{\boldsymbol{v}} f = \dfrac{1}{\tau_e}\,\mathrm{div}_{\boldsymbol{v}}\left(\boldsymbol{v} f + T_0\,\nabla_{\boldsymbol{v}} f\right), \\ -\varepsilon_0 \Delta\Phi = q_e \displaystyle\int_{\mathbb{R}^3} f \,\mathrm{d}\boldsymbol{v}, \end{array}\right.$$

where $\mathbf{E} = -\nabla_{\boldsymbol{x}}\Phi$ is the self-consistent electric field, ε_0 is the vacuum permittivity, q_e and m_e are elementary charge and mass of the electrons, whereas τ_e is the relaxation time due to the collisions of the particles with the surrounding bath and T_0 the background temperature. In the present article, we will not consider the coupling with the Poisson equation and suppose that the electric field $\mathbf{E}$ is given and only depends on the space variable. We refer to [109] and to the next chapter dedicated to the numerical approximation and analysis of the Vlasov-Poisson-Fokker-Planck system.

Considering $\varepsilon > 0$ as the square root of the ratio between the mass of electrons and ions and $\tau(\varepsilon) > 0$ the ratio between the elapsed time between two collisions of electrons and the observable time, it allows to identify different regimes and the Vlasov equation may be written in a adimensional form

$$\varepsilon \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + \mathbf{E}(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{v}} f = \frac{\varepsilon}{\tau(\varepsilon)} \operatorname{div}_{\boldsymbol{v}} (\boldsymbol{v} f + T_0 \nabla_{\boldsymbol{v}} f), \tag{2.1}$$

Our main purpose here is to build and analyze a numerical scheme able to capture two regimes of interest for equation (2.1) in a linear framework : the long time behavior $t \to \infty$ and the diffusive regime $\varepsilon \to 0$. In various situations, the scaling parameters at play may be non homogeneous across the system leading to intricate situations, where both processes may coexist. Thus, we aim at designing a scheme robust enough to capture simultaneously these different behaviors.

More precisely, we consider the one dimensional Vlasov-Fokker-Planck equation with periodic boundary conditions in space, which reads

$$\partial_t f + \frac{1}{\varepsilon} (v \, \partial_x f + E_\infty \, \partial_v f) = \frac{1}{\tau(\varepsilon)} \partial_v (v \, f + T_0 \, \partial_v f), \tag{2.2}$$

with $t \geq 0$, position $x \in \mathbb{T}$ and velocity $v \in \mathbb{R}$, whereas the electric field derives from a potential ϕ_∞ such that $E_\infty = -\partial_x \phi_\infty$, with the following regularity assumption

$$\phi_\infty \in W^{2,\infty}(\mathbb{T}). \tag{2.3}$$

We also define the density ρ by integrating the distribution function in velocity,

$$\rho(t,x) = \int_{\mathbb{R}} f(t,x,v) \, \mathrm{d}v. \tag{2.4}$$

It is worth to mention that there are already several works on preserving large-time behaviors of solutions to the Fokker-Planck equation or related kinetic models. On the one hand, a fully discrete finite difference scheme for the homogeneous Fokker-Planck equation has been proposed in the pioneering work of Chang and Cooper [65]. This scheme preserves the stationary solution and the entropy decay of the numerical solution. On the other hand, finite volume schemes preserving the exponential trend to equilibrium have been studied for non-linear convection-diffusion equations (see for example [205, 16, 42, 62, 124]). More recently, in [183], the authors investigate the question of describing correctly the equilibrium state of non-linear diffusion and kinetic models for high order schemes. Let us also mention some works on boundary value problems [108, 63] where non-homogeneous Dirichlet boundary conditions are dealt with.

In the case of space non homogeneous kinetic equations, the convergence to equilibrium becomes tricky because of the lack of coercivity since dissipation occurs only in the velocity variable whereas transport acts in the space variable. Therefore, only few results are available and a better understanding of hypocoercive structures at the discrete level is challenging. Let us mention a first rigorous work in this direction on the Kolmogorov equation [189, 116, 118]. In [116], a time-splitting scheme is applied and it is shown that solutions have polynomial decay in time. In [189, 118], a different approach has been used, based on the work of Hérau [135] and Villani [211], for finite difference and a finite element schemes. Later, Dujardin, Hérau and Lafitte [101] studied a finite difference scheme for the kinetic Fokker-Planck equation. Finally, in a more recent work [19], the authors established a discrete hypocoercivity framework based on the continuous approach provided in [97]. It is based on a modified discrete entropy, equivalent to a weighted L^2 norm involving macroscopic quantities and the authors show quantitative estimates on the numerical solution for large time and in the limit $\varepsilon \to 0$.

The present contribution can be considered as a continuation of this latter work in order to discretize the kinetic Fokker-Planck equation with an applied force field. On the one hand, we consider the case where the interactions associated to collisions and electrostatic effects have the same magnitude, that is, $\tau(\varepsilon) \sim \varepsilon$, hence the limit $t/\varepsilon \to +\infty$ corresponds to the long time behavior of equation (2.2). In this regime, the distribution function f relaxes towards the stationary solution to the Vlasov-Fokker-Planck equation $\rho_\infty \mathcal{M}$, where the Maxwellian $\mathcal{M}$ is given by

$$\mathcal{M}(v) \;=\; \frac{1}{\sqrt{2\pi\, T_0}} \exp\left(-\frac{|v|^2}{2\, T_0}\right),$$

whereas the density ρ_∞ is determined by

$$\rho_\infty \;=\; c_0 \exp\left(-\frac{\phi_\infty}{T_0}\right), \tag{2.5}$$

where the constant c_0 is fixed by the conservation of mass, that is,

$$\int_{\mathbb{T}} \rho_\infty \,\mathrm{d}x \;=\; \iint_{\mathbb{T}\times\mathbb{R}} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v\,.$$

Thus, we set f_∞ the stationary state of (2.2), defined as

$$f_\infty(x,v) \;=\; \rho_\infty(x)\,\mathcal{M}(v)$$

and we expect that $f \to f_\infty$ as $t/\varepsilon \to +\infty$.

On the other hand, the diffusive regime corresponds to a frontier where collisions dominate but still not enough to cancel completely the electrostatic effects. This situation occurs as $\varepsilon \to 0$ in the case where $\tau(\varepsilon) \sim \tau_0\,\varepsilon^2$, for some $\tau_0 > 0$. Due to collisions, the distribution of velocities also relaxes towards a Maxwellian equilibrium. However, in this case, the spatial distribution converges to a time dependent distribution ρ whose dynamics are driven by a drift-diffusion equation depending on the force field E_∞. Indeed, performing the change of variable $x \to x + \tau_0\,\varepsilon\, v$ in (2.2) and integrating with respect to v, we deduce that the quantity

$$\pi\,(t,x) \;=\; \int_{\mathbb{R}} f\,(t, x - \tau_0\,\varepsilon\, v, v)\,\mathrm{d}v\,,$$

solves the following equation

$$\partial_t \pi \,+\, \tau_0\, \partial_x \left(\int_{\mathbb{R}} E_\infty\, f\,(t, x - \tau_0\, \varepsilon\, v, v)\; \mathrm{d}v \,-\, T_0\, \partial_x\, \pi \right) \,=\, 0\,.$$

According to its definition, π verifies : $\rho \sim \pi$ in the limit $\varepsilon \to 0$. Therefore, we may formally replace π with ρ and ε with 0 in the latter equation. This yields

$$f(t,x,v) \underset{\varepsilon \to 0}{\longrightarrow} \rho_{\tau_0}(t,x)\, \mathcal{M}(v)\,,$$

where ρ_{τ_0} solves

$$\partial_t \rho_{\tau_0} \,+\, \tau_0\, \partial_x \left(E_\infty\, \rho_{\tau_0} \,-\, T_0\, \partial_x\, \rho_{\tau_0} \right) \,=\, 0\,. \tag{2.6}$$

To be noted that this regime is an intermediate situation which contains more information than the long time asymptotic since we have $\rho \to \rho_\infty$ by taking either $t \to +\infty$ or $\tau_0 \to +\infty$ in the latter equation.

At the discrete level, Asymptotic-Preserving schemes have been developed to capture in a discrete setting the diffusion limit, so that in the limit $\varepsilon \to 0$, the numerical discretization converges to the macroscopic model (see for instance [146, 159, 145, 157] on finite difference and finite volume schemes and [94, 76] on particle methods).

In the present article, our aim is to design a numerical scheme which is able to capture these two regimes but also all the intermediate situations where $\varepsilon^2 \lesssim \tau(\varepsilon) \lesssim \varepsilon$. More precisely, we suppose that

$$\sup_{\varepsilon > 0} \frac{\tau(\varepsilon)}{\varepsilon} \,\leq\, \overline{\tau}_0 \in (0\,,\, +\infty) \tag{2.7}$$

and distinguish two cases on $\tau(\varepsilon)$:

(*i*) either the diffusive regime assumption

$$\frac{\tau(\varepsilon)}{\varepsilon^2} \underset{\varepsilon \to 0}{\longrightarrow} \tau_0 \,<\, +\infty\,, \tag{2.8}$$

where collisional effects strongly dominate ;

(*ii*) or the intermediate regime assumption

$$\frac{\tau(\varepsilon)}{\varepsilon^2} \underset{\varepsilon \to 0}{\longrightarrow} +\infty\,, \tag{2.9}$$

which may for instance correspond to $\tau(\varepsilon) = \varepsilon^\beta$, with $1 \leq \beta < 2$. It describes all the intermediate situations between long time and diffusive regime.

The starting point of our analysis is the following estimate, obtained multiplying equation (2.2) by $f \,/\, f_\infty$, and balancing the transport term with the source term corresponding to the electric field thanks to the weight f_∞^{-1}

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}\times\mathbb{R}} |f - f_\infty|^2\, f_\infty^{-1} \mathrm{d}x\, \mathrm{d}v \,+\, \frac{T_0}{\tau(\varepsilon)} \int_{\mathbb{T}\times\mathbb{R}} \left| \partial_v \left(\frac{f}{f_\infty} \right) \right|^2 f_\infty\, \mathrm{d}x\, \mathrm{d}v \,=\, 0\,. \tag{2.10}$$

This estimate is important since it yields a L^2 stability result on the solution to the Vlasov-Fokker-Planck equation (2.2).

Our purpose is to design a numerical scheme for which such estimate occurs. To this aim, we split our approach in two steps : first we apply a Hermite spectral decomposition in velocity of f and then we apply a structure preserving finite volume scheme for the space discretization. In the next section (Section 2.2), we provide explicit convergence rates for the continuous model written in the Hermite basis (see Theorems 2.1 and 2.2). This first step allows us to present the general strategy and to highlight the main properties of the transport operator in order to design suitable numerical scheme. Therefore, in Section 2.3 we adapt these latter results without any loss to the fully discrete setting using a structure preserving finite volume scheme and an implicit Euler scheme for the time discretization (see Theorems 2.7 and 2.8). The variety of situations that we aim to cover may lead to various and intricate behaviors. Therefore, we successfully put great efforts into providing results which are uniform with respect to all parameters at play : time t, scaling parameters (ε, τ_0) and eventually the numerical discretization. The result is worth the pain, since we propose in the Section 2.4 various simulations, in which we are able to capture, at low computational cost, a rich variety of situations.

2.2 Hermite's decomposition for the velocity variable

The purpose of this section is to present a formulation of the Vlasov-Fokker-Planck equation (2.2) based on Hermite polynomials and to provide quantitative results on f when $\varepsilon \to 0$ and $t \to +\infty$. These results are identical to the ones obtained in the continuous case except that there are formulated on the corresponding Hermite's coefficients solution to a linear hyperbolic system. This formulation is well adapted to prepare the fully discrete setting in Section 2.3.

We first use Hermite polynomials in the velocity variable and write the Vlasov-Fokker-Planck equation (2.2) as an infinite hyperbolic system for the Hermite coefficients depending only on time and space. The idea is to apply a Galerkin method only keeping a small finite set of orthogonal polynomials rather than discretizing the distribution function in velocity [2, 148]. The merit to use orthogonal basis like the so-called scaled Hermite basis has been shown in [141, 206] or more recently for the Vlasov-Poisson system [88, 89, 112, 18] and for magnetized plasmas [66]. In this context the family of Hermite's functions $(\Psi_k)_{k\in\mathbb{N}}$ defined as

$$\Psi_k(v) \,=\, H_k\left(\frac{v}{\sqrt{T_0}}\right)\mathcal{M}(v)\,,$$

constitutes an orthonormal system for the inverse Gaussian weight, that is,

$$\int_{\mathbb{R}} \Psi_k(v)\,\Psi_l(v)\,\mathcal{M}^{-1}(v)\,\mathrm{d}v \,=\, \delta_{k,l}\,.$$

In the latter definition, $(H_k)_{k\in\mathbb{N}}$ stands for the family of Hermite polynomials defined recursively as follows $H_{-1} = 0$, $H_0 = 1$ and

$$\xi\, H_k(\xi) \;=\; \sqrt{k}\, H_{k-1}(\xi) \,+\, \sqrt{k+1}\, H_{k+1}(\xi)\,, \quad \forall\, k \,\geq\, 0\,.$$

Let us also point out that Hermite's polynomials verify the following relation

$$H_k'(\xi) \;=\; \sqrt{k}\, H_{k-1}(\xi)\,, \quad \forall\, k \,\geq\, 0\,.$$

Taking advantage of the latter relations, one can see why Hermite's functions arise naturally when studying the Vlasov-Poisson-Fokker-Planck model, especially in the diffusive regime, as they constitute an orthonormal basis which diagonalizes the Fokker-Planck operator :

$$\partial_v \left[\, v\, \Psi_k \,+\, T_0\, \partial_v\, \Psi_k \,\right] \;=\; -\,k\, \Psi_k\,.$$

Therefore, we consider the decomposition of f into its components $C = (C_k)_{k\in\mathbb{N}}$ in the Hermite basis

$$f\,(t,x,v) \;=\; \sum_{k\in\mathbb{N}} C_k\,(t,x)\, \Psi_k(v)\,. \tag{2.1}$$

It is worth mentioning that we also may consider a truncated series neglecting high order coefficient in order to construct a spectrally accurate approximation of f in the velocity variable.

As we have shown before, Hermite's decomposition with respect to the velocity variable is a suitable choice in our setting. When it comes to the space variable, we see from estimate (2.10) that the natural functional framework here is the L^2 space with weight ρ_∞^{-1}. Unfortunately, it is not very well adapted to the space discretization since it may generate additional spurious terms difficult to control when dealing with discrete integration by part. We bypass this difficulty by integrating the weight in the quantity of interest : instead of working directly with f, we consider the quantity $f \,/\, \sqrt{\rho_\infty}$ in order to get a well-balanced scheme in the same spirit to what has been already done in [63, 108] for well-balanced finite volume schemes. More precisely, we set

$$D_k \;:=\; \frac{C_k}{\sqrt{\rho_\infty}}$$

in (2.1), and inject this ansatz in (2.2). Using that $\rho_\infty\, E_\infty \;=\; T_0\, \partial_x \rho_\infty$, we get that $D \;=\; (D_k)_{k\in\mathbb{N}}$ satisfies the following system

$$\left\{\begin{array}{l} \partial_t D_k \,+\, \dfrac{1}{\varepsilon}\left(\sqrt{k}\, \mathcal{A}\, D_{k-1} \,-\, \sqrt{k+1}\, \mathcal{A}^\star D_{k+1}\right) \;=\; -\,\dfrac{k}{\tau(\varepsilon)}\, D_k\,, \\[2ex] D_k(t=0) = D_k^{\mathrm{in}}\,, \end{array}\right. \tag{2.2}$$

where operators $\mathcal{A}$ and $\mathcal{A}^\star$ are given by

$$\left\{\begin{array}{l} \mathcal{A}\, u \;=\; +\sqrt{T_0}\, \partial_x u \,-\, \dfrac{E_\infty}{2\sqrt{T_0}}\, u\,, \\[2ex] \mathcal{A}^\star\, u \;=\; -\sqrt{T_0}\, \partial_x u \,-\, \dfrac{E_\infty}{2\sqrt{T_0}}\, u\,. \end{array}\right.$$

In this framework, the equilibrium D_∞ to (2.2) is given by

$$D_{\infty,k} \;=\; \begin{cases} \sqrt{\rho_\infty}, \text{ if } k = 0\,, \\ 0, \text{ else}\,, \end{cases} \tag{2.3}$$

and estimate (2.10) simply rewrites

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|D(t) - D_\infty\|_{L^2}^2 \;+\; \frac{1}{\tau(\varepsilon)}\,\sum_{k\in\mathbb{N}^\star} k\,\|D_k(t)\|_{L^2(\mathbb{T})}^2 \;=\; 0\,, \tag{2.4}$$

where $\|\cdot\|_{L^2}$ stands for the overall L^2-norm **with no weight**

$$\|D\|_{L^2}^2 \;=\; \sum_{k\in\mathbb{N}} \|D_k\|_{L^2(\mathbb{T})}^2\,.$$

On top of that, the limit of the diffusive regime is given by $D_{\tau_0} = (D_{\tau_0,k})_{k\in\mathbb{N}}$ defined as follows

$$D_{\tau_0,k} = \begin{cases} D_{\tau_0,0}, \text{ if } k = 0\,, \\ 0, \text{ else}\,, \end{cases} \tag{2.5}$$

where the first Hermite coefficient $D_{\tau_0,0}$ solves the following drift-diffusion equation

$$\begin{cases} \partial_t D_{\tau_0,0} \;+\; \tau_0\,\mathcal{A}^\star\mathcal{A}D_{\tau_0,0} \;=\; 0\,, \\ D_{\tau_0,0}(t=0) \;=\; D_{\tau_0,0}^{\mathrm{in}}\,, \end{cases} \tag{2.6}$$

which is obtained substituting ρ_{τ_0} with $D_{\tau_0,0}\,\sqrt{\rho_\infty}$ in equation (2.6). We define $D_{\tau_0}^{\mathrm{in}} = \left(\delta_{k0}\,D_{\tau_0,0}^{\mathrm{in}}\right)_{k\in\mathbb{N}}$ where δ_{k0} is the Kronecker symbol.

To conclude this section, we introduce some additional norms which arise naturally along our analysis. In Section 2.2.3, we consider the following H^{-1} norm defined on the L^2 subspace orthogonal to $\sqrt{\rho_\infty}$: for all $g \in L^2(\mathbb{T})$ which meets the condition

$$\int_{\mathbb{T}} g\,\sqrt{\rho_\infty}\,\mathrm{d}x \;=\; 0\,, \tag{2.7}$$

we set

$$\|g\|_{H^{-1}} = \|\mathcal{A}\,u\|_{L^2(\mathbb{T})}\,,$$

where u solves the following elliptic equation

$$\begin{cases} \mathcal{A}^\star\mathcal{A}\,u \;=\; g\,, \\ \displaystyle\int_{\mathbb{T}} u\,\sqrt{\rho_\infty}\,\mathrm{d}x \;=\; 0\,. \end{cases} \tag{2.8}$$

The latter equation admits a unique solution in $H^2(\mathbb{T})$ for any data $g \in L^2(\mathbb{T})$ that meets the compatibility condition (2.7). This well-posedness result crucially relies on the Poincaré inequality (2.18).

In Section 2.2.3, we use the following H^1 norm, defined for all $D = (D_k)_{k\in\mathbb{N}}$ as follows

$$\|\mathcal{B}\, D\|_{L^2}^2 = \sum_{k\in\mathbb{N}} \|\mathcal{B}_k\, D_k\|_{L^2(\mathbb{T})}^2\,,$$

where the family of differential operator $\mathcal{B} = (\mathcal{B}_k)_{k\geq 0}$ is defined as follows

$$\mathcal{B}_k = \begin{cases} \mathcal{A}\,, \text{ if } k = 0\,, \\ \mathcal{A}^\star,\ \text{else}\,. \end{cases} \tag{2.9}$$

To end with, we introduce the notation $D_\perp = (D_{\perp,k})_{k\in\mathbb{N}}$, which corresponds to the Hermite coefficients of $f - \rho\,\mathcal{M}$, that is

$$D_{\perp,k} = \begin{cases} 0,\ \text{if } k = 0\,, \\ D_k,\ \text{else}\,, \end{cases} \tag{2.10}$$

so that

$$\|D_\perp\|_{L^2} = \|f - \rho\,\mathcal{M}\|_{L^2(f_\infty^{-1})}.$$

2.2.1 Main results

In this section, we present two results which aim at describing the dynamics of (2.2) in various regimes ranging from long time behavior to diffusive limit. We aim for estimates which capture simultaneously and quantitatively the limits $t \to +\infty$ and $\varepsilon \to 0$, in order to lay the groundwork for our upcoming numerical analysis, in which we will build a scheme capable of reproducing these estimates exactly.

Our first main result tackles the long time behavior of the solution $D = (D_k)_{k\in\mathbb{N}}$ to (2.2). It is uniform with respect ε and covers all the regimes of interests since we only impose assumption (2.7) on the scaling parameter $\tau(\varepsilon)$. This result is the first step towards its discrete analog, Theorem 2.7.

Theorem 2.1. *Suppose that condition* (2.7) *on* $\tau(\varepsilon)$ *is satisfied and let* $D = (D_k)_{k\in\mathbb{N}}$ *be the solution to* (2.2) *with an initial datum* D^{in}. *There exists some positive constant* C *depending only on* ϕ_∞ *and* T_0 *such that*

(i) under the condition $\left\|D^{\text{in}}\right\|_{L^2} < +\infty$, *it holds for all times* $t \geq 0$

$$\|D(t) - D_\infty\|_{L^2} \leq \sqrt{3}\,\left\|D^{\text{in}} - D_\infty\right\|_{L^2} \exp\left(-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\, t\right);$$

(ii) under the condition $\left\|\mathcal{B}D^{\text{in}}\right\|_{L^2} + \left\|D^{\text{in}}\right\|_{L^2} < +\infty$, *it holds for all times* $t \geq 0$

$$\|\mathcal{B}D(t)\|_{L^2} \leq \sqrt{3}\left(C\,(\overline{\tau}_0 + 1)\left\|D^{\text{in}} - D_\infty\right\|_{L^2} + \left\|\mathcal{B}D^{\text{in}}\right\|_{L^2}\right) \exp\left(-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\, t\right);$$

where $\kappa > 0$ *is given by*

$$\kappa = \frac{1}{C\,(\overline{\tau}_0^2 + 1)}\,.$$

The proof of this result is provided in Section 2.2.3. The main difficulty here consists in proving the convergence of the first coefficient D_0 in the Hermite decomposition of f towards the equilibrium $\sqrt{\rho_\infty}$. We adapt hypocoercivity methods developed in [211, 97] to the framework of Hermite decomposition. Instead of estimating directly the quantities of interest, we introduce modified entropy functionals (see (2.20) and (2.27)), in order to recover dissipation and thus a convergence rate on D_0. Then, the second item tackles the convergence in a H^1 setting. Though a bit more technical, this second convergence result contains no main additional difficulty in comparison to the L^2 convergence result. Actually this latter result is essentially motivated by the analysis of the regime $\varepsilon \to 0$ presented below.

This leads us to the second main result in this section, which describes the behavior of the system as ε vanishes. We distinguish the diffusive regime, which corresponds to the case where $\tau(\varepsilon)$ satisfies (2.8) and the intermediate situations between long time and diffusive regime where $\tau(\varepsilon)$ satisfies (2.9). We will adapt this result into the fully discrete setting in Theorem 2.8.

Theorem 2.2. *Suppose that* $\tau(\varepsilon)$ *meets assumption* (2.7)*. For all positive* ε*, consider* $D = (D_k)_{k\in\mathbb{N}}$ *the solution to* (2.2) *with an initial datum* D^{in} *such that*

$$\left\|D^{\mathrm{in}}\right\|_{H^1}^2 := \left\|\mathcal{B}D^{\mathrm{in}}\right\|_{L^2}^2 + \left\|D^{\mathrm{in}}\right\|_{L^2}^2 < +\infty\,.$$

The following statements hold true uniformly with respect to ε

(*i*) *suppose that* $\tau(\varepsilon)$ *satisfies* (2.8)*, that is* $\tau(\varepsilon) \sim \tau_0\,\varepsilon^2$ *and for simplicity, suppose*

$$\left|\frac{\tau(\varepsilon)}{\tau_0\,\varepsilon^2} - 1\right| \leq \frac{1}{2}\,, \quad \forall\,\varepsilon > 0 \tag{2.11}$$

and consider $D_{\tau_0} = (D_{\tau_0,k})_{k\in\mathbb{N}}$ *given by* (2.5)*. On the one hand, it holds for all time* $t \in \mathbb{R}^+$

$$\|D_\perp(t)\|_{L^2} \leq \left\|D_\perp^{\mathrm{in}}\right\|_{L^2} e^{-t/(4\tau_0\varepsilon^2)} + \tau_0\,\varepsilon\,C(\overline{\tau}_0 + 1)\left\|D^{\mathrm{in}} - D_\infty\right\|_{H^1} e^{-\tau_0\,\kappa\,t}\,,$$

where $D_\perp$ *is given in* (2.10)*; on the other hand, it holds*

$$\begin{aligned}\|D_0(t) - D_{\tau_0,0}(t)\|_{H^{-1}} \leq & \,C\left(\left\|D_0^{\mathrm{in}} - D_{\tau_0,0}^{\mathrm{in}}\right\|_{H^{-1}} + \varepsilon\,\tau_0\,(\overline{\tau}_0^3 + 1)\left\|D^{\mathrm{in}} - D_\infty\right\|_{H^1}\right) e^{-\tau_0\,\kappa\,t} \\ & + C\left|\frac{\tau_0\varepsilon^2}{\tau(\varepsilon)} - 1\right|\left\|D_{\tau_0}^{\mathrm{in}} - D_\infty\right\|_{L^2} e^{-\tau_0\kappa\,t}\,;\end{aligned}$$

(*ii*) *suppose that* $\tau(\varepsilon)$ *satisfies* (2.9)*, that is* $\tau(\varepsilon)/\varepsilon^2 \to +\infty$ *as* ε *vanishes. Then it holds for all time* $t \in \mathbb{R}^+$

$$\|D_\perp(t)\|_{L^2} \,\le\, \left\|D_\perp^{\mathrm{in}}\right\|_{L^2} e^{-t/(2\tau(\varepsilon))} \,+\, \frac{\tau(\varepsilon)}{\varepsilon} C(\overline{\tau}_0+1) \left\|D^{\mathrm{in}} - D_\infty\right\|_{H^1} e^{-\frac{\tau(\varepsilon)}{\varepsilon^2}\kappa\, t}\,,$$

as well as

$$\|D_0(t) - D_{\infty,0}\|_{H^{-1}} \,\le\, C\left(\left\|D_0^{\mathrm{in}} - D_{\infty,0}\right\|_{H^{-1}} \,+\, \frac{\tau(\varepsilon)}{\varepsilon}\left(\overline{\tau}_0^3+1\right)\left\|D^{\mathrm{in}} - D_\infty\right\|_{H^1}\right) e^{-\frac{\tau(\varepsilon)}{\varepsilon^2}\kappa\, t}\,.$$

In the latter estimate, constant C *only depends on* ϕ_∞ *and* T_0 *and exponent* κ *is given by*

$$\kappa \,=\, \frac{1}{C\left(\overline{\tau}_0^2+1\right)}\,.$$

The proof of this result is provided in Section 2.2.4, it showcases two major difficulties. The first one is similar to the one encountered in Theorem 2.1 ; instead of estimating directly the H^{-1} norm between the first Hermite coefficient D_0 and its limit, we find the right intermediate quantity in order to recover dissipation (see (2.29)). However, unlike in the case of Theorem 2.1, we crucially need to incorporate derivatives of the solution D to (2.2) in this quantity in order to obtain some convergence rates. This leads us to the second difficulty, which is that we propagate some regularity. Furthermore, since Theorem 2.2 describes simultaneously the large time behavior and the asymptotic $\varepsilon \to 0$, it is not sufficient to propagate derivative globally nor uniformly with respect to time, we need instead to prove a convergence result in regular norms. This motivates item (*ii*) in Theorem 2.1, which will play a key role in our proof. This regularity issue explains why we prove H^{-1} convergence with respect to the first Hermite coefficient whereas we achieve strong L^2 convergence with respect to other coefficients. To be noted that strong L^2 convergence for the first coefficient may be achieved with our method at the price of loosing pointwise estimate with respect to time and thus considering integrated norms with respect to the time variable.

Theorems 2.1 and 2.2 fully answer their purpose, which is to describe the dynamics of (2.2) in the regime of interests, uniformly with respect to all parameters at play here.

2.2.2 Preliminary results

Let us first emphasize the important properties satisfied by $\mathcal{A}$, which we will need to recover later on, in the discrete setting. First, $\mathcal{A}^\star$ is its dual operator in $L^2(\mathbb{T})$, indeed for all u, $v \in H^1(\mathbb{T})$ it holds

$$\langle \mathcal{A}^\star u,\, v\rangle = \langle \mathcal{A} v,\, u\rangle\,, \tag{2.12}$$

where $\langle .,\, .\rangle$ denotes the classical scalar product in $L^2(\mathbb{T})$. Furthermore, $D_{\infty,0}$ lies in the kernel of $\mathcal{A}$, indeed

$$\mathcal{A}\, D_{\infty,0} \,=\, 0\,; \tag{2.13}$$

in this setting, conservation of mass is ensured by the following property

$$\int_{\mathbb{T}} \mathcal{A}^\star u \sqrt{\rho_\infty}\, \mathrm{d}x \,=\, 0\,, \tag{2.14}$$

indeed, considering equation (2.2) with index $k = 0$ integrated over $\mathbb{T}$ and applying the latter relation with $u = D_1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} D_0(t) \sqrt{\rho_\infty}\, \mathrm{d}x \,=\, 0\,,$$

and therefore

$$\int_{\mathbb{T}} D_0(t) \sqrt{\rho_\infty}\, \mathrm{d}x \,=\, \int_{\mathbb{T}} D_{\infty,0} \sqrt{\rho_\infty}\, \mathrm{d}x\,; \tag{2.15}$$

we also point out that since

$$\sqrt{T_0}\,(\mathcal{A} + \mathcal{A}^\star) \,=\, \partial_x \phi_\infty\,,$$

it holds

$$\|\,(\mathcal{A} + \mathcal{A}^\star)\, u\|_{L^2} \,\leq\, \frac{1}{\sqrt{T_0}}\, \|\phi_\infty\|_{W^{1,\infty}} \|u\|_{L^2}\,, \tag{2.16}$$

on top of that, operators $\mathcal{A}$ and $\mathcal{A}^\star$ do not commute and we have

$$[\mathcal{A},\, \mathcal{A}^\star] \,=\, \mathcal{A}\,\mathcal{A}^\star - \mathcal{A}^\star\,\mathcal{A} \,=\, \partial_{xx}\phi_\infty\,,$$

which yields

$$\|\,[\mathcal{A},\, \mathcal{A}^\star]\, u\|_{L^2} \,\leq\, \|\phi_\infty\|_{W^{2,\infty}}\, \|u\|_{L^2}\,; \tag{2.17}$$

the last key property verified by operator $\mathcal{A}$ is the following Poincaré-Wirtinger inequality : under the compatibility condition (2.7) on $u \in H^1(\mathbb{T})$ it holds

$$\|u\|_{L^2} \,\leq\, C_P \sqrt{T_0} \left(\int_{\mathbb{T}} \left| \partial_x \left(\frac{u}{\sqrt{\rho_\infty}} \right) \right|^2 \rho_\infty\, \mathrm{d}x \right)^{1/2} \,=\, C_P \|\mathcal{A}\, u\|_{L^2}\,, \tag{2.18}$$

for some positive constant C_P depending only on the potential ϕ_∞ and T_0. A proof of this result will be given in the discrete setting (see Lemma 2.9), we do not detail it in the continuous case since it is not our main interest here.

2.2.3 Proof of Theorem 2.1

It is worth to mention that estimate (2.4) itself is not sufficient to conclude on the rate of convergence of D to the equilibrium D_∞, since there is no dissipation with respect to the zero-th Hermite coefficient D_0. Therefore, it does not provide quantitative estimates when it comes to its convergence towards $D_{\infty,0}$. Recovering this dissipation is the key feature of hypocoercivity [211, 97]. In our setting it is done by combining the equations on D_0 and D_1, to remove stiff terms

$$\partial_t \left(D_0 + \frac{\tau(\varepsilon)}{\varepsilon} \mathcal{A}^\star D_1 \right) + \frac{\tau(\varepsilon)}{\varepsilon^2} \left(\mathcal{A}^\star \mathcal{A}\, D_0 \,-\, \sqrt{2}\, (\mathcal{A}^\star)^2\, D_2 \right) \,=\, 0\,. \tag{2.19}$$

To prove quantitative estimates on the solution to (2.2), we therefore introduce the "modified entropy functional" [211, 97] : for any $\alpha_0 > 0$, which will be specified later, we define $\mathcal{H}_0$ as

$$\mathcal{H}_0[D|D_\infty] \,=\, \frac{1}{2}\, \|D(t) - D_\infty\|_{L^2}^2 \,+\, \alpha_0 \left\langle \frac{\tau(\varepsilon)}{\varepsilon}\, \mathcal{A}^\star D_1, u^\varepsilon \right\rangle, \tag{2.20}$$

where u^ε is the particular solution to equation (2.8) with source term $g \,=\, D_0 \,-\, D_{\infty,0}$. To be noted that $g \,=\, D_0 \,-\, D_{\infty,0}$ fulfills the compatibility condition (2.7), thanks to the conservation of mass property (2.14).

The first step consists in proving some intermediate results on the solutions u^ε to (2.8)

Lemma 2.3. *Consider any $g \in L^2(\mathbb{T})$ which meets condition* (2.7) *and u the corresponding solution to* (2.8). *Then, u satisfies the following estimate*

$$\|\mathcal{A}\, u\|_{L^2} \,\leq\, C_P\, \|g\|_{L^2}\,, \tag{2.21}$$

and

$$\|\mathcal{A}^2\, u\|_{L^2} \,\leq\, \left(1 \,+\, \frac{C_P}{\sqrt{T_0}}\, \|\phi_\infty\|_{W^{1,\infty}}\right) \|g\|_{L^2}\,, \tag{2.22}$$

where C_P is the Poincaré constant in (2.18).
Moreover, considering now the solution D to (2.2) *and u^ε the solution to* (2.8) *with source term $g \,=\, D_0 \,-\, D_{\infty,0}$, it holds for all time $t \geq 0$*

$$\varepsilon\, \|\mathcal{A}\, \partial_t u^\varepsilon(t)\|_{L^2} \,\leq\, \|D_1(t)\|_{L^2}\,. \tag{2.23}$$

Proof. The first estimate is obtained by testing the elliptic equation (2.8) against u and applying (2.12)

$$\|\mathcal{A}\, u\|_{L^2}^2 \,\leq\, \|g\|_{L^2}\, \|u\|_{L^2}\,,$$

hence the Wirtinger-Poincaré inequality (2.18) yields,

$$\|\mathcal{A}\, u\|_{L^2} \,\leq\, C_P\, \|g\|_{L^2}\,.$$

For the second estimate, we rewrite $\mathcal{A}^2 u$ as follows

$$\mathcal{A}^2\, u \,=\, -\mathcal{A}^\star \mathcal{A}\, u \,+ (\mathcal{A} + \mathcal{A}^\star)\, \mathcal{A}\, u\,,$$

then we replace $\mathcal{A}^\star \mathcal{A}\, u$ according to equation (2.8), take the L^2 norm on both sides of the relation and apply in turn (2.16) to estimate operator $\mathcal{A} + \mathcal{A}^\star$ and item (2.21) to estimate the norm of $\mathcal{A}\, u$, it yields

$$\|\mathcal{A}^2\, u\|_{L^2} \,\leq\, \left(1 \,+\, \frac{C_P}{\sqrt{T_0}}\, \|\phi_\infty\|_{W^{1,\infty}}\right) \|g\|_{L^2}\,.$$

For the third estimate we consider now that D is solution to (2.2) and first take the time derivative of the elliptic equation (2.8) and use the equation (2.2) on D_0 to get

$$\varepsilon\, \partial_t(\mathcal{A}^\star \mathcal{A}\, u^\varepsilon) \;=\; \varepsilon\, \partial_t(D_0 - D_{\infty,0}) \;=\; -\mathcal{A}^\star D_1\,.$$

Then multiply by $\partial_t u^\varepsilon$ and use (2.12) to get

$$\|\partial_t \mathcal{A}\, u^\varepsilon\|_{L^2}^2 \;=\; -\frac{1}{\varepsilon}\,\langle D_1, \partial_t \mathcal{A}\, u^\varepsilon\rangle \;\leq\; \frac{1}{\varepsilon}\,\|D_1\|_{L^2}\,\|\partial_t \mathcal{A}\, u^\varepsilon\|_{L^2}\,.$$

□

Thanks to the latter result we now prove that for small enough $\alpha_0 > 0$, the square root of the modified entropy is equivalent to the L^2 norm of $D - D_\infty$.

Lemma 2.4. *Suppose that condition* (2.7) *on* $\tau(\varepsilon)$ *is satisfied. Then for all* $\alpha_0 \in (0, \overline{\alpha}_0)$, *with* $\overline{\alpha}_0 = 1/(4\,\overline{\tau}_0\, C_P)$ *and* $D \in L^2(\mathbb{T})$ *such that* $D_0 - D_\infty$ *satisfies the compatibility condition* (2.7), *one has*

$$\|D - D_\infty\|_{L^2}^2 \leq\; 4\,\mathcal{H}_0[D|D_\infty] \;\leq\; 3\,\|D - D_\infty\|_{L^2}^2\,. \tag{2.24}$$

Proof. We estimate the additional term in the expression of $\mathcal{H}_0$ by applying the duality formula (2.12) and then Cauchy-Schwarz inequality

$$|\,\langle \mathcal{A}^\star D_1, u^\varepsilon\rangle\,| \;=\; |\,\langle D_1, \mathcal{A}\, u^\varepsilon\rangle_{L^2}\,| \;\leq\; \|D_1\|_{L^2}\,\|\mathcal{A}\, u^\varepsilon\|_{L^2}\,.$$

Then, we apply item (2.21) of Lemma 2.3 with u^ε and $g = D_0 - D_{\infty,0}$ and upper bound the norm of $\mathcal{A}\, u^\varepsilon$ accordingly

$$\|D_1\|_{L^2}\,\|\mathcal{A}\, u^\varepsilon\|_{L^2} \;\leq\; C_P\,\|D - D_\infty\|_{L^2}^2\,,$$

hence, applying assumption (2.7), we deduce

$$\alpha_0\,\frac{\tau(\varepsilon)}{\varepsilon}\,|\,\langle \mathcal{A}^\star D_1, u^\varepsilon\rangle\,| \;\leq\; \alpha_0\,\overline{\tau}_0\, C_P\,\|D - D_\infty\|_{L^2}^2\,.$$

Choosing $\overline{\alpha}_0 = 1/(4\,\overline{\tau}_0\, C_P)$, the result follows for $\alpha_0 \in (0, \overline{\alpha}_0)$. □

Relying on the previous lemmas, we are now able to carry out the proof of the first item (i) of Theorem 2.1. We compute the time derivative of the modified relative entropy and split into three terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_0[D(t)|D_\infty] \;=\; \mathcal{I}_1(t) \;+\; \alpha_0\,\mathcal{I}_2(t) \;+\; \alpha_0\,\mathcal{I}_3(t)\,,$$

where the first one corresponds to the dissipation of the L^2 norm (2.4),

$$\mathcal{I}_1 \;=\; -\frac{1}{\tau(\varepsilon)}\,\sum_{k\in\mathbb{N}} k\,\|D_k\|_{L^2}^2\;,$$

whereas the other ones correspond to the additional term of the modified relative entropy,

$$\begin{cases} \mathcal{I}_2 \; := \; -\dfrac{\tau(\varepsilon)}{\varepsilon^2} \left\langle \mathcal{A}^\star \mathcal{A} \left(D_0 - D_{\infty,0}\right) \, - \, \sqrt{2}\, (\mathcal{A}^\star)^2 \, D_2, \, u^\varepsilon \right\rangle \, - \, \dfrac{1}{\varepsilon} \left\langle \mathcal{A}^\star D_1, \, u^\varepsilon \right\rangle, \\ \mathcal{I}_3 \; := \; +\dfrac{\tau(\varepsilon)}{\varepsilon} \left\langle \mathcal{A}^\star D_1, \, \partial_t u^\varepsilon \right\rangle. \end{cases}$$

On the one hand, the term $\mathcal{I}_2$ gives the expected dissipation on $(D_0 - D_{\infty,0})$ since u^ε solves (2.8) with source term $(D_0 - D_{\infty,0})$. On the other hand we get some additional terms which can be estimated thanks to (2.21) and (2.22) in Lemma 2.3, it yields,

$$\begin{aligned} \mathcal{I}_2 \;\; &\leq \;\; -\frac{\tau(\varepsilon)}{\varepsilon^2} \, \|D_0 - D_{\infty,0}\|_{L^2}^2 \, + \, \frac{\tau(\varepsilon)}{\varepsilon^2} \, \sqrt{2} \left(1 + \frac{C_P}{\sqrt{T_0}} \, \|\phi_\infty\|_{W^{1,\infty}}\right) \|D_0 - D_{\infty,0}\|_{L^2} \|D_2\|_{L^2} \\ &\quad + \frac{C_P}{\varepsilon} \, \|D_0 - D_{\infty,0}\|_{L^2} \, \|D_1\|_{L^2} \, , \\ &\leq \;\; -\frac{\tau(\varepsilon)}{\varepsilon^2} \, (1 \, - \, C\,\eta) \, \|D_0 - D_{\infty,0}\|_{L^2}^2 \, + \, \frac{C}{2\,\eta} \left(\frac{\tau(\varepsilon)}{\varepsilon^2} \|D_2\|_{L^2}^2 \, + \, \frac{1}{\tau(\varepsilon)} \, \|D_1\|_{L^2}^2 \right), \end{aligned}$$

for any positive η and for some positive constant C depending only on T_0 and ϕ_∞. The term $\mathcal{I}_3$ is estimated directly by applying (2.23) of Lemma 2.3,

$$\mathcal{I}_3 \, \leq \, \frac{\tau(\varepsilon)}{\varepsilon^2} \|D_1\|_{L^2}^2 \, .$$

From these latter estimates and taking $\eta = 1/(2C)$, we get the following inequality

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}_0[D|D_\infty] \\ &\leq -\frac{\tau(\varepsilon)}{\varepsilon^2} \left(\frac{\alpha_0}{2} \, \|D_0 - D_{\infty,0}\|_{L^2}^2 + \left(\frac{\varepsilon^2}{\tau(\varepsilon)^2} \, - \, C^2 \left(1 \, + \, \frac{\varepsilon^2}{\tau(\varepsilon)^2}\right) \alpha_0 \right) \sum_{k \in \mathbb{N}} k \, \|D_k\|_{L^2}^2 \right). \end{aligned}$$

We choose α_0 sufficiently small such that

$$\frac{\alpha_0}{2} \, \leq \, \left(\frac{\varepsilon^2}{\tau(\varepsilon)^2} \, - \, C^2 \left(1 \, + \, \frac{\varepsilon^2}{\tau(\varepsilon)^2}\right) \alpha_0 \right),$$

which, according to assumption (2.7) on $\tau(\varepsilon)$, is fulfilled as long as

$$\alpha_0 \, \leq \, \frac{1}{C\,(\overline{\tau}_0^2 + 1)} \, ,$$

for some constant C depending only on ϕ_∞ and T_0, and taking κ_0 such that $3\,\kappa_0/4 \, = \, \alpha_0/2$, we derive the following estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}_0[D|D_\infty] + \frac{\tau(\varepsilon)}{\varepsilon^2} \, \frac{3\,\kappa_0}{4} \, \|D - D_\infty\|_{L^2}^2 \, \leq \, 0 \, .$$

Then applying Lemma 2.4 and taking $\alpha_0 \leq \overline{\alpha}_0$, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_0[D|D_\infty] + \frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa_0\,\mathcal{H}_0[D|D_\infty] \,\leq\, 0\,,$$

which yields after applying Gronwall's lemma, for any $t \geq 0$,

$$\mathcal{H}_0[D(t)|D_\infty] \,\leq\, \mathcal{H}_0[D^{\mathrm{in}}|D_\infty]\,\exp\left(-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa_0\,t\right).$$

We conclude this proof by applying Lemma 2.4 in order to substitute $\mathcal{H}_0$ with the L^2 norm of $D - D_\infty$ in the latter estimate.

We now turn to the proof of the second item *(ii)* of Theorem 2.1. To estimate the norm of $\mathcal{B}D$, we apply the operator $\mathcal{B}_k$ to (2.2) and next multiply by $\mathcal{B}_k D_k$, integrate with respect to $x \in \mathbb{T}$ and sum over $k \in \mathbb{N}$, after re-indexing the sum with respect to k, it yields

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathcal{B}D(t)\|_{L^2}^2 \,=\, \mathcal{J}_1(t)\,,$$

where $\mathcal{J}_1$ is defined as follows

$$\mathcal{J}_1 \,=\, \sum_{k\in\mathbb{N}^\star} -\frac{k}{\tau(\varepsilon)}\,\|\mathcal{B}_k D_k\|_{L^2}^2 \,+\, \frac{\sqrt{k}}{\varepsilon}\left(\langle \mathcal{B}_{k-1}\mathcal{A}^\star D_k,\, \mathcal{B}_{k-1}D_{k-1}\rangle \,-\, \langle \mathcal{B}_k \mathcal{A}\,D_{k-1},\, \mathcal{B}_k D_k\rangle\right).$$

Hence applying an integration by part and from the specific choice (2.9) of $\mathcal{B}$, we have

$$\mathcal{J}_1 \,=\, -\frac{1}{\tau(\varepsilon)}\sum_{k\in\mathbb{N}^\star} k\,\|\mathcal{B}_k D_k\|_{L^2}^2 \,-\, \frac{1}{\varepsilon}\sum_{k\geq 2}\sqrt{k}\,\langle\,[\mathcal{A}^\star,\mathcal{A}]\,D_{k-1},\, \mathcal{A}^\star D_k\rangle\,. \tag{2.25}$$

Applying Young inequality and property (2.17) on the commutator $[\mathcal{A}^\star,\mathcal{A}]$, we get that

$$\mathcal{J}_1 \,\leq\, \frac{1}{\tau(\varepsilon)}\left(\frac{\eta}{2}\,\|\phi_\infty\|_{W^{2,\infty}}^2 - 1\right)\sum_{k\in\mathbb{N}^\star} k\,\|\mathcal{B}_k D_k\|_{L^2}^2 \,+\, \frac{1}{2\,\eta}\,\frac{\tau(\varepsilon)}{\varepsilon^2}\sum_{k\geq 1}\|D_k\|_{L^2}^2\,.$$

Therefore, choosing $\eta \leq 1/\|\phi_\infty\|_{W^{2,\infty}}^2$, it yields

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathcal{B}D\|_{L^2}^2 \,+\, \frac{1}{2\,\tau(\varepsilon)}\sum_{k\in\mathbb{N}^\star} k\,\|\mathcal{B}_k D_k\|_{L^2}^2 \,\leq\, C\,\frac{\tau(\varepsilon)}{\varepsilon^2}\sum_{k\geq 1}\|D_k\|_{L^2}^2\,. \tag{2.26}$$

Again since there is no dissipation on the zero-th Hermite coefficient of $\mathcal{B}D$, we proceed as for the L^2 estimate and introduce a correction $\mathcal{H}_1$ given by

$$\mathcal{H}_1[D|D_\infty] \,=\, \frac{1}{2}\,\|\mathcal{B}D\|_{L^2}^2 \,+\, \alpha_1\left\langle \frac{\tau(\varepsilon)}{\varepsilon}\,\mathcal{A}\,D_0, D_1\right\rangle, \tag{2.27}$$

where α_1 has to be determined. First, we point out that for small enough $\alpha_1 > 0$, the modified entropy $\mathcal{H}_1$ is controlled by the squares of the L^2 norms of $D - D_\infty$ and $\mathcal{B}D$.

Lemma 2.5. *Suppose that condition* (2.7) *on* $\tau(\varepsilon)$ *is satisfied. Then for all* $\alpha_1 \in (0, \overline{\alpha}_1)$, *with* $\overline{\alpha}_1 = 1/(2\,\overline{\tau}_0)$ *and* $D \in L^2(\mathbb{T})$, *one has*

$$\|\mathcal{B}D\|_{L^2}^2 - \|D - D_\infty\|_{L^2}^2 \leq 4\,\mathcal{H}_1[D|D_\infty] \;\leq\; 3\,\|\mathcal{B}D\|_{L^2}^2 + \|D - D_\infty\|_{L^2}^2 \,. \tag{2.28}$$

Proof. The result is obtained applying the Young inequality to the additional term in the definition (2.27) of $\mathcal{H}_1$ and using that $\mathcal{A}D_{\infty,0} \;=\; 0$. ☐

To complete the proof of the second item (ii) in Theorem 2.1, we compute the time derivative of the modified relative entropy and split into two terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_1[D|D_\infty] \;=\; \mathcal{J}_1 + \alpha_1\,\mathcal{J}_2\,,$$

where the first one corresponds to the dissipation of the L^2 norm of $\mathcal{B}\,(D - D_\infty)$ for which we already have an estimate (2.26), that is,

$$\mathcal{J}_1 \;\leq\; -\frac{1}{2\,\tau(\varepsilon)}\sum_{k\in\mathbb{N}^\star} k\,\|\mathcal{B}_k D_k\|_{L^2}^2 \;+\; C\,\frac{\tau(\varepsilon)}{\varepsilon^2}\sum_{k\geq 1}\|D_k\|_{L^2}^2\,,$$

whereas the other one corresponds to the additional term of the modified relative entropy,

$$\mathcal{J}_2 \;:=\; \frac{\tau(\varepsilon)}{\varepsilon^2}\left(\langle \mathcal{A}\mathcal{A}^\star D_1,\, D_1\rangle \;-\; \|\mathcal{A}\,D_0\|_{L^2}^2 \;+\; \sqrt{2}\,\langle \mathcal{A}\,D_0,\, \mathcal{A}^\star D_2\rangle\right) \;-\; \frac{1}{\varepsilon}\,\langle D_1,\, \mathcal{A}\,D_0\rangle\,.$$

From properties (2.12) and (2.13) of operators $(\mathcal{A},\, \mathcal{A}^\star)$, we have

$$\frac{1}{\varepsilon}\,\langle D_1,\, \mathcal{A}\,D_0\rangle \;=\; \left\langle \frac{1}{\tau(\varepsilon)^{1/2}}\,\mathcal{A}^\star D_1,\; \frac{\tau(\varepsilon)^{1/2}}{\varepsilon}\,(D_0 - D_{\infty,0})\right\rangle.$$

Applying Young inequality on the third term in the definition of $\mathcal{J}_2$ and on the latter term, it yields

$$\mathcal{J}_2 \;\leq\; -\frac{\tau(\varepsilon)}{\varepsilon^2}\left[\frac{1}{2}\,\|\mathcal{A}\,D_0\|_{L^2}^2 \;-\; \left(1 + \frac{\varepsilon^2}{\tau(\varepsilon)^2}\right)\sum_{k\in\mathbb{N}^\star} k\,\|\mathcal{B}_k D_k\|_{L^2}^2 \;-\; \|D_0 - D_{\infty,0}\|_{L^2}^2\right].$$

Therefore, from these estimates, we get the following inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_1[D|D_\infty] \;&\leq\; (C + \alpha_1)\,\frac{\tau(\varepsilon)}{\varepsilon^2}\,\|D - D_\infty\|_{L^2}^2 \\ &-\; \frac{\tau(\varepsilon)}{2\,\varepsilon^2}\left[\alpha_1\,\|\mathcal{A}\,D_0\|_{L^2}^2 \;+\; \left(\frac{\varepsilon^2}{\tau(\varepsilon)^2} - 2\,\alpha_1\left(1 + \frac{\varepsilon^2}{\tau(\varepsilon)^2}\right)\right)\sum_{k\in\mathbb{N}^\star} k\,\|\mathcal{B}_k D_k\|_{L^2}^2\right],\end{aligned}$$

choosing α_1 sufficiently small such that

$$\alpha_1 \;\leq\; \left(\frac{\varepsilon^2}{\tau(\varepsilon)^2} - 2\,\alpha_1\left(1 + \frac{\varepsilon^2}{\tau(\varepsilon)^2}\right)\right),$$

which is verified under the condition

$$\alpha_1 \,\leq\, \frac{1}{2+3\,\overline{\tau}_0^2}\,,$$

we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_1[D|D_\infty] \,+\, \frac{\tau(\varepsilon)}{\varepsilon^2}\,\frac{\alpha_1}{2}\,\|\mathcal{B}D\|_{L^2}^2 \,\leq\, C\,\frac{\tau(\varepsilon)}{\varepsilon^2}\,\|D-D_\infty\|_{L^2}^2\,.$$

Furthermore, taking $\alpha_1 \leq 1/(2\,\overline{\tau}_0)$ and applying Lemma 2.5, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_1[D|D_\infty] \,+\, \frac{\tau(\varepsilon)}{\varepsilon^2}\,\frac{2\,\alpha_1}{3}\,\mathcal{H}_1[D|D_\infty] \,\leq\, C\,\frac{\tau(\varepsilon)}{\varepsilon^2}\,\|D-D_\infty\|_{L^2}^2\,.$$

Then we set

$$\kappa_1 \,=\, \min\left(\frac{2\,\alpha_1}{3}\,,\,\kappa_0\right)$$

and multiply the latter inequality by $\exp\left(\frac{\tau(\varepsilon)}{\varepsilon^2}\,\frac{2\,\alpha_1}{3}\,t\right)$, integrate in time and apply the first item (i) of Theorem 2.1 to estimate the right hand side, this yields

$$\mathcal{H}_1[D(t)|D_\infty] \,\leq\, \left(C\left(\overline{\tau}_0^2+1\right)\left\|D^{\mathrm{in}}\,-\,D_\infty\right\|_{L^2}^2 \,+\, \mathcal{H}_1[D^{\mathrm{in}}|D_\infty]\right)\exp\left(-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa_1\,t\right).$$

We conclude this proof by substituting $\mathcal{H}_1$ with the norm of $\mathcal{B}D$ in the latter estimate according to Lemma 2.5.

2.2.4 Proof of Theorem 2.2

Once again, instead of estimating directly the H^{-1} norm of $D_0 - D_{\tau_0}$, we introduce the following quantity, meant to recover dissipation on the zero-th Hermite coefficient

$$\mathcal{E}(t) \,=\, \frac{1}{2}\,\|\mathcal{A}\,v^\varepsilon(t)\|_{L^2}^2\,, \tag{2.29}$$

where $v^\varepsilon(t)$ solves the elliptic equation (2.8) with source term given by

$$g(t) \,=\, D_0(t) \,+\, \frac{\tau(\varepsilon)}{\varepsilon}\,\mathcal{A}^\star D_1(t) \,-\, D_{\tau_0,0}(t)\,,$$

where $D_0(t)$ and $D_1(t)$ are the first two components of the solution $D(t)$ of (2.2) and $D_{\tau_0,0}(t)$ is either the unique solution to the convection-diffusion equation (2.6) when $\tau(\varepsilon)$ satisfies (2.8), that is $\tau(\varepsilon)/\varepsilon^2 \to \tau_0 < +\infty$ or the stationary solution $D_{\infty,0}$ given by (2.3) when $\tau(\varepsilon)$ satisfies (2.9), that is $\tau(\varepsilon)/\varepsilon^2 \to +\infty$. The latter right hand side is motivated by equation (2.19) since it is given by the difference between $D_0 + \frac{\tau(\varepsilon)}{\varepsilon}\,\mathcal{A}^\star D_1$ and $D_{\tau_0,0}$. We point out that the latter source term meets the compatibility condition (2.7) thanks to property (2.14), which ensures that $\mathcal{A}^\star D_1(t)$ is orthogonal to $\sqrt{\rho_\infty}$ in $L^2\,(\mathbb{T})$.

Before proving the first item of Theorem 2.2, let us present some preliminary results. On the one hand, the following Lemma ensures that $\mathcal{E}(t)$ is controlled by the squares of the L^2 norm of $\mathcal{B}D(t)$ and the H^{-1} norm of $D_0(t) - D_{\tau_0,0}(t)$

Lemma 2.6. *We consider $\mathcal{E}(t)$ defined by (2.29). It holds uniformly with respect to ε*

$$\mathcal{E}(t) \,\leq\, \|D_0(t) - D_{\tau_0,0}(t)\|^2_{H^{-1}} \,+\, C_P^2\, \frac{\tau(\varepsilon)^2}{\varepsilon^2}\, \|\mathcal{B}D(t)\|^2_{L^2}\,, \tag{2.30}$$

and

$$\frac{1}{4}\, \|D_0(t) - D_{\tau_0,0}(t)\|^2_{H^{-1}} \,-\, C_P^2\, \frac{\tau(\varepsilon)^2}{2\,\varepsilon^2}\, \|\mathcal{B}D(t)\|^2_{L^2} \,\leq\, \mathcal{E}(t)\,. \tag{2.31}$$

Proof. Defining w^ε and u_{τ_0} as the respective solutions to (2.8) with source term $g = \mathcal{A}^\star D_1$ and $D_{\tau_0,0} - D_{\infty,0}$, it holds

$$v^\varepsilon \,=\, u^\varepsilon \,-\, u_{\tau_0} \,+\, \frac{\tau(\varepsilon)}{\varepsilon}\, w^\varepsilon\,.$$

We apply operator $\mathcal{A}$ to the latter relation, take the L^2 norm, and apply the triangular inequality, it yields

$$\sqrt{2\,\mathcal{E}} \,\leq\, \|\mathcal{A}\,(u^\varepsilon - u_{\tau_0})\|_{L^2} \,+\, \frac{\tau(\varepsilon)}{\varepsilon}\, \|\mathcal{A}\,w^\varepsilon\|_{L^2}\,,$$

and

$$\|\mathcal{A}\,(u^\varepsilon - u_{\tau_0})\|_{L^2} \,-\, \frac{\tau(\varepsilon)}{\varepsilon}\, \|\mathcal{A}\,w^\varepsilon\|_{L^2} \,\leq\, \sqrt{2\,\mathcal{E}}\,.$$

We estimate $\|\mathcal{A}\,w^\varepsilon\|_{L^2}$ applying (2.21) in Lemma 2.3 with source term $g = \mathcal{A}^\star D_1$, this yields

$$\sqrt{2\,\mathcal{E}} \,\leq\, \|D_0 - D_{\tau_0,0}\|_{H^{-1}} \,+\, \frac{\tau(\varepsilon)}{\varepsilon}\, C_P\, \|\mathcal{B}D\|_{L^2}\,,$$

and

$$\|D_0 - D_{\tau_0,0}\|_{H^{-1}} \,-\, \frac{\tau(\varepsilon)}{\varepsilon}\, C_P\, \|\mathcal{B}D\|_{L^2} \,\leq\, \sqrt{2\,\mathcal{E}}\,.$$

We obtain the result taking the square of the latter inequalities and applying Young's inequality.

□

On the other hand, when τ_0 is finite, we observe that the long time behavior of $D_{\tau_0,0}$ may be easily investigated. Indeed, since $\mathcal{A}\,D_{\infty,0} = 0$, we have that $D_{\tau_0,0} - D_{\infty,0}$ also solves (2.6). Therefore, multiplying (2.6) by $D_{\tau_0,0} - D_{\infty,0}$, integrating over $\mathbb{T}$ and applying the Poincaré inequality (2.18), we obtain the following estimate after applying Gronwall lemma

$$\|D_{\tau_0}(t) - D_\infty\|_{L^2} \,\leq\, \|D^{\text{in}}_{\tau_0} - D_\infty\|_{L^2}\, \exp\left(-\frac{\tau_0}{C_P^2}\, t\right), \quad \forall\, t \in \mathbb{R}^+\,. \tag{2.32}$$

We are now able to prove the first item (i) of Theorem 2.2, which treats the case where $\tau(\varepsilon) \sim \tau_0\,\varepsilon^2$, when $\varepsilon \to 0$ where $\tau_0 \in \mathbb{R}_\star^+$. To derive the first estimate in item (i) of Theorem 2.2, our starting point is the L^2 estimate (2.4) which ensures

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D_\perp(t)\right\|_{L^2}^2 + \frac{1}{\tau(\varepsilon)}\left\|D_\perp(t)\right\|_{L^2}^2 &\leq -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D_0(t) - D_{\infty,0}\right\|_{L^2}^2\\ &\leq -\frac{1}{\varepsilon}\left\langle \mathcal{A}^\star D_1(t),\, D_0(t) - D_{\infty,0}\right\rangle\\ &= -\frac{1}{\varepsilon}\left\langle D_1(t),\, \mathcal{A}\left(D_0(t) - D_{\infty,0}\right)\right\rangle,\end{aligned}$$

hence it gives from the Young inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|D_\perp(t)\right\|_{L^2}^2 + \frac{1}{\tau(\varepsilon)}\left\|D_\perp(t)\right\|_{L^2}^2 \leq \frac{\tau(\varepsilon)}{\varepsilon^2}\left\|\mathcal{B}D(t)\right\|_{L^2}^2.$$

We bound $\|\mathcal{B}D(t)\|_{L^2}^2$ applying item (ii) of Theorem 2.1. After multiplying the latter estimate by $e^{t/\tau(\varepsilon)}$ and integrating with respect to time, it yields

$$\begin{aligned}\left\|D_\perp(t)\right\|_{L^2}^2 &\leq \left\|D_\perp^{\mathrm{in}}\right\|_{L^2}^2 \exp\left(-\frac{t}{\tau(\varepsilon)}\right)\\ &+ \left(C(\overline{\tau}_0^2+1)\left\|D^{\mathrm{in}} - D_\infty\right\|_{L^2}^2 + \left\|\mathcal{B}D^{\mathrm{in}}\right\|_{L^2}^2\right)\frac{3\,\tau(\varepsilon)^2}{\varepsilon^2 - \kappa\,\tau(\varepsilon)^2}\exp\left(-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,t\right),\end{aligned}$$

where C is a positive constant depending only on ϕ_∞ and T_0 and $\kappa = \left(C(\overline{\tau}_0^2+1)\right)^{-1}$. Then we apply condition (2.7) on $\tau(\varepsilon)$, which ensures that taking C greater than 2 in the definition of κ, it holds $1/2 \leq 1 - \kappa\,\tau(\varepsilon)^2/\varepsilon^2$ uniformly with respect to ε. Therefore, we deduce the following estimate, which yields the first result in (i) of Theorem 2.1, after taking its square root and applying assumption (2.11) in order to substitute $\tau(\varepsilon)$ with $\tau_0\,\varepsilon^2$

$$\left\|D_\perp(t)\right\|_{L^2}^2 \leq \left\|D_\perp^{\mathrm{in}}\right\|_{L^2}^2 e^{-\frac{t}{\tau(\varepsilon)}} + 6\left(C(\overline{\tau}_0^2+1)\left\|D^{\mathrm{in}} - D_\infty\right\|_{L^2}^2 + \left\|\mathcal{B}D^{\mathrm{in}}\right\|_{L^2}^2\right)\frac{\tau(\varepsilon)^2}{\varepsilon^2}\,e^{-\frac{\tau(\varepsilon)}{\varepsilon^2}\kappa\,t}.$$

We now prove the second result in item (i) of Theorem 2.2. To do so, we evaluate $\mathcal{E}$ observing that

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \left\langle \partial_t\left(D_0 + \frac{\tau(\varepsilon)}{\varepsilon}\,\mathcal{A}^\star D_1 - D_{\tau_0,0}\right),\, v^\varepsilon\right\rangle.$$

Therefore, relying on equations (2.19) and (2.6) we deduce

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\frac{\tau(\varepsilon)}{\varepsilon^2}\left\|D_0 + \frac{\tau(\varepsilon)}{\varepsilon}\,\mathcal{A}^\star D_1 - D_{\tau_0,0}\right\|_{L^2}^2 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3,$$

where

$$\begin{cases}\mathcal{E}_1 = \left(\tau_0 - \dfrac{\tau(\varepsilon)}{\varepsilon^2}\right)\left\langle \mathcal{A}^\star\mathcal{A}\,D_{\tau_0,0}\,,\, v^\varepsilon\right\rangle,\\[2ex] \mathcal{E}_2 = \dfrac{\tau(\varepsilon)^2}{\varepsilon^3}\left\langle \mathcal{A}^\star\mathcal{A}\,D_1\,,\, v^\varepsilon\right\rangle,\\[2ex] \mathcal{E}_3 = \sqrt{2}\,\dfrac{\tau(\varepsilon)}{\varepsilon^2}\left\langle (\mathcal{A}^\star)^2\,D_2\,,\, v^\varepsilon\right\rangle.\end{cases}$$

We rewrite $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ according to the following considerations : first, we notice that $D_{\infty,0}$ solves (2.13) and therefore add $D_{\infty,0}$ to the left hand side of the bracket in $\mathcal{E}_1$, second we apply the duality formula (2.12) in $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ and then replace v^ε in $\mathcal{E}_1$ and $\mathcal{E}_2$ according to the relation

$$\mathcal{A}^\star \mathcal{A}\, v^\varepsilon \;=\; D_0 + \frac{\tau(\varepsilon)}{\varepsilon}\, \mathcal{A}^\star D_1 - D_{\tau_0,0}\,.$$

Hence, we obtain

$$\begin{cases} \mathcal{E}_1 \;=\; \left(\tau_0 - \dfrac{\tau(\varepsilon)}{\varepsilon^2}\right) \left\langle D_{\tau_0,0} - D_{\infty,0}\,,\, D_0 + \dfrac{\tau(\varepsilon)}{\varepsilon}\, \mathcal{A}^\star D_1 - D_{\tau_0,0} \right\rangle, \\ \mathcal{E}_2 \;=\; \dfrac{\tau(\varepsilon)^2}{\varepsilon^3} \left\langle D_1\,,\, D_0 + \dfrac{\tau(\varepsilon)}{\varepsilon}\, \mathcal{A}^\star D_1 - D_{\tau_0,0} \right\rangle, \\ \mathcal{E}_3 \;=\; \sqrt{2}\, \dfrac{\tau(\varepsilon)}{\varepsilon^2} \left\langle D_2\,,\, \mathcal{A}^2\, v^\varepsilon \right\rangle. \end{cases}$$

To estimate $\mathcal{E}_1$, we apply Young's inequality, which yields

$$\mathcal{E}_1 \;\leq\; \frac{\eta}{2} \frac{\tau(\varepsilon)}{\varepsilon^2}\, \|D_0 + \frac{\tau(\varepsilon)}{\varepsilon}\, \mathcal{A}^\star D_1 - D_{\tau_0,0}\|^2_{L^2} \;+\; \frac{1}{2\eta} \frac{\varepsilon^2}{\tau(\varepsilon)} \left|\tau_0 - \frac{\tau(\varepsilon)}{\varepsilon^2}\right|^2 \|D_{\tau_0} - D_\infty\|^2_{L^2}\,,$$

for all positive η. To estimate $\mathcal{E}_2$, we apply Young's inequality and then assumption (2.7) which ensures that $\tau(\varepsilon)^3/\varepsilon^4 \;\leq\; \left(\overline{\tau}_0^2\, \tau(\varepsilon)\right)/\varepsilon^2$, this gives

$$\mathcal{E}_2 \;\leq\; \frac{\eta}{2} \frac{\tau(\varepsilon)}{\varepsilon^2}\, \|D_0 + \frac{\tau(\varepsilon)}{\varepsilon}\, \mathcal{A}^\star D_1 - D_{\tau_0,0}\|^2_{L^2} \;+\; \frac{1}{\eta} \frac{\tau(\varepsilon)}{\varepsilon^2}\, \overline{\tau}_0^2\, \|D_\perp\|^2_{L^2}\;,$$

for all positive η. To estimate $\mathcal{E}_3$, we apply Young's inequality and then bound the norm of $\mathcal{A}^2\, v^\varepsilon$ by applying item (2.22) in Lemma 2.3 with source term

$$g \;=\; D_0 + \frac{\tau(\varepsilon)}{\varepsilon}\, \mathcal{A}^\star D_1 - D_{\tau_0,0}\,,$$

it yields

$$\mathcal{E}_3 \;\leq\; \eta\, \frac{\tau(\varepsilon)}{\varepsilon^2}\, \|D_0 + \frac{\tau(\varepsilon)}{\varepsilon}\, \mathcal{A}^\star D_1 - D_{\tau_0,0}\|^2_{L^2} \;+\; \frac{C}{\eta} \frac{\tau(\varepsilon)}{\varepsilon^2}\, \|D_\perp\|^2_{L^2}\;,$$

for some constant C depending only on ϕ_∞ and T_0. We gather the latter estimates, take $\eta = 1/4$ and apply item (2.21) in Lemma 2.3, which ensures that

$$\mathcal{E} \;\leq\; \frac{C_P^2}{2}\, \|D_0 + \frac{\tau(\varepsilon)}{\varepsilon}\, \mathcal{A}^\star D_1 - D_{\tau_0,0}\|^2_{L^2}\,.$$

Therefore, we obtain

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} + \frac{\tau(\varepsilon)}{C_P^2\, \varepsilon^2}\, \mathcal{E} \;\leq\; C\, \frac{\tau(\varepsilon)}{\varepsilon^2} \left(1 + \overline{\tau}_0^2\right) \|D_\perp\|^2_{L^2} \;+\; C\, \frac{\varepsilon^2}{\tau(\varepsilon)} \left|\tau_0 - \frac{\tau(\varepsilon)}{\varepsilon^2}\right|^2 \|D_{\tau_0} - D_\infty\|^2_{L^2}\,,$$

for some constant C depending only on ϕ_∞ and T_0. Then we multiply the latter estimate by $\exp\left(\dfrac{\tau(\varepsilon)}{C_P^2\,\varepsilon^2}\,t\right)$ and integrate with respect to time. After applying (2.32) to estimate $\|D_{\tau_0}-D_\infty\|_{L^2}$ and the first result in item (i) of Theorem 2.2 to estimate the norm of $D_\perp$, it yields

$$\mathcal{E}(t)\;\leq\;\left(\mathcal{E}(0)\,+\,C\,\frac{\tau(\varepsilon)^2}{\varepsilon^2}\,(\overline{\tau}_0^6+1)\left\|D^{\rm in}-D_\infty\right\|_{H^1}^2\right)\exp\left(-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,t\right)$$
$$+\,C\left|\frac{\tau_0\,\varepsilon^2}{\tau(\varepsilon)}-1\right|^2\,\|D_{\tau_0}^{\rm in}-D_\infty\|_{L^2}^2\,\left(\frac{2\,\tau_0\,\varepsilon^2}{\tau(\varepsilon)}-1\right)^{-1}\,\exp\left(-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,t\right).$$

To conclude, we substitute $\mathcal{E}(t)$ (resp. $\mathcal{E}(0)$) in the latter estimate according to (2.31) (resp. (2.30)) in Lemma 2.6 and then apply assumption (2.11) on $\tau(\varepsilon)$, which ensures $\left(\dfrac{2\,\tau_0\,\varepsilon^2}{\tau(\varepsilon)}-1\right)^{-1}\leq 3$, this yields

$$\|D_0(t)-D_{\tau_0,0}(t)\|_{H^{-1}}^2\;\leq$$
$$C\left(\|D_0^{\rm in}-D_{\tau_0,0}^{\rm in}\|_{H^{-1}}^2\,+\,\frac{\tau(\varepsilon)^2}{\varepsilon^2}\,(\overline{\tau}_0^6+1)\left\|D^{\rm in}-D_\infty\right\|_{H^1}^2\right)e^{-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,t}\,+$$
$$C\left|\frac{\tau_0\,\varepsilon^2}{\tau(\varepsilon)}-1\right|^2\|D_{\tau_0}^{\rm in}-D_\infty\|_{L^2}^2\,e^{-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,t}\,.$$

We obtain the second estimate provided in (i) of Theorem 2.2 taking the square root in the latter estimate and applying assumption (2.11) in order to substitute $\tau(\varepsilon)$ with $\tau_0\,\varepsilon^2$.

To prove the second item (ii) of Theorem 2.2, we follow the same lines as the ones for item (i) replacing D_{τ_0} by D_∞ and observing that D_∞ also solves the equation (2.6) since it is a stationary solution. Therefore, computations are even simpler since the term $\mathcal{E}_1$ vanishes in this case. As a consequence the estimate provided in item (ii) follows.

2.3 Finite volume discretization for the space variable

In this section we present a finite volume scheme for (2.2). Then we prove discrete hypocoercive estimates on the discrete solution to investigate the long time behavior and the speed of convergence to the steady state. Finally, we prove an asymptotic preserving property for the diffusive limit taking $\tau(\varepsilon)\sim\tau_0\,\varepsilon^2$ with error estimates with respect to ε. Thanks to the groundwork laid in the previous Section, we are able to propose a scheme which describes all the variety of regimes that we aim to capture in this article.

2.3.1 Numerical scheme

For simplicity purposes, we consider the problem in one space dimension. It will be straightforward to generalize this construction for Cartesian meshes in multidimensional case. In a one-dimensional setting, we consider an interval (a, b) of $\mathbb{R}$ and for $N_x \in \mathbb{N}^\star$, we introduce the set $\mathcal{J} = \{1, \dots, N_x\}$ and a family of control volumes $(K_j)_{j\in\mathcal{J}}$ such that $K_j = \left]x_{j-1/2}, x_{j+1/2}\right[$ with x_j the middle of the interval K_j and

$$a = x_{1/2} < x_1 < x_{3/2} < ... < x_{j-1/2} < x_j < x_{j+1/2} < ... < x_{N_x} < x_{N_x+1/2} = b\,.$$

Let us set

$$\begin{cases} \Delta x_j = x_{j+1/2} - x_{j-1/2}, & \text{for } j \in \mathcal{J}\,, \\ \Delta x_{i+1/2} = x_{j+1} - x_j, & \text{for } 1 \le j \le N_x - 1\,. \end{cases}$$

We also introduce the parameter h such that

$$h \,=\, \max_{j\in\mathcal{J}} \Delta x_j\,.$$

Let Δt be the time step. We set $t^n = n\Delta t$ with $n \in \mathbb{N}$. A time discretization of $\mathbb{R}^+$ is then given by the increasing sequence of $(t^n)_{n\in\mathbb{N}}$. In the sequel, we will denote by D_k^n the approximation of $D_k(t^n)$, where the index k represents the k-th mode of the Hermite decomposition, whereas $\mathcal{D}_{k,j}^n$ is an approximation of the mean value of D_k over the cell K_j at time t^n.

First of all, the initial condition is discretized on each cell K_j by :

$$\mathcal{D}_{k,j}^0 = \frac{1}{\Delta x_j}\int_{K_j} D_k^{\text{in}}(x)\,dx, \quad j \in \mathcal{J}\,.$$

The finite volume scheme is obtained by integrating the equation (2.2) over each control volume K_j and over each time step. Concerning the time discretization, we can choose any implicit method (backward Euler, Implicit Runge-Kutta,...). Since in this paper we are interested in the spatial discretization, we will only consider a backward Euler method afterwards. Let us now focus on the spatial discretization.

By integrating equation (2.2) on K_j for $j \in \mathcal{J}$, we obtain the numerical scheme : for $D_k^n = (\mathcal{D}_{k,j}^n)_{j\in\mathcal{J}}$

$$\frac{D_k^{n+1} - D_k^n}{\Delta t} + \frac{1}{\varepsilon}\left(\sqrt{k}\,\mathcal{A}_h\,D_{k-1}^{n+1} - \sqrt{k+1}\,\mathcal{A}_h^\star\,D_{k+1}^{n+1}\right) = -\frac{k}{\tau(\varepsilon)}\,D_k^{n+1}\,, \tag{2.1}$$

where $\mathcal{A}_h$ (resp. $\mathcal{A}_h^\star$) is an approximation of the operator $\mathcal{A}$ (resp. $\mathcal{A}^\star$) given by

$$\mathcal{A}_h = (\mathcal{A}_j)_{j\in\mathcal{J}} \quad \text{and} \quad \mathcal{A}_h^\star = (\mathcal{A}_j^\star)_{j\in\mathcal{J}}\,. \tag{2.2}$$

and where for $D = (\mathcal{D}_j)_{j\in\mathcal{J}}$ it holds

$$\begin{cases} \mathcal{A}_j D \,=\, +\sqrt{T_0}\left(\dfrac{\mathcal{D}_{j+1} - \mathcal{D}_{j-1}}{2\Delta x_j} \,-\, \dfrac{E_{\infty,j}}{2\,T_0}\,\mathcal{D}_j\right), & j \in \mathcal{J}\,, \\ \mathcal{A}_j^\star D \,=\, -\sqrt{T_0}\left(\dfrac{\mathcal{D}_{j+1} - \mathcal{D}_{j-1}}{2\Delta x_j} \,+\, \dfrac{E_{\infty,j}}{2\,T_0}\,\mathcal{D}_j\right), & j \in \mathcal{J}\,, \end{cases} \tag{2.3}$$

whereas the discrete electric field $E_{\infty,j}$ is given by

$$E_{\infty,j} \;=\; -\frac{\phi_{\infty,j+1}-\phi_{\infty,j-1}}{2\Delta x_j} \;=\; \frac{2\,T_0}{\sqrt{\rho}_{\infty,j}}\,\frac{\sqrt{\rho}_{\infty,j+1}-\sqrt{\rho}_{\infty,j-1}}{2\,\Delta x_j}\,, \tag{2.4}$$

where $\rho_{\infty,j}$ is an approximation of the stationary density ρ_∞ on the cell K_j. This latter formula is consistent with the definition of $\sqrt{\rho}_\infty = c_0\,e^{-\phi_\infty/(2T_0)}$ and the fact that

$$\frac{1}{2\,T_0}\,\partial_x\phi_\infty \;=\; -\frac{1}{\sqrt{\rho}_\infty}\,\partial_x\sqrt{\rho}_\infty\,.$$

This choice of discretization is motivated by preserving at the discrete level the key properties (2.12)-(2.18). In the end, we propose the following approximation of the continuous solution f to (2.2)

$$f^n(x,v) \;=\; \sum_{k\in\mathbb{N}} \sqrt{\rho}_\infty(x)\,D_k^n(x)\,\Psi_k(v)\,,$$

where for each $k \geq 0$ and $n \geq 0$, we define a piecewise constant function D_k^n from the numerical values $(\mathcal{D}_{k,j}^n)_{j\in\mathcal{J}}$ as

$$D_k^n(x) = \mathcal{D}_{k,j}^n, \qquad x\in K_j\,.$$

In this context the equilibrium D_∞ is given by

$$D_{\infty,k} \;=\; \begin{cases} \sqrt{\rho}_\infty, & \text{if } k=0\,,\\[2mm] 0, & \text{else}\,; \end{cases} \tag{2.5}$$

as for the limit in the diffusive regime $D_{\tau_0}^n = (D_{\tau_0,k}^n)_{k\in\mathbb{N}}$, it is given by

$$D_{\tau_0,k}^n = \begin{cases} D_{\tau_0,0}^n, & \text{if } k=0\,,\\[2mm] 0, & \text{else}\,, \end{cases} \tag{2.6}$$

where $D_{\tau_0,0}^n$ solves the following discrete version of equation (2.6)

$$\frac{D_{\tau_0,0}^{n+1}-D_{\tau_0,0}^n}{\Delta t} \;+\; \tau_0\,\mathcal{A}_h^\star\mathcal{A}_h D_{\tau_0,0}^{n+1} \;=\; 0\,. \tag{2.7}$$

We now introduce the norms we will work with in this section. We denote by $\langle .,.\rangle$ the L^2 scalar product for any $u=(u_j)_{j\in\mathcal{J}}$ and $v=(v_j)_{j\in\mathcal{J}}$,

$$\langle u,\,v\rangle \;=\; \sum_{j\in\mathcal{J}} \Delta x_j\,u_j\,v_j$$

and

$$\|u\|_{L^2} \;=\; \left(\sum_{j\in\mathcal{J}}\Delta x_j\,u_j^2\right)^{1/2}.$$

As in the (2.7), we consider the following H^{-1} norm defined on the L^2 subspace orthogonal to $\sqrt{\rho}_\infty$: for all $g_h = (g_j)_{j\in\mathcal{J}}$ which meets the condition

$$\sum_{j\in\mathcal{J}} \Delta x_j\, g_j\, \sqrt{\rho}_{\infty,j} \;=\; 0\,, \tag{2.8}$$

we set

$$\|g_h\|_{H^{-1}} = \|\mathcal{A}\, u_h\|_{L^2(\mathbb{T})}\;,$$

where $u_h = (u_j)_{j\in\mathcal{J}}$ is the solution to the discrete equivalent of equation (2.8)

$$\begin{cases} (\mathcal{A}_h^\star\, \mathcal{A}_h)\, u_h \;=\; g\,, \\[2ex] \displaystyle\sum_{j\in\mathcal{J}} \Delta x_j\, u_j\, \sqrt{\rho}_{\infty,j} \;=\; 0\,. \end{cases} \tag{2.9}$$

We also use the H^1 norm, analog to the one given in (2.9), defined for all $D = (D_k)_{k\in\mathbb{N}}$ as follows

$$\|\mathcal{B}_h\, D\|_{L^2}^2 \;=\; \sum_{k\in\mathbb{N}} \|\mathcal{B}_k\, D_k\|_{L^2}^2\,,$$

where the family of discrete operator $\mathcal{B}_h = (\mathcal{B}_{h,k})_{k\,\geq\,0}$ is given as follows

$$\mathcal{B}_{h,k} = \begin{cases} \mathcal{A}_h\,, \text{ if } k \;=\; 0\,, \\[2ex] \mathcal{A}_h^\star\,, \text{ else}\,. \end{cases} \tag{2.10}$$

To conclude with this section, we take the same definition of $D_\perp$ as in the continuous setting.

2.3.2 Main results

We can now release the two results that constitute the core of this article. Thanks to our choice of discretization, they are an exact translation of their continuous analogs, Theorems 2.1 and 2.2, into the discrete setting, without any loss of accuracy nor uniformity with respect to the parameters at play in our analysis. On top of that, the results are also uniform with respect to the discretization parameters.

This first result is the continuous analog of Theorem 2.1, it ensures that our scheme has the same long time behavior as the continuous model

Theorem 2.7. *Suppose that condition* (2.7) *on* $\tau(\varepsilon)$ *is satisfied and Let* $D^n = (D_k^n)_{k\in\mathbb{N}}$ *be the solution to* (2.1)*. The following statements hold true*

(*i*) *there exists some positive constant* C_0 *depending only on* ϕ_∞ *and* T_0 *such that for all* $\varepsilon > 0$ *and all* $n \geq 0$*, we have*

$$\|D^n - D_\infty\|_{L^2} \;\leq\; \sqrt{3}\,\left\|D^0 - D_\infty\right\|_{L^2}\left(1 + \frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa_0\,\Delta t\right)^{-n/2}\;;$$

(ii) suppose in addition that the mesh is regular enough so that the quantity

$$R_h = \sup_{(i,j)\in\mathcal{J}^2} \left|\Delta x_j \Delta x_i^{-1} - 1\right| \tag{2.11}$$

stays uniformly bounded with respect to the discretization parameter h. Then there exists a positive constant C_1 (depending only on ϕ_∞, T_0 and R_h) such that that for all $\varepsilon > 0$ and all $n \geq 0$, we have

$$\|\mathcal{B}_h D^n\|_{L^2} \leq \sqrt{3}\left(C_1\,(\overline{\tau}_0+1)\left\|\mathcal{B}_h D^0\right\|_{L^2} + \left\|D^0 - D_\infty\right\|_{L^2}\right)\left(1 + \frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa_1\,\Delta t\right)^{-\frac{n}{2}},$$

In the previous estimates $\kappa_i > 0$ is given by

$$\kappa_i = \frac{1}{C_i\,(\overline{\tau}_0^2+1)}.$$

Our second result deals with the asymptotic $\varepsilon \to 0$, it is the discrete analog of Theorem 2.2

Theorem 2.8. *Suppose that $\tau(\varepsilon)$ meets assumption* (2.7) *and that the mesh meets assumption* (2.11)*. Consider the solution $D^n = (D^n_k)_{k\in\mathbb{N}}$ to* (2.1)*. The following statements hold true uniformly with respect to ε*

(i) suppose that $\tau(\varepsilon)$ satisfies (2.8) *and* (2.11) *and consider $D^n_{\tau_0} = (D^n_{\tau_0,k})_{k\in\mathbb{N}}$ given by* (2.6)*. Then it holds for all $n \geq 0$,*

$$\|D^n_\perp\|_{L^2} \leq \left\|D^0_\perp\right\|_{L^2}\left(1+\frac{\Delta t}{2\,\tau_0\,\varepsilon^2}\right)^{-\frac{n}{2}} + \tau_0\,\varepsilon\, C(\overline{\tau}_0+1)\left\|D^0 - D_\infty\right\|_{H^1}(1+\tau_0\,\kappa\,\Delta t)^{-\frac{n}{2}},$$

and

$$\left\|D^n_0 - D^n_{\tau_0,0}\right\|_{H^{-1}} \leq C\left(\left\|D^0_0 - D^0_{\tau_0,0}\right\|_{H^{-1}} + \varepsilon\,\tau_0\,(\overline{\tau}_0^3+1)\left\|D^0 - D_\infty\right\|_{H^1}\right)(1+\tau_0\kappa\Delta t)^{-\frac{n}{2}},$$
$$C\left|\frac{\tau_0\varepsilon^2}{\tau(\varepsilon)} - 1\right|\left\|D^0_{\tau_0} - D_\infty\right\|_{L^2}(1+\tau_0\kappa\Delta t)^{-\frac{n}{2}};$$

(ii) suppose that $\tau(\varepsilon)$ satisfies (2.9)*. Then it holds for any $n \geq 0$*

$$\|D_\perp\|^2_{L^2} \leq$$
$$\|D_\perp\|^2_{L^2}\left(1+\frac{\Delta t}{\tau(\varepsilon)}\right)^{-\frac{n}{2}} + \frac{\tau(\varepsilon)}{\varepsilon}\,C(\overline{\tau}_0+1)\left\|D^0 - D_\infty\right\|_{H^1}\left(1+\frac{\tau(\varepsilon)}{\varepsilon^2}\kappa\Delta t\right)^{-\frac{n}{2}},$$

and

$$\left\|D^n_0 - D^n_{\infty,0}\right\|_{H^{-1}} \leq$$
$$C\left(\left\|D^0_0 - D^0_{\infty,0}\right\|_{H^{-1}} + \frac{\tau(\varepsilon)}{\varepsilon}\,(\overline{\tau}_0^3+1)\left\|D^0 - D_\infty\right\|_{H^1}\right)\left(1+\frac{\tau(\varepsilon)}{\varepsilon^2}\kappa\Delta t\right)^{-\frac{n}{2}}.$$

In the latter estimate, constant C only depends on ϕ_∞, T_0 and R_h and exponent κ is given by

$$\kappa = \frac{1}{C\,(\overline{\tau}_0^2 + 1)}.$$

Furthermore the shorthand notation $\|\cdot\|_{H^1}$ stands for

$$\|D\|_{H^1}^2 := \|\mathcal{B}D\|_{L^2}^2 + \|D\|_{L^2}^2 .$$

The proof of these results follows almost exactly the same lines as the proof of Theorems 2.1 and 2.2 thanks to the Lemma 2.9, which constitutes the keystone of our analysis and which ensures that our discretization $\mathcal{A}_h$ of operator $\mathcal{A}$ shares all the important properties (2.12)-(2.18) of its continuous analog. The only difference comes down to some numerical remainder terms that we easily control applying methods already developed in the continuous section.

2.3.3 Preliminary properties

This section is dedicated to the following fundamental Lemma, which ensures that the key properties (2.12)-(2.18) of the continuous operator $\mathcal{A}$ are preserved by its discrete analog $\mathcal{A}_h$. Thanks to this Lemma, all the computations carried in Section 2.2 directly translate into the discrete framework.

Lemma 2.9. *Consider the discrete operators $\mathcal{A}_h$ and $\mathcal{A}_h^\star$ given in* (2.2)*. Then we have for any $u = (u_j)_{j\in\mathcal{J}}$ and $v = (v_j)_{j\in\mathcal{J}}$*

1. *preservation of the duality formula*

$$\langle \mathcal{A}_h u,\, v\rangle = \langle u,\, \mathcal{A}_h^\star v\rangle\,;$$

2. *preservation of the kernel of operator $\mathcal{A}_h$*

$$\mathcal{A}_h D_{\infty,0} = 0\,,$$

   where the equilibrium D_∞ is given by (2.5) *;*

3. *preservation of the mass conservation properties*

$$\sum_{j\in\mathcal{J}} \Delta x_j\, \mathcal{A}_j^\star u\, \sqrt{\rho}_{\infty,j} = 0\,, \tag{2.12}$$

   and for all $n \geq 0$, the solution $D_0^n = (\mathcal{D}_{0,j}^n)_{j\in\mathcal{J}}$ to (2.1) *with index $k = 0$ verifies*

$$\sum_{j\in\mathcal{J}} \Delta x_j\, \mathcal{D}_{0,j}^n\, \sqrt{\rho}_{\infty,j} = \sum_{j\in\mathcal{J}} \Delta x_j\, \rho_{\infty,j}\,; \tag{2.13}$$

4. *preservation of the sum property*

$$\| (\mathcal{A}_h + \mathcal{A}_h^\star)\, u\|_{L^2} \;\leq\; \frac{1}{\sqrt{T_0}}\, \|\phi_\infty\|_{W^{1,\infty}} \|u\|_{L^2}\,;$$

5. *preservation with the commutator property*

$$\| [\mathcal{A}_h,\, \mathcal{A}_h^\star]\, u\|_{L^2} \;\leq\; C\, \|\phi_\infty\|_{W^{2,\infty}} \|u\|_{L^2}\,,$$

   where constant C depends only on R_h (see (2.11)*), it is explicitly given by*

$$C \;=\; 2 \,+\, R_h\,;$$

6. *conservation of the Poincaré-Wirtinger inequality : under condition* (2.8) *on u there exists a constant $C_d > 0$ depending only on ϕ_∞ and T_0 such that*

$$\|u\|_{L^2} \;\leq\; C_d\, \|\mathcal{A}_h\, u\|_{L^2}\,. \tag{2.14}$$

Remark 2.10. *When the mesh is regular, item* (5) *in Lemma 2.9 may be improved into a consistent estimate compared to its continuous analog* (2.17)*, indeed we easily obtain*

$$\| [\mathcal{A}_h,\, \mathcal{A}_h^\star]\, u\|_{L^2} \;\leq\; \left(\|\phi_\infty\|_{W^{2,\infty}} \,+\, \frac{h}{2}\, \|\phi_\infty\|_{W^{3,\infty}} \right) \|u\|_{L^2}\,,$$

for any $u = (u_j)_{j\in\mathcal{J}}$, following the same method as in the proof.

Proof. To prove item (1), we consider any $(u_j)_{j\in\mathcal{J}}$ and $(v_j)_{j\in\mathcal{J}}$, we have after a discrete integration by part and using periodic boundary conditions

$$\begin{aligned}
\langle \mathcal{A}_h u,\, v\rangle &= \sum_{j\in\mathcal{J}} \Delta x_j\, \mathcal{A}_j u\, v_j \\
&= \sum_{j\in\mathcal{J}} \sqrt{T_0} \left(\frac{u_{j+1} - u_{j-1}}{2}\, v_j \,-\, \Delta x_j\, \frac{E_{\infty,j}}{2\, T_0}\, u_j\, v_j \right) \\
&= \sum_{j\in\mathcal{J}} -\sqrt{T_0} \left(\frac{v_{j+1} - v_{j-1}}{2}\, u_j \,+\, \Delta x_j\, \frac{E_{\infty,j}}{2\, T_0}\, v_j\, u_j \right) \;=\; \langle u\,, \mathcal{A}_h^\star\, v\rangle\,.
\end{aligned}$$

To prove item (2), we look for $D = (D_k)_{k\in\mathbb{N}}$ such that $\mathcal{A}_h\, D_0 \;=\; 0$, that is,

$$0 \;=\; \mathcal{A}_i\, D_0 \;=\; \frac{\sqrt{T_0}}{2\,\Delta x_j} \left(\mathcal{D}_{0,\,j+1} - \mathcal{D}_{0,\,j-1} \,+\, \frac{\phi_{\infty,j+1} - \phi_{\infty,j-1}}{2\, T_0}\, \mathcal{D}_{0,\,j} \right).$$

Hence, from the particular choice of the discrete electric field (2.4), we have that

$$\frac{\mathcal{D}_{0,\,j+1} - \mathcal{D}_{0,\,j-1}}{\mathcal{D}_{0,\,j}} \,-\, \frac{\sqrt{\rho}_{\infty,\,j+1} - \sqrt{\rho}_{\infty,\,j-1}}{\sqrt{\rho}_{\infty,\,j}} \;=\; 0\,,$$

which yields to definition (2.5).

We turn to the mass conservation property (3). According to the definition (2.3) of $\mathcal{A}_h^\star$, it holds

$$\mathcal{A}_j^\star u\,\sqrt{\rho}_{\infty,j}\,\Delta x_j \;=\; -\sqrt{T_0}\left(\sqrt{\rho}_{\infty,j}\,\frac{u_{j+1}-u_{j-1}}{2} \;+\; \frac{\sqrt{\rho}_{\infty,j+1}-\sqrt{\rho}_{\infty,j-1}}{2}\,u_j\right).$$

Therefore, relation (2.12) is obtained summing the latter over $j \in \mathcal{J}$ and performing a discrete integration by part. Relation (2.13) is obtained evaluating equation (2.1) with index $k = 0$ and $j \in \mathcal{J}$, multiplying by $\sqrt{\rho}_{\infty,j}\,\Delta x_j$, then summing over $j \in \mathcal{J}$ and applying relation (2.12) with $u = D_1^{n+1}$.

We prove item (4) taking the L^2 norm in the following relation

$$\sqrt{T_0}\left(\mathcal{A}_j + \mathcal{A}_j^\star\right) u \;=\; -\,\frac{2\,T_0}{\sqrt{\rho}_{\infty,j}}\,\frac{\sqrt{\rho}_{\infty,j+1}-\sqrt{\rho}_{\infty,j-1}}{2\,\Delta x_j}\,u_j\,,$$

which holds for any $u = (u_j)_{j\in\mathcal{J}}$.

We turn to item (5) and compute the commutator for the discrete operator $[\mathcal{A}_h,\,\mathcal{A}_h^\star]$ as

$$\begin{aligned}
[\mathcal{A}_h,\,\mathcal{A}_h^\star]_j\,u &= (\mathcal{A}_h\,\mathcal{A}_h^\star - \mathcal{A}_h^\star\,\mathcal{A}_h)_j u \\
&= -\frac{E_{\infty,j+1}-E_{\infty,j-1}}{4\,\Delta x_j}\,(u_{j+1}+u_{j-1}) \\
&\quad - \frac{E_{\infty,j+1}-2\,E_{\infty,j}+E_{\infty,j-1}}{4\,\Delta x_j}\,(u_{j+1}-u_{j-1})\,,
\end{aligned}$$

and therefore, we deduce item (5) taking the L^2 norm in the latter result.

Finally, we prove the Poincaré inequality (2.14). Consider $u = (u_j)_{j\in\mathcal{J}}$ which meets condition (2.8) and let us denote by $\overline{\rho}_\infty$ the mean of ρ_∞

$$\overline{\rho}_\infty = \sum_{j\in\mathcal{J}} \Delta x_j\,\rho_{\infty,j}\,.$$

First using the zero weighted average assumption (2.8) on u, we remark that the cross term vanishes and

$$\begin{aligned}
\|u\|_{L^2}^2 &= \sum_{j\in\mathcal{J}} \Delta x_j\left(\frac{u_j}{\sqrt{\rho}_{\infty,j}}\right)^2 \rho_{\infty,j}\,,\\
&= \frac{1}{2\,\overline{\rho}_\infty}\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{J}} \Delta x_j\,\Delta x_k\left(\frac{u_k}{\sqrt{\rho}_{\infty,k}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right)^2 \rho_{\infty,j}\,\rho_{\infty,k}\,,\\
&= \frac{1}{\overline{\rho}_\infty}\sum_{k\in\mathcal{J}}\sum_{j<k} \Delta x_j\,\Delta x_k\left(\frac{u_k}{\sqrt{\rho}_{\infty,k}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right)^2 \rho_{\infty,j}\,\rho_{\infty,k}\,.
\end{aligned}$$

For $j<k$, we have

$$\frac{u_k}{\sqrt{\rho_{\infty,k}}} - \frac{u_j}{\sqrt{\rho_{\infty,j}}} = \sum_{l=j}^{k-1} \frac{u_{l+1}}{\sqrt{\rho_{\infty,l+1}}} - \frac{u_l}{\sqrt{\rho_{\infty,l}}}\,,$$

which yields

$$\|u\|_{L^2}^2 \le \overline{\rho}_\infty \left(\sum_{l\in\mathcal{J}} \frac{u_{l+1}}{\sqrt{\rho_{\infty,l+1}}} - \frac{u_l}{\sqrt{\rho_{\infty,l}}}\right)^2. \tag{2.15}$$

On the other hand, we set for any $j\in\mathcal{J}$

$$\sqrt{\overline{\rho}}_{\infty,j} = \frac{\sqrt{\rho}_{\infty,j-1}+\sqrt{\rho}_{\infty,j+1}}{2}, \quad \text{and} \quad \eta_j = \frac{\sqrt{\rho}_{\infty,j+1}-\sqrt{\rho}_{\infty,j-1}}{2\sqrt{\overline{\rho}}_{\infty,j}},$$

and observe that the discrete operator $\mathcal{A}_h u$ may be written as

$$\frac{\Delta x_j}{\sqrt{\overline{\rho}}_{\infty,j}}\mathcal{A}_j u = \frac{\sqrt{T_0}}{2}\left[\left(\frac{u_{j+1}}{\sqrt{\rho}_{\infty,j+1}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right)(1+\eta_j) + \left(\frac{u_j}{\sqrt{\rho}_{\infty,j}} - \frac{u_{j-1}}{\sqrt{\rho}_{\infty,j-1}}\right)(1-\eta_j)\right].$$

Then we have using periodic boundary conditions

$$\begin{aligned}\sqrt{T_0}\sum_{j\in\mathcal{J}}\left(\frac{u_{j+1}}{\sqrt{\rho}_{\infty,j+1}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right) &= \frac{\sqrt{T_0}}{2}\sum_{j\in\mathcal{J}}\left(\frac{u_{j+1}}{\sqrt{\rho}_{\infty,j+1}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right) + \left(\frac{u_j}{\sqrt{\rho}_{\infty,j}} - \frac{u_{j-1}}{\sqrt{\rho}_{\infty,j-1}}\right)\\ &= \sum_{j\in\mathcal{J}}\frac{\Delta x_j}{\sqrt{\overline{\rho}}_{\infty,j}}\mathcal{A}_j u - \sqrt{T_0}\left(\frac{u_{j+1}}{\sqrt{\rho}_{\infty,j+1}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right)\frac{\eta_j-\eta_{j+1}}{2}\end{aligned}$$

Hence using that ϕ_∞ is Lipschitzian, we have

$$|\eta_{j+1}-\eta_j| \le C_\phi\, h,$$

which yields that

$$\sqrt{T_0}\sum_{j\in\mathcal{J}}\left|\frac{u_{j+1}}{\sqrt{\rho}_{\infty,j+1}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right| \le \sum_{j\in\mathcal{J}}\frac{\Delta x_j}{\sqrt{\overline{\rho}}_{\infty,j}}|\,\mathcal{A}_j\, u\,| + C_\phi\, h\sqrt{T_0}\sum_{j\in\mathcal{J}}\left|\frac{u_{j+1}}{\sqrt{\rho}_{\infty,j+1}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right|.$$

On the one hand, we consider the case when h is small enough such that $1-C_\phi h \ge 1/2$, we get that

$$\sum_{j\in\mathcal{J}}\left|\frac{u_{j+1}}{\sqrt{\rho}_{\infty,j+1}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right| \le \frac{2}{\sqrt{T_0}}\sum_{j\in\mathcal{J}}\frac{\Delta x_j}{\sqrt{\overline{\rho}}_{\infty,j}}|\,\mathcal{A}_j\, u\,|$$

On the other hand, when $1-C_\phi\, h \le 1/2$ (the space step h is large), we use the fact that in finite dimension, both semi-norms are equivalent. Thus, there exists a constant $C'_\phi>0$, independent of h, such that

$$\sum_{j\in\mathcal{J}}\left|\frac{u_{j+1}}{\sqrt{\rho}_{\infty,j+1}} - \frac{u_j}{\sqrt{\rho}_{\infty,j}}\right| \le \frac{C'_\phi}{\sqrt{T_0}}\sum_{j\in\mathcal{J}}\frac{\Delta x_j}{\sqrt{\overline{\rho}}_{\infty,j}}|\,\mathcal{A}_j\, u\,|\,.$$

Gathering the latter result with (2.15), it yields

$$\|u\|_{L^2}^2 \;\leq\; \frac{\left(C'_\phi\right)^2\,\overline{\rho}_\infty}{T_0}\left(\sum_{j\in\mathcal{J}} \frac{\Delta x_j}{\sqrt{\overline{\rho}_{\infty,j}}}\,|\,\mathcal{A}_j\,u\,|\right)^2 .$$

Using the Cauchy-Schwarz inequality, we obtain the result

$$\|u\|_{L^2}^2 \;\leq\; C_d^2\,\|\mathcal{A}_h\,u\|_{L^2}^2\,,$$

where C_d^2 is given by

$$C_d^2 \;=\; \frac{\left(C'_\phi\right)^2\,\overline{\rho}_\infty}{T_0}\sum_{j\in\mathcal{J}}\frac{\Delta x_j}{|\sqrt{\overline{\rho}}_{\infty,j}|^2}.$$

□

From the latter results, we may now get estimates on the solution u_h to (2.9) as in Lemma 2.3 in the continuous setting.

Lemma 2.11. *Let us consider the solution u_h to* (2.9) *with source term $g = (g_j)_{j\in\mathcal{J}}$ satisfying the compatibility assumption* (2.8)*. Then, u_h satisfies the following estimate*

$$\|\mathcal{A}_h\,u_h\|_{L^2} \;\leq\; C_d\,\|g\|_{L^2}\,, \tag{2.16}$$

and

$$\|\mathcal{A}_h^2\,u_h\|_{l^2} \;\leq\; \left(1 \,+\, \frac{C_d}{\sqrt{T_0}}\,\|\partial_x\phi_\infty\|_{L^\infty}\right)\|g\|_{L^2}\,. \tag{2.17}$$

Moreover, consider now $(D_k^n)_{k\in\mathbb{N}}$ solution to (2.1) *and $u_h^n = (u_j^n)_{j\in\mathcal{J}}$ the corresponding solution to* (2.9) *with the source term $D_0^n - \sqrt{\rho}_\infty$. Then we define $d_t u_h^{n+1}$ as*

$$d_t u_h^{n+1} \;=\; \frac{u_h^{n+1} - u_h^n}{\Delta t}\,, \tag{2.18}$$

which satisfies

$$\varepsilon\left\|\mathcal{A}_h\,d_t u_h^{n+1}\right\|_{L^2} \;\leq\; \|D_1^{n+1}\|_{L^2}\,. \tag{2.19}$$

Proof. We follow the proof of Lemma 2.3, we multiply (2.9) by $\Delta x_i\,u_i$, sum over $i \in \mathcal{J}$ and apply item (1) of Lemma 2.9, it yields

$$\|\mathcal{A}_h\,u_h\|_{L^2}^2 \;\leq\; \|D - \sqrt{\rho}_\infty\|_{L^2}\,\|u_h\|_{L^2}\,,$$

hence the discrete Wirtinger-Poincaré inequality, obtained in Lemma 2.11, gives,

$$\|\mathcal{A}_h\,u_h\|_{L^2} \;\leq\; C_d\,\|D - \sqrt{\rho}_\infty\|_{L^2}\,.$$

For the second estimate, we observe that

$$(\mathcal{A}_h \,+\, \mathcal{A}_h^\star)_j\,u_h \;=\; \frac{\sqrt{\rho}_{\infty,j+1} - \sqrt{\rho}_{\infty,j-1}}{2\,\Delta x_j\,\sqrt{\rho}_{\infty,j}}\,u_j$$

hence we obtain

$$\begin{aligned}(\mathcal{A}_h^2)_j\, u_h &= -\,(\mathcal{A}_h^\star\,\mathcal{A}_h)_j\, u_h + \frac{\sqrt{\rho}_{\infty,j+1}-\sqrt{\rho}_{\infty,j-1}}{2\,\Delta x_j\,\sqrt{\rho}_{\infty,j}}\,\mathcal{A}_j\,u_h \\ &= -\left(D_{0,j}-\sqrt{\rho}_{\infty,j}\right) + \frac{\sqrt{\rho}_{\infty,j+1}-\sqrt{\rho}_{\infty,j-1}}{2\,\Delta x_j\,\sqrt{\rho}_{\infty,j}}\,\mathcal{A}_j\,u_h\,.\end{aligned}$$

Since ϕ_∞ is Lipschitzian and applying (2.16), we obtain the result

$$\|\mathcal{A}_h^2\,u_h\|_{L^2} \,\leq\, C\,\|D(t)-\sqrt{\rho}_\infty\|_{L^2}\,.$$

For the third estimate we consider now the solution $D^n = (D_k^n)_{k\in\mathbb{N}}$ to (2.1) and u_h^n the solution to (2.9) with source term $D_0^n - \sqrt{\rho}_\infty$. We get for any $j \in \mathcal{J}$,

$$(\mathcal{A}_h^\star\mathcal{A}_h)_j\; d_t u_h^{n+1} \;=\; \frac{\mathcal{D}_{0,j}^{n+1}-\mathcal{D}_{0,j}^n}{\Delta t} \;=\; -\frac{1}{\varepsilon}\,\mathcal{A}_j^\star\,D_1^{n+1}\,.$$

Then we multiply by $\Delta x_j\, d_t u_h^{n+1}$, sum over $j \in \mathcal{J}$ and use (2.12) to get

$$\left\|\mathcal{A}_h\,d_t u_h^{n+1}\right\|_{L^2}^2 \;=\; -\frac{1}{\varepsilon}\left\langle D_1^{n+1},\,\mathcal{A}_h\,d_t u_h^{n+1}\right\rangle \;\leq\; \frac{1}{\varepsilon}\,\|D_1^{n+1}\|_{L^2}\left\|\mathcal{A}_h\,d_t u_h^{n+1}\right\|_{L^2}\,.$$

□

2.3.4 Proof of Theorem 2.7

We split the proof of Theorem 2.7 into two steps corresponding to the L^2 and H^1 convergence result. Thanks to Lemma 2.11, the method followed in Section 2.2 to prove the continuous analog to this result (Theorem 2.1) directly applies here, excepted for some additional numerical remainders for which we give a detailed method in order to get control over.

We define $\mathcal{H}_0^n$ as

$$\mathcal{H}_0^n \;=\; \frac{1}{2}\,\|D^n - D_\infty\|_{L^2}^2 \;+\; \alpha_0\left\langle \frac{\tau(\varepsilon)}{\varepsilon}\,\mathcal{A}_h^\star D_1^n,\, u_h^n\right\rangle, \tag{2.20}$$

where u^n is solution to (2.9) with $D_0^n - \sqrt{\rho}_\infty$ as a source term. First let us point out that $\mathcal{H}_0^n$ shares the same properties as its continuous analog, indeed it holds

Lemma 2.12. *Suppose that condition* (2.7) *on* $\tau(\varepsilon)$ *is satisfied. Then for all* $\alpha_0 \in (0, \overline{\alpha}_0)$, *with* $\overline{\alpha}_0 = 1/(4\,\overline{\tau}_0\,C_d)$ *and* $D^n = (\mathcal{D}_{k,j}^n)_{j\in\mathcal{J},\,k\in\mathbb{N}}$, *one has*

$$\frac{1}{4}\,\|D^n - D_\infty\|_{L^2}^2 \leq\, \mathcal{H}_0^n \,\leq\, \frac{3}{4}\,\|D^n - D_\infty\|_{L^2}^2\,. \tag{2.21}$$

Proof. The proof follows the same lines as the one of Lemma 2.4. □

We are now able to proceed to the proof of the first item (i) of Theorem 2.7. On the one hand, proceeding as the proof of item (i) in Theorem 2.1, it yields from Lemma 2.9

$$\frac{\mathcal{H}_0^{n+1} - \mathcal{H}_0^n}{\Delta t} = \mathcal{I}_1^{n+1} + \alpha_0\, \mathcal{I}_2^{n+1} + \alpha_0\, \mathcal{I}_3^{n+1} - \mathcal{R}_0^{n+1}\,, \tag{2.22}$$

where

$$\mathcal{I}_1^{n+1} = -\frac{1}{\tau(\varepsilon)} \sum_{k\in\mathbb{N}^\star} k \left\| D_k^{n+1} \right\|_{L^2}^2$$

whereas the other terms correspond to the additional term of the modified relative entropy,

$$\begin{cases} \mathcal{I}_2^{n+1} := -\dfrac{\tau(\varepsilon)}{\varepsilon^2} \left\langle \mathcal{A}_h^\star \mathcal{A}_h \left(D_0^{n+1} - \sqrt{\rho_\infty} \right) - \sqrt{2}\, (\mathcal{A}_h^\star)^2\, D_2^{n+1}, u_h^{n+1} \right\rangle - \dfrac{1}{\varepsilon} \left\langle \mathcal{A}_h^\star\, D_1^{n+1}, u_h^{n+1} \right\rangle, \\ \mathcal{I}_3^{n+1} := +\dfrac{\tau(\varepsilon)}{\varepsilon} \left\langle \mathcal{A}_h^\star\, D_1^{n+1},\, d_t u_h^{n+1} \right\rangle, \end{cases}$$

where $d_t u_h^{n+1}$ is given in (2.18) and $\mathcal{R}_0$ is a purely numerical remainder given by

$$\mathcal{R}_0^{n+1} = \frac{1}{2\,\Delta t} \| D^{n+1} - D^n \|_{L^2}^2 + \alpha_0\, \frac{\tau(\varepsilon)}{\varepsilon} \left\langle \mathcal{A}_h^\star \left(D_1^{n+1} - D_1^n \right),\, d_t u_h^{n+1} \right\rangle. \tag{2.23}$$

Both terms $\mathcal{I}_2^{n+1}$ and $\mathcal{I}_3^{n+1}$ can be estimated as in the proof of item (i) in Theorem 2.1, which yields

$$\mathcal{I}_2^{n+1} \le -\frac{\tau(\varepsilon)}{\varepsilon^2} (1 - C\,\eta)\, \| D_0^{n+1} - D_{\infty,0} \|_{L^2}^2 + \frac{C}{2\,\eta} \left(\frac{\tau(\varepsilon)}{\varepsilon^2} \| D_2^{n+1} \|_{L^2}^2 + \frac{1}{\tau(\varepsilon)}\, \| D_1^{n+1} \|_{L^2}^2 \right),$$

for any positive η and for some positive constant C depending only on T_0 and ϕ_∞ and

$$\mathcal{I}_3^{n+1} \le \frac{\tau(\varepsilon)}{\varepsilon^2} \| D_1^{n+1} \|_{L^2}^2\,.$$

From these latter estimates and taking $\eta = 1/(2C)$ and as long as

$$\alpha_0 < \frac{1}{C\,(\overline{\tau}_0^2 + 1)}\,,$$

for C great enough and taking κ_0 such that $3\,\kappa_0/4 = \alpha_0/2$, we get that

$$\frac{\mathcal{H}_0^{n+1} - \mathcal{H}_0^n}{\Delta t} + \frac{\tau(\varepsilon)}{\varepsilon^2}\, \kappa_0\, \mathcal{H}_0^{n+1} \le -\mathcal{R}_0^{n+1}\,.$$

Now we treat the remainder term $\mathcal{R}_0^{n+1}$, observing that

$$\left| \left\langle \mathcal{A}_h^\star \left(D_1^{n+1} - D_1^n \right),\, d_t u_h^{n+1} \right\rangle \right| \le \frac{1}{2\,\Delta t} \left(\| D_1^{n+1} - D_1^n \|_{L^2}^2 + \| \mathcal{A}_h \left(u_h^{n+1} - u_h^n \right) \|_{L^2}^2 \right).$$

Therefore, applying (2.16) in Lemma 2.11 with source term $D_0^{n+1} - D_0^n$, we obtain

$$\left|\left\langle \mathcal{A}_h^\star \left(D_1^{n+1} - D_1^n\right),\, d_t u_h^{n+1} \right\rangle\right| \leq \frac{1+C_d^2}{2\,\Delta t}\, \|D^{n+1} - D^n\|_{L^2}^2 \,.$$

Since $\tau(\varepsilon)$ meets assumption (2.7), the latter estimate ensures that, as long as $\alpha_0 \leq \left(\overline{\tau}_0\,(1+C_d^2)\right)^{-1}$, it holds

$$0 \,\leq\, \mathcal{R}_0^{n+1}\,,$$

which yields

$$\frac{\mathcal{H}_0^{n+1} - \mathcal{H}_0^n}{\Delta t} \,+\, \frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa_0\,\mathcal{H}_0^{n+1} \,\leq\, 0\,.$$

The result follows by applying a discrete Gronwall's lemma and then applying Lemma 2.12 in order to substitute $\mathcal{H}_0^n$ with the L^2 norm of $D^n - D_\infty$ in the latter estimate.

Now we turn to the proof of the second item (ii) of Theorem 2.7. Following Section 2.2.3, we introduce $\mathcal{H}_1^n$ given by

$$\mathcal{H}_1^n \,=\, \frac{1}{2}\,\|\mathcal{B}_h D^n\|_{L^2}^2 \,+\, \alpha_1 \left\langle \frac{\tau(\varepsilon)}{\varepsilon}\,\mathcal{A}_h D_0^n, D_1^n \right\rangle, \tag{2.24}$$

where α_1 has to be determined. Once again, $\mathcal{H}_1^n$ shares the same properties as its continuous analog

Lemma 2.13. *Suppose that condition* (2.7) *on* $\tau(\varepsilon)$ *is satisfied. Then for all* $\alpha_1 \in (0, \overline{\alpha}_1)$, *with* $\overline{\alpha}_1 = 1/(2\,\overline{\tau}_0)$ *and* $D^n = (D_k^n)_{k\in\mathbb{N}}$, *one has*

$$\|\mathcal{B}_h D^n\|_{L^2}^2 - \|D^n - D_\infty\|_{L^2}^2 \leq\, 4\,\mathcal{H}_1^n \,\leq\, 3\,\|\mathcal{B}_h D^n\|_{L^2}^2 + \|D^n - D_\infty\|_{L^2}^2\,.$$

Proof. The result is obtained applying the same method as in the proof of Lemma 2.5. □

We now compute the variation of the modified relative entropy between one time step from t^n to t^{n+1} and split it into three terms

$$\frac{\mathcal{H}_1^{n+1} - \mathcal{H}_1^n}{\Delta t} \,=\, \mathcal{J}_1^{n+1} + \alpha_1\,\mathcal{J}_2^{n+1} \,-\, \mathcal{R}_1^{n+1},$$

where $\mathcal{J}_1^{n+1}$ is given by

$$\mathcal{J}_1^{n+1} \,:=\, -\frac{1}{\varepsilon}\sum_{k\geq 2}\sqrt{k}\,\left\langle [\mathcal{A}_h^\star, \mathcal{A}_h]\, D_{k-1}^{n+1},\, \mathcal{A}_h^\star D_k^{n+1} \right\rangle \,-\, \frac{1}{\tau(\varepsilon)}\sum_{k\in\mathbb{N}^\star} k\,\left\|\mathcal{B}_{h,k} D_k^{n+1}\right\|_{L^2}^2$$

and

$$\begin{aligned}\mathcal{J}_2^{n+1} \;&:=\; \frac{\tau(\varepsilon)}{\varepsilon^2}\left(\left\langle \mathcal{A}_h\mathcal{A}_h^\star D_1^{n+1},\, D_1^{n+1}\right\rangle - \left\|\mathcal{A}_h D_0^{n+1}\right\|_{L^2}^2 + \sqrt{2}\left\langle \mathcal{A}_h D_0^{n+1},\, \mathcal{A}_h^\star D_2^{n+1}\right\rangle\right) \\ &-\;\; \frac{1}{\varepsilon}\left\langle D_1^{n+1},\, \mathcal{A}_h D_0^{n+1}\right\rangle\end{aligned}$$

whereas $\mathcal{R}_1^n$ is given by

$$\mathcal{R}_1^{n+1} \;=\; \frac{1}{\Delta t}\left(\frac{1}{2}\|\mathcal{B}_h\left(D^{n+1}-D^n\right)\|_{L^2}^2 \;+\; \alpha_1\,\frac{\tau(\varepsilon)}{\varepsilon}\left\langle \mathcal{A}_h\left(D_0^{n+1}-D_0^n\right),\, D_1^{n+1}-D_1^n\right\rangle\right). \tag{2.25}$$

On the one hand we estimate the terms $\mathcal{J}_1^{n+1}$ and $\mathcal{J}_2^{n+1}$ following the same method as the one presented to estimate their continuous analogs $\mathcal{J}_1(t)$ and $\mathcal{J}_2(t)$ (see the proof item (ii) in Theorem 2.1). On the other hand, the remainder term $\mathcal{R}_1^{n+1}$ can be treated as $\mathcal{R}_0^{n+1}$ in the proof of (i) of Theorem 2.7. Indeed,

$$\left|\left\langle \mathcal{A}_h\left(D_0^{n+1}-D_0^n\right), D_1^{n+1}-D_1^n\right\rangle\right| \;\leq\; \frac{1}{2}\left(\|D_0^{n+1}-D_0^n\|_{L^2}^2+\|\mathcal{A}^\star\left(D_1^{n+1}-D_1^n\right)\|_{L^2}^2\right).$$

According to the mass conservation property (2.13), $D_0^{n+1}-D_0^n$ meets condition (2.8). Therefore we may apply the discrete Poincaré inequality (2.14) to bound $\|D_0^{n+1}-D_0^n\|_{L^2}^2$ in the latter estimate, this yields

$$\left|\left\langle \mathcal{A}_h\left(D_0^{n+1}-D_0^n\right), D_1^{n+1}-D_1^n\right\rangle\right| \;\leq\; \frac{1+C_d^2}{2}\,\|\mathcal{B}_h\left(D^{n+1}-D^n\right)\|_{L^2}^2\,.$$

As in the case of $\mathcal{R}_0^{n+1}$ in the former section, the latter estimate ensures that, as long as $\alpha_0 \;\leq\; \left(\overline{\tau}_0\,(1+C_d^2)\right)^{-1}$, it holds

$$0 \;\leq\; \mathcal{R}_1^{n+1}\,.$$

Hence, we obtain the result by adapting at the discrete level the proof of item (ii) in Theorem 2.1 to bound $\mathcal{J}_1^{n+1}$ and $\mathcal{J}_2^{n+1}$ and applying a discrete Gronwall lemma.

2.3.5 Proof of Theorem 2.8

As in the continuous setting, we prove that the solution $D^n = (D_k^n)_{k\in\mathbb{N}}$ to (2.1) converges to $D_{\tau_0}^n = (D_{\tau_0,k}^n)_{k\in\mathbb{N}}$ given by (2.6)-(2.7), whose long time behavior is easily obtained relying on the discrete Poincaré inequality (2.14)

$$\|D_{\tau_0}^n - D_\infty\|_{L^2} \;\leq\; \|D_{\tau_0}^0 - D_\infty\|_{L^2}\left(1\,+\,\frac{2\,\tau_0}{C_d^2}\,\Delta t\right)^{-\frac{n}{2}}, \quad \forall\, t\in\mathbb{R}^+\,. \tag{2.26}$$

We estimate $\left\|D_0^n - D_{\tau_0,0}^n\right\|_{H^{-1}}$ by introducing the intermediate quantity $\mathcal{E}$, meant to recover coercivity with respect to the first coefficient D_0^n

$$\mathcal{E}^n \;=\; \frac{1}{2}\,\|\mathcal{A}_h\, v_h^n\|_{L^2}^2\,, \tag{2.27}$$

where v_h^n solves (2.9) with source term

$$g \;=\; D_0^n \;+\; \frac{\tau(\varepsilon)}{\varepsilon}\,\mathcal{A}_h^\star D_1^n \;-\; D_{\tau_0,0}^n\,.$$

The following lemma ensures that the quantity $\mathcal{E}^n$ shares the same properties as its continuous analog. Indeed it holds

Lemma 2.14. *We consider $\mathcal{E}^n$ defined by* (2.27)*. It holds uniformly with respect to ε*

$$\mathcal{E}^n \leq \|D^n - D^n_{\tau_0}\|^2_{H^{-1}} + C_d^2\,\frac{\tau(\varepsilon)^2}{\varepsilon^2}\,\|\mathcal{B}_h D^n\|^2_{L^2}\,, \tag{2.28}$$

and

$$\frac{1}{4}\,\|D^n - D^n_{\tau_0}\|^2_{H^{-1}} - C_d^2\,\frac{\tau(\varepsilon)^2}{2\,\varepsilon^2}\,\|\mathcal{B}_h D^n\|^2_{L^2} \leq \mathcal{E}^n\,. \tag{2.29}$$

Proof. Defining w_h^n and u_{τ_0} as the respective solutions to (2.9) with source term $g = \mathcal{A}_h^\star D_1^n$ and $D_{\tau_0,0} - D_{\infty,0}$, it holds

$$v_h^n = u_h^n - u_{\tau_0}^n + \frac{\tau(\varepsilon)}{\varepsilon}\,w_h^n\,.$$

Applying operator $\mathcal{A}_h$ to the latter relation, taking the L^2 norm, and applying the triangular inequality, it yields

$$\sqrt{2\,\mathcal{E}^n} \leq \left\|\mathcal{A}_h\left(u_h^n - u_{\tau_0}^n\right)\right\|_{L^2} + \frac{\tau(\varepsilon)}{\varepsilon}\,\|\mathcal{A}_h\,w_h^n\|_{L^2}\,,$$

and

$$\left\|\mathcal{A}_h\left(u_h^n - u_{\tau_0}^n\right)\right\|_{L^2} - \frac{\tau(\varepsilon)}{\varepsilon}\,\|\mathcal{A}_h\,w_h^n\|_{L^2} \leq \sqrt{2\,\mathcal{E}^n}\,.$$

We estimate $\|\mathcal{A}_h\,w_h^n\|_{L^2}$ applying (2.16) in Lemma 2.11, this yields

$$\sqrt{2\,\mathcal{E}^n} \leq \|D^n - D^n_{\tau_0}\|_{H^{-1}} + \frac{\tau(\varepsilon)}{\varepsilon}\,C_d\,\|\mathcal{B}_h D^n\|_{L^2}\,,$$

and

$$\|D^n - D^n_{\tau_0}\|_{H^{-1}} - \frac{\tau(\varepsilon)}{\varepsilon}\,C_d\,\|\mathcal{B}_h D^n\|_{L^2} \leq \sqrt{2\,\mathcal{E}^n}\,.$$

We obtain the result taking the square of the latter inequalities and applying Young's inequality. □

We now treat the asymptotic limit $\varepsilon \to 0$ corresponding to the case of (i) in Theorem 2.8 and therefore suppose that $\tau(\varepsilon)$ fulfills the assumptions (2.7), (2.8) and (2.11). As in the continuous setting, we start by deriving the first result in (i) of Theorem 2.8. We already know from the L^2 estimate (2.22) that

$$\begin{aligned}\frac{\left\|D_\perp^{n+1}\right\|^2_{L^2} - \|D_\perp^n\|^2_{L^2}}{2\,\Delta t} \quad &+ \quad \frac{1}{\tau(\varepsilon)}\left\|D_\perp^{n+1}\right\|^2_{L^2} \\ &\leq \; -\left\langle \frac{D_0^{n+1} - D_0^n}{\Delta t},\, D_0^{n+1} - D_0^n \right\rangle - \frac{1}{2\,\Delta t}\sum_{k\in\mathbb{N}^*}\|D_k^{n+1} - D_k^n\|^2_{L^2} \\ &\leq \; -\left\langle \frac{D_0^{n+1} - D_0^n}{\Delta t},\, D_0^{n+1} - D_{\infty,0} \right\rangle.\end{aligned}$$

Therefore, we replace $D_0^{n+1} - D_0^n$ according to equation (2.1), and after applying the duality formula of Lemma 2.9-(1), we obtain

$$\frac{\left\|D_\perp^{n+1}\right\|_{L^2}^2 - \|D_\perp^n\|_{L^2}^2}{\Delta t} + \frac{1}{\tau(\varepsilon)} \left\|D_\perp^{n+1}\right\|_{L^2}^2 \leq -\frac{1}{\varepsilon} \left\langle D_1^{n+1}, \mathcal{A}_h D_0^{n+1} \right\rangle,$$

Hence, after multiplying by Δt and applying the Young inequality to bound the right hand side of the latter inequality, it yields

$$\left(1 + \frac{\Delta t}{\tau(\varepsilon)}\right) \left\|D_\perp^{n+1}\right\|_{L^2}^2 \leq \|D_\perp^n\|_{L^2}^2 + \Delta t \, \frac{\tau(\varepsilon)}{\varepsilon^2} \|\mathcal{B}_h D^{n+1}\|_{L^2}^2 .$$

To achieve the proof, it remains to bound $\|\mathcal{B}_h D^{n+1}\|_{L^2}^2$ by applying Theorem 2.7-(ii) and again following the line of the proof of Theorem 2.2, we deduce

$$\begin{aligned} \|D_\perp^n\|_{L^2}^2 \leq & \left\|D_\perp^0\right\|_{L^2}^2 \left(1 + \frac{\Delta t}{\tau(\varepsilon)}\right)^{-n} \\ & + 6\left(C(\overline{\tau}_0^2 + 1)\,\|D^0 - D_\infty\|_{L^2}^2 + \|\mathcal{B}_h D^0\|_{L^2}^2\right) \frac{\tau(\varepsilon)^2}{\varepsilon^2} \left(1 + \frac{\tau(\varepsilon)}{\varepsilon^2}\kappa\Delta t\right)^{-n} . \end{aligned}$$

Therefore we obtain the result taking the square root in the latter estimate and substituting $\tau(\varepsilon)$ with $\tau_0\,\varepsilon^2$ according to assumption (2.11).

To prove the second result of (i) in Theorem 2.8 we evaluate $\mathcal{E}^n$ as in the proof of Theorem 2.2 observing that

$$\|\mathcal{A}_h\, v_h^n\|_{L^2}^2 = \left\langle D_0^n + \frac{\tau(\varepsilon)}{\varepsilon} \mathcal{A}_h^\star D_1^n - D_{\tau_0,0}^n,\, v_h^n \right\rangle$$

hence, relying on equations (2.1) and (2.7) we deduce

$$\frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} = -\frac{\tau(\varepsilon)}{\varepsilon^2} \|D_0^n + \frac{\tau(\varepsilon)}{\varepsilon} \mathcal{A}_h^\star D_1^n - D_{\tau_0}^n\|_{L^2}^2 + \mathcal{E}_1^{n+1} + \mathcal{E}_2^{n+1} + \mathcal{E}_3^{n+1} - \mathcal{R}_3^{n+1},$$

where $\mathcal{E}_1^{n+1}$, $\mathcal{E}_2^{n+1}$ and $\mathcal{E}_3^{n+1}$ are the numerical equivalents of the terms $\mathcal{E}_1(t)$, $\mathcal{E}_2(t)$ and $\mathcal{E}_3(t)$ in the proof of Theorem 2.2

$$\begin{cases} \mathcal{E}_1^{n+1} = \left(\tau_0 - \frac{\tau(\varepsilon)}{\varepsilon^2}\right) \left\langle \mathcal{A}_h^\star \mathcal{A}_h\, D_{\tau_0,0}^{n+1} \,,\, v_h^{n+1} \right\rangle, \\ \mathcal{E}_2^{n+1} = \frac{\tau(\varepsilon)^2}{\varepsilon^3} \left\langle \mathcal{A}_h^\star \mathcal{A}_h\, D_1^{n+1} \,,\, v_h^{n+1} \right\rangle, \\ \mathcal{E}_3^{n+1} = \sqrt{2}\,\frac{\tau(\varepsilon)}{\varepsilon^2} \left\langle (\mathcal{A}_h^\star)^2\, D_2^{n+1} \,,\, v_h^{n+1} \right\rangle, \end{cases}$$

and $\mathcal{R}_3^{n+1}$ is a numerical dissipation term

$$\mathcal{R}_3^{n+1} = \frac{1}{2\Delta t} \left\|\mathcal{A}_h \left(v_h^{n+1} - v_h^n\right)\right\|_{L^2}^2 .$$

Since $\mathcal{R}_3^{n+1}$ is positive, we apply the same method as the one presented in the proof of Theorem 2.2 and therefore we obtain the following estimate for $\mathcal{E}^n$

$$\left(1+\frac{\tau(\varepsilon)\Delta t}{C_d^2\,\varepsilon^2}\right)\mathcal{E}^{n+1} \leq \mathcal{E}^n + C\,\Delta t\,\frac{\tau(\varepsilon)}{\varepsilon^2}\left(1+\overline{\tau}_0^2\right)\left\|D_\perp^{n+1}\right\|_{L^2}^2$$
$$+ C\,\Delta t\,\frac{\varepsilon^2}{\tau(\varepsilon)}\left|\tau_0-\frac{\tau(\varepsilon)}{\varepsilon^2}\right|^2 \|D_{\tau_0}^{n+1}-D_\infty\|_{L^2}^2\,,$$

for some constant C depending only on ϕ_∞ and T_0. In the latter inequality, we bound $\|D_{\tau_0}^{n+1}-D_\infty\|_{L^2}^2$ according to (2.26) and the norm of $D_\perp$ according to the first estimate of (i) in Theorem 2.8. Then we multiply the inequality by $\left(1+\frac{\tau(\varepsilon)\Delta t}{C_d^2\,\varepsilon^2}\right)^n$ and sum for k ranging from 0 to $n-1$, it yields

$$\mathcal{E}^n \leq \left(\mathcal{E}^0 + C\,\frac{\tau(\varepsilon)^2}{\varepsilon^2}\,(\overline{\tau}_0^6+1)\|D^0-D_\infty\|_{H^1}^2\right)\left(1+\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,\Delta t\right)^{-n}$$
$$+ C\left|\frac{\tau_0\,\varepsilon^2}{\tau(\varepsilon)}-1\right|^2\|D_{\tau_0}^0-D_\infty\|_{L^2}^2\left(\frac{2\,\tau_0\,\varepsilon^2}{\tau(\varepsilon)}-1\right)^{-1}\left(1+\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,\Delta t\right)^{-n}.$$

To conclude, we substitute $\mathcal{E}^n$ (resp. $\mathcal{E}^0$) in the latter estimate according to (2.29) (resp. (2.28)) in Lemma 2.6 and then apply assumption (2.11) on $\tau(\varepsilon)$, which ensures

$$\left(\frac{2\,\tau_0\,\varepsilon^2}{\tau(\varepsilon)}-1\right)^{-1} \leq 3,$$

this yields

$$\|D_0^n-D_{\tau_0,0}^n\|_{H^{-1}}^2 \leq C\left(\|D_0^0-D_{\tau_0,0}^0\|_{H^{-1}}^2 + \frac{\tau(\varepsilon)^2}{\varepsilon^2}\,(\overline{\tau}_0^6+1)\|D^0-D_\infty\|_{H^1}^2\right)\left(1+\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,\Delta t\right)^{-n}$$
$$+ C\left|\frac{\tau_0\,\varepsilon^2}{\tau(\varepsilon)}-1\right|^2\|D_{\tau_0}^0-D_\infty\|_{L^2}^2\left(1+\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\,\Delta t\right)^{-n}.$$

We obtain the result taking the square root in the latter estimate and substituting $\tau(\varepsilon)$ with $\tau_0\,\varepsilon^2$according to assumption (2.11).
Finally the proof of the second item follows the same lines replacing $D_{\tau_0}^n$ by D_∞ in the discrete functional $\mathcal{E}^n$.

2.4 Numerical simulations

We performed several numerical simulations which confirm the accuracy of the scheme (2.1). We do not detail this process here and rather focus on the physical interpretation and the quantitative results obtained in our experiments. We refer to [17] for a precise discussion on that matter.

In this section, we want to illustrate the quantitative estimates of the solution obtained using the Hermite Spectral method in velocity and finite volume scheme in space for the one-dimensional Vlasov-Fokker-Planck equation. We choose $\tau(\varepsilon) = \tau_0\,\varepsilon^2$ with $\tau_0 = 5$ and consider the Vlasov-Fokker-Planck equation (2.1) with $E_\infty = -\partial_{\boldsymbol{x}}\phi_\infty$ and

$$\phi_\infty(x) = 0.1\,\cos\left(\frac{2\pi\,x}{L}\right) + 0.9\,\cos\left(\frac{4\pi\,x}{L}\right),$$

The stationary state is given by the Maxwell-Boltzmann distribution

$$f_\infty(x,v) = \frac{c_0}{\sqrt{2\pi}}\,\exp\left(-\left(\phi_\infty + \frac{|v|^2}{2}\right)\right),$$

where c_0 is given by mass conservation

$$\int_{\mathbb{T}\times\mathbb{R}} f_\infty \mathrm{d}v\mathrm{d}x \;=\; \int_{\mathbb{T}\times\mathbb{R}} f_0 \mathrm{d}v\mathrm{d}x,$$

where f_0 is the initial datum.

In our simulation, we take a time step $\Delta t = 10^{-3}$, a number of modes $N_H = 200$ and $N_x = 64$. It is worth to mention that all the numerical simulations presented in this section are not affected by the numerical parameters, which allows us to focus our discussion on the quantitative results on the diffusive limit $\varepsilon \to 0$ and large time behavior.

2.4.1 Test 1 : centered Maxwellian

For the first test, we choose the following initial condition

$$f_0(x,v) \;=\; \frac{1}{\sqrt{2\pi}}\left(1 + \delta\cos\left(\frac{2\pi\,x}{L}\right)\right)\exp\left(-\frac{|v|^2}{2}\right),$$

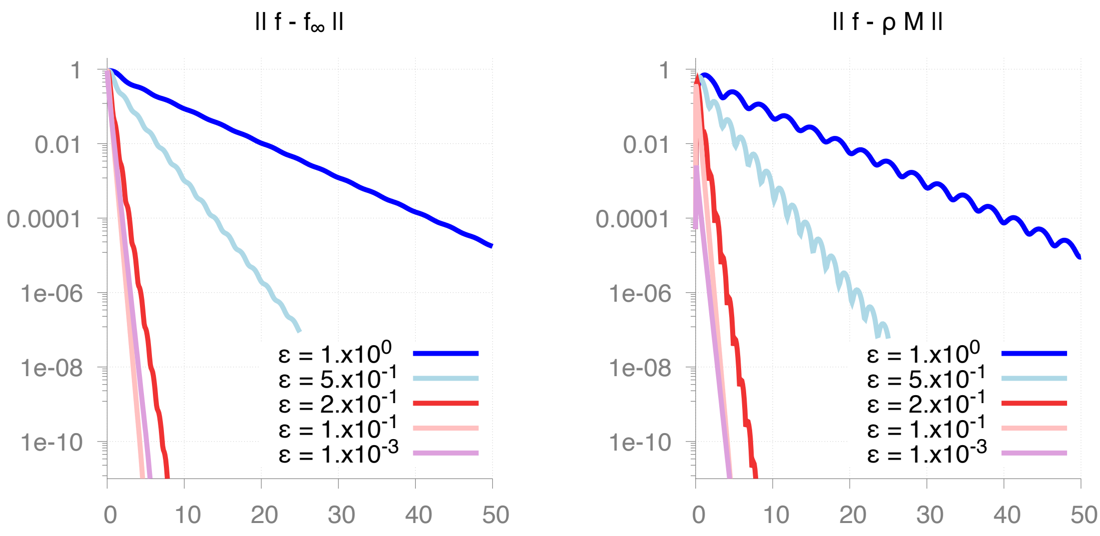
with $\delta = 0.5$ and $L\; =\; 10$.
On the one hand, we present in Figure 2.1 the time evolution of $\|f - f_\infty\|_{L^2(f_\infty^{-1})}$ and the relative entropy on f,

$$\|f - \rho\,\mathcal{M}\|_{L^2(f_\infty^{-1})} \;=\; \|D_\perp(t)\|_{L^2}.$$

The most striking feature in this test consists in the oscillatory behavior of the relative entropy which unfolds in the relaxation of f towards its equilibrium. These oscillations may be observed in Figure 2.1-(b) and occur for various values of ε ranging from 1 represented by blue curves to 2.10^{-1} represented by red curves.

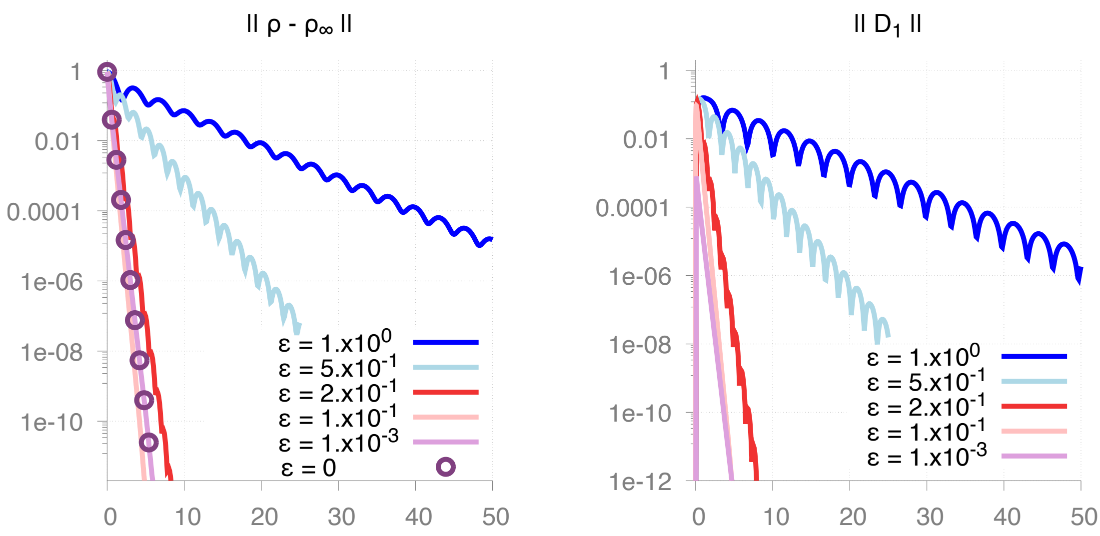
We also present in Figure 2.2 the relaxation to equilibrium of macroscopic quantities

$$\|D_0 - D_{\infty,0}\|_{L^2} = \|\rho - \rho_\infty\|_{L^2(f_\infty^{-1})}$$



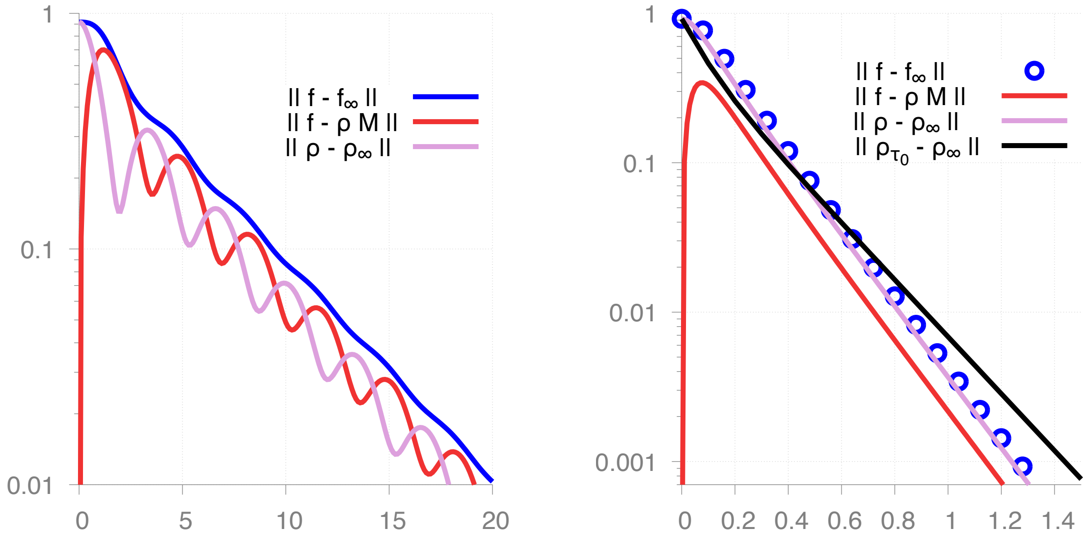
(a) (b)

FIGURE 2.1 – **Test 1 : centered Maxwellian.** time evolution in log scale of (a) $\|f - f_\infty\|_{L^2(f_\infty^{-1})}$, (b) $\|f - \rho\,\mathcal{M}\|_{L^2(f_\infty^{-1})}$.



(a) (b)

FIGURE 2.2 – **Test 1 : centered Maxwellian.** time evolution in log scale of (a) $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ and (b) $\|D_1\|_{L^2}$.



$(a)\ \varepsilon = 1$ $(b)\ \varepsilon = 10^{-1}$

FIGURE 2.3 – **Test 1 : centered Maxwellian.** time evolution in log scale of $\|f - f_\infty\|_{L^2(f_\infty^{-1})}$ (blue), $\|f - \rho\,\mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink) and $\|\rho_{\tau_0} - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (black) for (a) $\varepsilon = 1$ and (b) $\varepsilon = 10^{-1}$.

and the norm of the first moment D_1. Time oscillations, observed on the distribution function, seem to affect macroscopic quantities associated to the solution as moments D_0 and D_1.

On the other hand, we provide In Figure 2.3, a detailed description in the case $\varepsilon = 1$, where we see that the oscillations of the spatial density and the ones of the higher modes in velocity are asynchronous, this may be interpreted as a transfer of information between these two quantities. This phenomenon has already been investigated for non-linear kinetic models (see [111]) but we show through these experiments that even the simple model at play here captures this phenomena.

These oscillations stay visible for surprisingly small values of ε, up to 10^{-1}. It showcases the robustness of our scheme, which is still able to capture them at low computational cost. To be noted that our numerical experiments indicate that a non zero external force field seems to be mandatory to observe this oscillatory behavior. We also emphasize that these oscillations seem to be quite sensitive to the choice of the initial data and the external field (see the second numerical test with a different initial data, where such oscillations disappear for large time).

This leads us to the second feature of this test, which is the asymptotic preserving property of the scheme for various values of ε. The method is accurate on large time intervals in the situation where $\varepsilon = 1$ (see Figure 2.3-(a)), which corresponds to the long time behavior of the model but it is also accurate when $\varepsilon \ll 1$. Indeed, as it is shown in Figure 2.2-(a), the purple error curve of the density ρ corresponds exactly to the circled error curve of the macroscopic model ρ_{τ_0} when $\varepsilon = 10^{-3}$ and even smaller (not shown since the curves coincide).

Finally we focus on the intermediate value $\varepsilon = 10^{-1}$, for which we observe in Figures 2.1-(a), 2.2-(a) and 2.3-(b), a somehow surprising phenomenon : the kinetic model relaxes faster towards equilibrium than the macroscopic one. This appears to be a consequence of our choice of initial data which is already at local equilibrium at time $t = 0$. This aspect of the experiment justifies our efforts to cover a wide range of values for the scaling parameter ε : it enables to capture intermediate regimes which may display peculiar phenomena. As we will see in the next section, the reverse situation is possible as well, when the initial condition is far from equilibrium.

We conclude this section by drawing the readers attention towards Figure 2.4, which features the graph of the solution f at different times, in the case $\varepsilon = 1$ and on which we witness its intricate relaxation towards equilibrium.

2.4.2 Test 2 : shifted Maxwellian

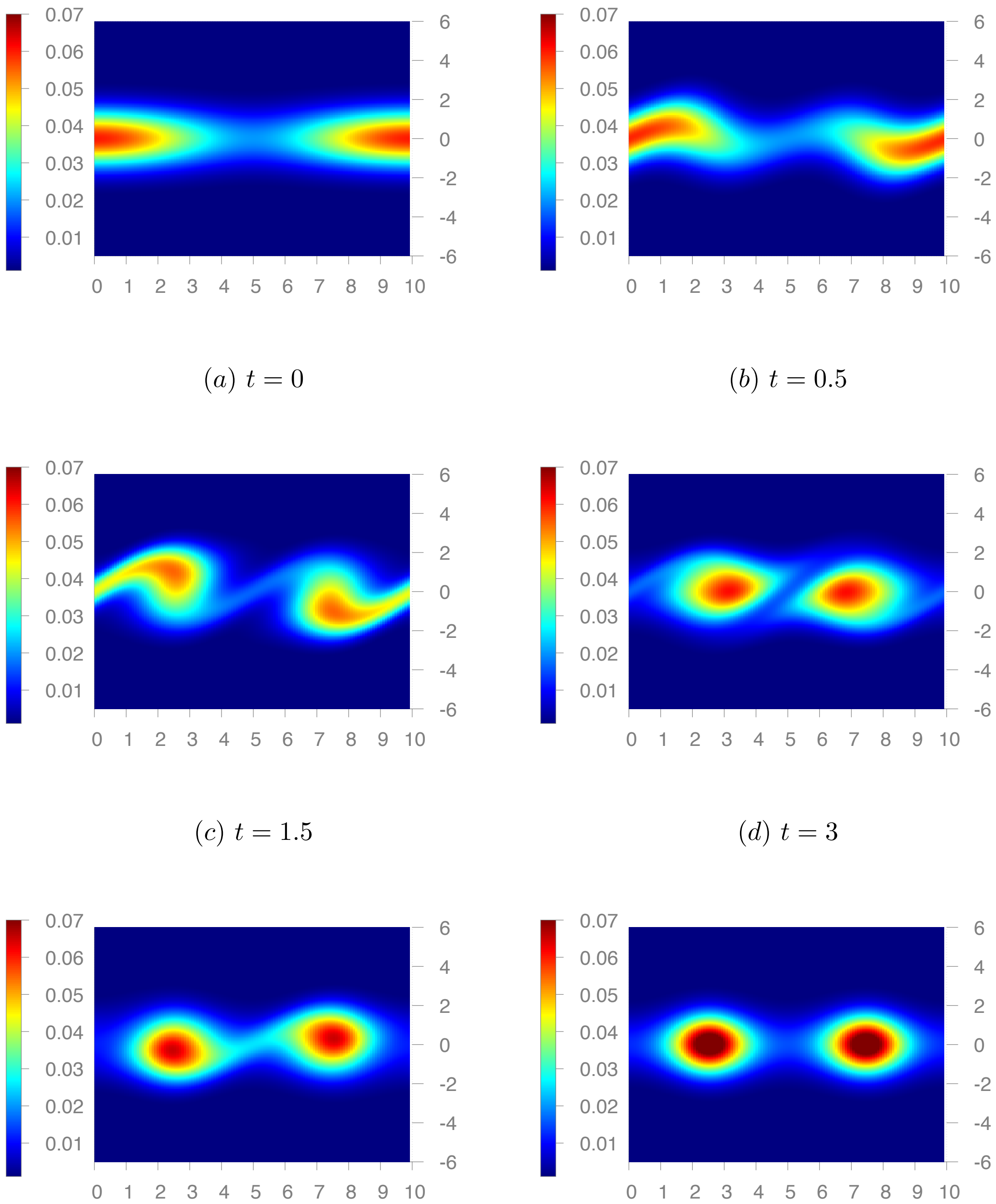
We now choose the same parameter as before excepted that the initial condition is a shifted Maxwellian

$$f_0(x,v) \;=\; \frac{1}{\sqrt{2\pi}}\left(1+\delta\cos\left(\frac{2\pi\, x}{L}\right)\right)\exp\left(-\frac{|v-u_0|^2}{2}\right),$$

with $u_0 = 1$, which is far from equilibrium.

First, we focus on the case $\varepsilon = 1$ displayed in Figure 2.5, where we observe that unlike in the previous test, the oscillatory relaxation stops after a short time and is replaced by a slower but straight relaxation towards equilibrium. Another interesting comment on Figure 2.5 is that all the curves associated to value of ε below 5.10^{-2} (red, beige, pink and purple) are parallel. These two features might be explained by a fine spectral analysis of the model at play.
We now zoom in to focus on smaller time intervals and propose a detailed description of these dynamics in Figure 2.6, where we distinguish three phases constituting a great illustration for the result presented in item (i) of Theorem 2.8 :

1. the first phase is the initial time layer, it occurs on negligible time intervals compared to the time scale chosen in Figure 2.6 but it is still visible if we focus on the red curves, representing the norm of $D_\perp$, in plots (a) to (d). As predicted by the first result in (i) of Theorem 2.8, higher Hermite modes gathered in the quantity $D_\perp$ undergo a steep exponential descent with theoretical rate of order $(\varepsilon^2\,\tau_0)^{-1}$, until they reach a critical level of order ε ;
2. the second phase corresponds to the diffusive regime where f is close to $\rho_{\tau_0}\,\mathcal{M}$. Indeed we see that for times ranging from ~ 0 up to $t = 1$ in the case $\varepsilon = 10^{-2}$



(a) $t = 0$ (b) $t = 0.5$

(c) $t = 1.5$ (d) $t = 3$

(e) $t = 5$ (f) $t = 20$

FIGURE 2.4 – **Test 1 : centered Maxwellian.** snapshots of the distribution function for $\varepsilon = 1$ at time $t = 0$, 0.5, 1.5, 3, 5 and 20.

and increasing up to $t = 3$ in the case $\varepsilon = 10^{-5}$, the red curve, which represents the norm of $D_\perp$, is parallel to the pink line corresponding to the norm of $\rho - \rho_{\tau_0}$ which itself coincides with the black curve representing the norm of $\rho_{\tau_0} - \rho_\infty$. It indicates that, for a finite amount of time which increases as ε goes to zero, the kinetic model behaves like the macroscopic one;

3. the last phase is the long time behavior, it starts as the error between ρ_{τ_0} and ρ is of the same order as the error between ρ and ρ_∞. In Figure 2.6 (a)-(d), it corresponds to the intersection between circled blue and black lines. As predicted by the second result in (i) of Theorem 2.8, this circled curve, representing the error $\|\rho - \rho_{\tau_0}\|$, starts with an ordinate of order ε at time $t = 0$, then it decays with a rate proportional to τ_0 but smaller than the relaxation rate of the macroscopic model. This constitutes a striking illustration of "hypocoercivity" phenomenon induced by the transport term proper to kinetic equations. During this final phase, the solution f to (2.2) slowly relaxes towards equilibrium. A surprising and unexpected fact is that the transition from diffusive regime to long time behavior occurs in a synchronized fashion for the spatial density and higher modes in velocity. Indeed, as it can be observed in plots (a) to (c) of Figure 2.6, the inflections points of the red and the pink curves are almost aligned.

2.5 Conclusion and perspectives

In the present article, we design a numerical method capable to capture a rich variety of regimes for a Vlasov-Fokker-Planck equation with external force field. We prove quantitative estimates for all the regimes of interest, and do this uniformly with respect to all parameter at play. We illustrate the robustness of our scheme by proposing several numerical tests in which we capture a wide variety of situations (exponential decay with oscillations, transition phase between diffusive regime an long time behavior, initial time layer, etc ...). Furthermore, we built the method such that it should be easily adaptable in any dimension, at least for cartesian mesh.

Two questions arise naturally from this work. The first one is to build on the groundwork laid in this article in order to design a scheme which takes into account non-linear coupling with Poisson for the electric force field. This challenging perspective would be a great improvement since even for the continuous model, there exists to our knowledge very few results which treat the longtime behavior and the diffusive regime with the accuracy proposed in this article. Up to our knowledge, all the works on this subject have restrictions on the dimension of the phase-space and therefore, it would naturally be interesting to propose a method which applies in the physical case $d = 3$.
Another interesting question arose from our numerical tests, in which we witnessed oscilla-

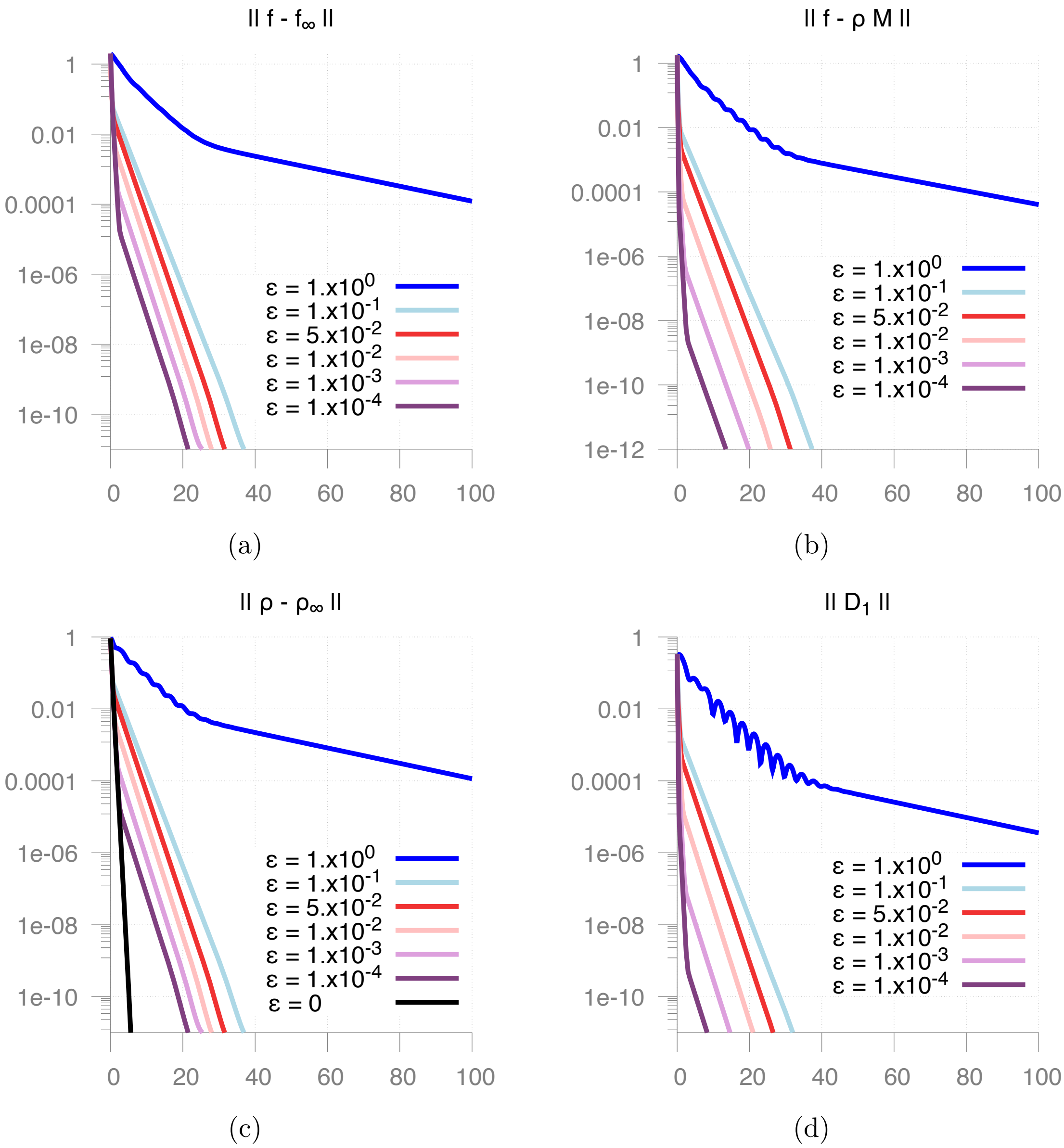
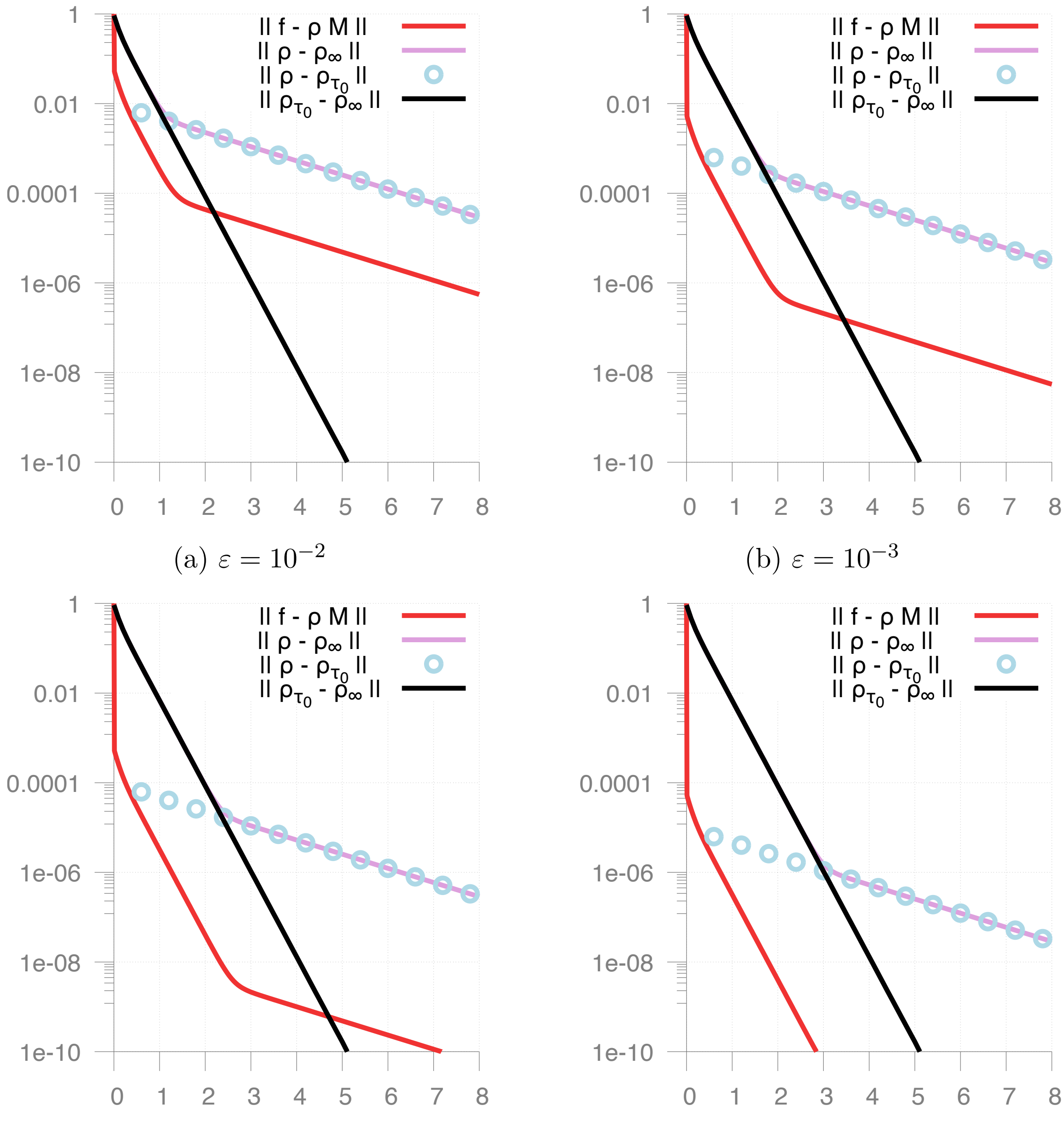


FIGURE 2.5 – **Test 2 : shifted Maxwellian.** time evolution in log scale of (a) $\|f - f_\infty\|_{L^2(f_\infty^{-1})}$, (b) $\|f - \rho\,\mathcal{M}\|_{L^2(f_\infty^{-1})}$, (c) $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ and (d) $\|D_1\|_{L^2}$.



(a) $\varepsilon = 10^{-2}$

(b) $\varepsilon = 10^{-3}$

(c) $\varepsilon = 10^{-4}$

(d) $\varepsilon = 10^{-5}$

FIGURE 2.6 – **Test 2 : shifted Maxwellian.** time evolution in log scale of $\|f - \rho\,\mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink), $\|\rho - \rho_{\tau 0}\|_{L^2(\rho_\infty^{-1})}$ (blue points) and $\|\rho_{\tau 0} - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (black) for $\varepsilon = 10^{-2}$, 10^{-3}, 10^{-4} and 10^{-5}.

ting behaviors in the solution's relaxation towards equilibrium as well as transition phase between diffusive regime and longtime behavior. It would be of great interest to carry out a fine spectral analysis of the model both at the continuous and the discrete level in order to provide a quantitative description of these phenomena : we may hope for precise and enlightening results due to the simplicity of our model.

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Chapitre 3

An asymptotic-preserving scheme for the Vlasov-Poisson-Fokker-Planck model

We propose a fully discrete finite volume scheme for the Vlasov-Poisson-Fokker-Planck system written as an hyperbolic system thanks to a spectral decomposition in the basis of Hermite function with respect to the velocity variable and a structure preserving finite volume scheme for the space variable. On the one hand, we show that this scheme naturally preserves the stationary solution and the weighted L^2 relative entropy for its linearized version. On the other hand, we adapt the arguments developed in [25] based on hypocoercivity methods to get quantitative estimates on the convergence to equilibrium of the discrete solution for the linearized Vlasov-Poisson-Fokker-Planck system. Finally, we perform extensive numerical simulations for the nonlinear system to illustrate the efficiency of this approach for a large variety of collisional regimes (plasma echos for weakly collisional regimes and trend to equilibrium for collisional plasmas).

Contents

3.1 Introduction

We consider the one dimensional Vlasov-Poisson-Fokker-Planck system describing the dynamics of an electron distribution function $f \equiv f(t,x,v)$ and an electrical potential $\phi \equiv \phi(t,x)$ in a plasma where $v \in \mathbb{R}$ stands for the velocity, $x \in \mathbb{T}$ is the position and $t \geq 0$ represents the time variable. This couple (f,ϕ) is solution to the following system

$$\begin{cases} \varepsilon\,\partial_t f \,+\, v\,\partial_x f \,-\, \partial_x\phi\,\partial_v f \,=\, \dfrac{\varepsilon}{\tau(\varepsilon)}\partial_v\left(v\,f \,+\, T_0\,\partial_v f\right)\,, \\ -\partial_x^2\phi \,=\, \rho-\rho_i\,, \quad \rho(t,x) \,=\, \displaystyle\int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v\,, \end{cases} \tag{3.1}$$

completed with the following condition which ensures uniqueness of the electrical potential ϕ

$$\int_{\mathbb{T}} \phi(t,x)\,\mathrm{d}x \,=\, 0\,. \tag{3.2}$$

The scaling parameter ε corresponds to the square root of the ratio between the mass of electrons and ions. Since we focus on the situation where the electron to ion mass ratio is very small, it allows to describe ions as a steady thermal bath. More precisely, ions are supposed to be fixed with a prescribed temperature $T_0 > 0$ and a density $\rho_i > 0$, which is an integrable function over $\mathbb{T}$. Furthermore, the following quasi-neutrality assumption is satisfied for all time $t \geq 0$

$$\int_{\mathbb{T}} \rho(t,x)\,\mathrm{d}x \,=\, \int_{\mathbb{T}} \rho_i\,\mathrm{d}x$$

as soon as this condition is initially verified. The other scaling parameter $\tau(\varepsilon) > 0$ stands for the ratio between the time which separates two collisions of an electron with the ionic background and the timescale of observation. In this work, we suppose

$$\tau(\varepsilon) = \tau_0\, \varepsilon.$$

Therefore, as ε goes to zero, we expect that the couple (f, ϕ), solution to (3.1)-(3.2), converges to (f_∞, ϕ_∞) given by

$$\begin{cases} v\,\partial_x f_\infty \,-\, \partial_x\phi_\infty\,\partial_v f_\infty \,=\, \dfrac{1}{\tau_0}\partial_v\left(v\, f_\infty \,+\, T_0\,\partial_v f_\infty\right)\,, \\ -\partial_x^2\phi_\infty \,=\, \rho_\infty - \rho_i\,, \quad \rho_\infty(t,x) \,=\, \displaystyle\int_{\mathbb{R}} f_\infty(t,x,v)\,\mathrm{d}v\,. \end{cases}$$

Actually, the equilibrium state f_∞ is uniquely determined as follows

$$f_\infty(x,v) \,=\, \rho_\infty(x)\,\mathcal{M}(v)\,,$$

where $\mathcal{M}$ denotes the Maxwellian with temperature T_0

$$\mathcal{M}(v) \,=\, \frac{1}{\sqrt{2\pi\,T_0}}\,\exp\left(-\frac{|v|^2}{2\,T_0}\right)\,. \tag{3.3}$$

The latter equality means that the electrons' temperature relaxes to the background temperature and ρ_∞ is given by the Maxwell-Boltzmann equilibrium, that is,

$$\begin{cases} \rho_\infty \,=\, \exp\left(-\dfrac{\phi_\infty}{T_0}\right)\,, \\ -\partial_x^2\phi_\infty \,=\, \rho_\infty - \rho_i\,, \end{cases} \tag{3.4}$$

completed with the following condition

$$\int_{\mathbb{T}} \exp\left(-\frac{\phi_\infty}{T_0}\right)\mathrm{d}x \,=\, \int_{\mathbb{T}} \rho_i\,\mathrm{d}x\,.$$

Let us point out that there is an attractive version of the Vlasov-Poisson-Fokker-Planck system (3.1)-(3.2) which is also widely used in stellar physics. For that case the repulsive electrostatic force is replaced by the attractive gravitational force, responsible for a change of sign in the Poisson equation. In this paper, we only consider the repulsive case.

In order to design a well-balanced approximation for (3.1)-(3.2), we consider an equivalent reformulation where ρ_i is replaced with the quantities at equilibrium and the electrical potential ϕ is replaced by $\psi = \phi - \phi_\infty$, hence we have

$$\begin{cases} \varepsilon\,\partial_t f \,+\, v\,\partial_x f \,-\, \partial_x\phi_\infty\,\partial_v f \,-\, \partial_x\psi\,\partial_v f \,=\, \dfrac{1}{\tau_0}\partial_v\left(v\, f \,+\, T_0\,\partial_v f\right)\,, \\ -\partial_x^2\psi \,=\, \rho - \rho_\infty\,, \quad \rho(t,x) \,=\, \displaystyle\int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v\,, \end{cases} \tag{3.5}$$

coupled with the condition on ψ,

$$\int_{\mathbb{T}} \psi(t,x)\,\mathrm{d}x \;=\; 0\,. \tag{3.6}$$

The equilibrium to (3.5)-(3.6) is now characterized by (f_∞, ψ_∞) where $\psi_\infty \equiv 0$.

The key-estimate to prove the trend to equilibrium of solutions to (3.5)-(3.6) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f, f_\infty) \;=\; -\frac{1}{\varepsilon\,\tau_0}\mathcal{I}(f, f_\infty),$$

where $\mathcal{H}(f, f_\infty)$ denotes the free energy

$$\mathcal{H}(f, f_\infty) \;:=\; \int_{\mathbb{R}^2} \ln\left(\frac{f}{f_\infty}\right) f\,\mathrm{d}x\,\mathrm{d}v \;+\; \frac{1}{2\,T_0}\,\|\,\partial_x\psi\,\|_{L^2}^2\;,$$

and $\mathcal{I}(f, f_\infty)$ is the entropy dissipation

$$\mathcal{I}(f, f_\infty) \;:=\; 4\,T_0 \int_{\mathbb{R}^2} \left|\partial_v\sqrt{\frac{f}{f_\infty}}\right|^2 f_\infty\,\mathrm{d}x\,\mathrm{d}v\,.$$

For the linear Vlasov-Fokker-Planck equation ($\psi \equiv 0$), L. Desvillettes and C. Villani have proposed a general method, based on the latter relative entropy estimate and logarithmic Sobolev inequalities, to overcome the problem due to the degeneracy in the position variable [91]. This approach has been widely explored in the last decade [211, 97, 135]. However, when the Vlasov-Fokker-Planck equation is coupled with the Poisson equation for the electrical potential, a different functional framework, based on weighted L^2 spaces, is applied motivated by the following estimate when f is near equilibrium [136, 1, 138, 155]. Indeed, plugging the following formal expansion into the free energy

$$f\,\ln\left(\frac{f}{f_\infty}\right) \underset{f\to f_\infty}{\sim} f - f_\infty \;+\; \frac{|f-f_\infty|^2}{2\,f_\infty}$$

and using that mass is conserved for solutions to (3.5)-(3.6), we define the linearized free energy as

$$\mathcal{E}(t) \;=\; \|\,f(t) - f_\infty\,\|_{L^2(f_\infty^{-1})}^2 \;+\; \frac{1}{T_0}\,\|\,\partial_x\psi(t)\,\|_{L^2}^2\;. \tag{3.7}$$

Unfortunately, this functional is not dissipated for the solution to the nonlinear system (3.5)-(3.6), but only for its linearized version given by

$$\begin{cases} \varepsilon\,\partial_t f \;+\; v\,\partial_x f \;-\; \partial_x\phi_\infty\,\partial_v f \;-\; \partial_x\psi\,\partial_v f_\infty \;=\; \dfrac{1}{\tau_0}\partial_v\left(v\,f \;+\; T_0\,\partial_v f\right)\,, \\[2ex] -\partial_x^2\psi \;=\; \rho - \rho_\infty\,, \quad \rho(t,x) \;=\; \displaystyle\int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v\,, \end{cases} \tag{3.8}$$

coupled with the condition on ψ,

$$\int_{\mathbb{T}} \psi(t,x)\,\mathrm{d}x \;=\; 0\,. \tag{3.9}$$

This yields for the solution (f, ψ) to (3.8)-(3.9) (see Proposition 3.1 for a complete proof)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{E}(t) \;=\; -\,\frac{T_0}{\varepsilon\,\tau_0}\int_{\mathbb{R}^2}\left|\partial_v\left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty\,\mathrm{d}x\,\mathrm{d}v\,. \tag{3.10}$$

The purpose of this paper is to design a numerical scheme for the nonlinear Vlasov-Poisson-Fokker-Planck system (3.5)-(3.6) for which such estimate occurs on its linearized version (3.8)-(3.9). To this aim, we propose a simple time splitting scheme, where the first stage consists in applying a Hermite decomposition in velocity and a structure preserving finite volume scheme for the space discretization for f coupled with a suitable reformulated Poisson equation for ψ to the linearized system (3.8)-(3.9) whereas the second stage solves the remaining quadratic part for f

$$\varepsilon\,\partial_t f \;-\; \partial_x\psi\,\partial_v(f-f_\infty) \;=\; 0\,,$$

for which ψ is unchanged.

This approach has several advantages from the computational and stability point of view. Indeed, both steps are fully implicit in time allowing to use a large time step uniform with respect to the parameter ε. Moreover, the linearized equation is autonomous, hence it requires to solve the **same** linear system at each time step, which can be done using a direct solver. Furthermore, solving the time dependent implicit second step, is in fact negligible in terms of computational costs since the associated system is trivially invertible due to the Hermite discretization. Finally, the numerical approximation of the linearized system allows to capture a consistent asymptotic profile when $\varepsilon \to 0$ and it also preserves the free energy estimate as in [25], which treats the case of a Vlasov-Fokker-Planck equation without a coupling with Poisson.

In Section 3.2 we propose a numerical discretization of the full model (3.5)-(3.6) based on a Hermite's decomposition in the velocity space and finite volume scheme for the space discretization. Then, in Section 3.3, we prove quantitative properties on its linearized version (3.8)-(3.9). More precisely, we first prove a discrete version of the free energy estimate (3.10) and then using discrete hypocoercive estimates, we get an exponential trend to equilibrium with rate $1/\varepsilon$. Finally in Section 3.4, we carry out numerical experiments which illustrate the robustness of our scheme in a wide variety of situations including near-to-collisionless regime and inhomogeneous ionic background. In particular, we observe formation of non-linear echoes and study their suppression in weakly collisional settings as well as simultaneous vortex/filamentation formation for inhomogeneous ionic background. These phenomena have drawn intense mathematical interest in the kinetic community over the past decade [10, 9, 11, 128, 68]. With this work, we aim at taking part in these efforts by proposing numerical methods capable to capture these phenomena efficiently.

3.2 Numerical scheme

We follow the lines of our previous work for the linear Vlasov-Fokker-Planck equation [25] and then propose a discretization of the Poisson equation allowing to preserve the energy estimate for the linearized problem (3.8)-(3.9). Finally, the nonlinear part is discretized using an implicit scheme.

3.2.1 Hermite's decomposition for the velocity variable

Let us first focus on the discretization of the velocity variable. It consists in performing a spectral decomposition of the distribution f into its Hermite modes $(\Psi_k)_{k\in\mathbb{N}}$ defined as

$$\Psi_k(v) \,=\, H_k\left(\frac{v}{\sqrt{T_0}}\right)\mathcal{M}(v)\,,$$

and which constitute an orthonormal system for the inverse Gaussian weight since it holds

$$\int_{\mathbb{R}} \Psi_k(v)\,\Psi_l(v)\,\mathcal{M}^{-1}(v)\mathrm{d}v \,=\, \delta_{k,l}\,,$$

where $\mathcal{M}$ is the Maxwellian corresponding to the stationary state of the Fokker-Planck operator (3.3). In the latter definition, $(H_k)_{k\in\mathbb{N}}$ stands for the family of Hermite polynomials defined recursively as follows $H_{-1}=0$, $H_0=1$ and

$$\xi\, H_k(\xi) \,=\, \sqrt{k}\, H_{k-1}(\xi) \,+\, \sqrt{k+1}\, H_{k+1}(\xi)\,, \quad \forall\, k \,\geq\, 0\,.$$

Let us also point out that Hermite's polynomials verify the following relation

$$H_k'(\xi) \,=\, \sqrt{k}\, H_{k-1}(\xi)\,, \quad \forall\, k \,\geq\, 0\,.$$

The Hermite system arises naturally in our context since it offers a simple discrete reformulation of the $L^2\left(f_\infty^{-1}\right)$-norm which appears in the key estimate (3.10), indeed it holds

$$\|f(t)\|^2_{L^2\left(f_\infty^{-1}\right)} \,=\, \sum_{k\in\mathbb{N}^*} \|C_k(t)\|^2_{L^2\left(\rho_\infty^{-1}\right)}\,,$$

where $C \,=\, (C_k)_{k\in\mathbb{N}}$ stand for the Hermite components of f

$$f\,(t,x,v) \,=\, \sum_{k\in\mathbb{N}} C_k\,(t,x)\;\Psi_k(v)\,.$$

As one can see in the latter relation, each term of the sequence $C \,=\, (C_k)_{k\in\mathbb{N}}$ naturally belongs to the weighted space $L^2\left(\rho_\infty^{-1}\right)$. From the numerical point of view, working in weighted spaces induces difficulties when it comes to integro/differential manipulation such as integration by part. This is the reason why rather than discretizing coefficients $C \,=\, (C_k)_{k\in\mathbb{N}}$, we consider their re-normalized versions $D \,=\, (D_k)_{k\in\mathbb{N}}$ defined as

$$f\,(t,x,v) \,=\, \sqrt{\rho_\infty}(x)\sum_{k\in\mathbb{N}} D_k\,(t,x)\;\Psi_k(v)\,. \tag{3.11}$$

According to latter considerations, renormalized Hermite coefficients $D \,=\, (D_k)_{k\in\mathbb{N}}$ verify

$$\|f(t)\|^2_{L^2\left(f_\infty^{-1}\right)} \,=\, \sum_{k\in\mathbb{N}} \|D_k(t)\|^2_{L^2(\mathbb{T})}\,.$$

To sum up, renormalized Hermite coefficients play a fundamental role in our analysis for two reasons : they offer a discrete reformulation of the key quantity $\mathcal{E}(t)$ given by (3.7) and they belong to the **unweighted** L^2-Lebesgue space over $\mathbb{T}$. However, there is one more benefit coming out of this choice : thanks to the properties of Hermite polynomials, one can see that Hermite functions diagonalize the Fokker-Planck operator since it holds

$$\partial_v \left[v\, \Psi_k \,+\, T_0\, \partial_v\, \Psi_k \right] \,=\, -\,k\, \Psi_k\,.$$

Therefore, following [25], we substitute the decomposition (3.11) in the first line of (3.5) : using the identities $E_\infty = -\partial_x \phi_\infty$ and $\rho_\infty\, E_\infty = T_0\, \partial_x \rho_\infty$, we get that $D = (D_k)_{k\in\mathbb{N}}$ satisfies the following system

$$\begin{cases} \varepsilon\, \partial_t D_k \,+\, \sqrt{k}\, \mathcal{A}\, D_{k-1} \,-\, \sqrt{k+1}\, \mathcal{A}^\star D_{k+1} \,+\, \sqrt{\dfrac{k}{T_0}}\, \partial_x \psi\, D_{k-1} \,=\, -\,\dfrac{k}{\tau_0}\, D_k\,, \\ D_k(t=0) = D_k^{0,\varepsilon}\,, \end{cases} \tag{3.12}$$

where the operators $\mathcal{A}$ and $\mathcal{A}^\star$ are given by

$$\begin{cases} \mathcal{A}\, u \,=\, +\sqrt{T_0}\, \partial_x u \,-\, \dfrac{E_\infty}{2\sqrt{T_0}}\, u\,, \\ \mathcal{A}^\star\, u \,=\, -\sqrt{T_0}\, \partial_x u \,-\, \dfrac{E_\infty}{2\sqrt{T_0}}\, u\,. \end{cases}$$

Notice that both operators $\mathcal{A}$ and $\mathcal{A}^\star$ may be rewritten as follows

$$\begin{cases} \mathcal{A}\, u \,=\, +\sqrt{T_0\, \rho_\infty}\, \partial_x \left(\dfrac{u}{\sqrt{\rho_\infty}} \right)\,, \\ \mathcal{A}^\star\, u \,=\, -\sqrt{\dfrac{T_0}{\rho_\infty}}\, \partial_x \left(\sqrt{\rho_\infty}\, u \right)\,. \end{cases}$$

To conclude, we denote by D_∞ the Hermite decomposition of the equilibrium f_∞. It is determined by

$$D_{\infty,k} \,=\, \begin{cases} \sqrt{\rho_\infty}, \ \text{if } k=0\,, \\ 0, \ \text{else}\,. \end{cases} \tag{3.13}$$

3.2.2 Poisson equation formulated in the Hermite framework

To compute the electrical potential ψ, we will reformulate the Poisson equation in such a way that the free energy estimate for the linearized system (3.8)-(3.9) is satisfied. To this aim, we introduce a modified potential v given by

$$v \,=\, \frac{\sqrt{\rho_\infty}}{T_0}\, \psi\,,$$

hence using the definition of the operator $\mathcal{A}$, the electric field $E = -\partial_x\psi$ is given by

$$E\,\sqrt{\rho_\infty} \;=\; -\partial_x\psi\,\sqrt{\rho_\infty} \;=\; -\sqrt{T_0}\,\mathcal{A}\,v$$

and the modified potential v solves the modified Poisson equation

$$\left(\mathcal{A}^\star\,\rho_\infty^{-1}\,\mathcal{A}\right)\,v \;=\; D_0 - \sqrt{\rho_\infty}\,. \tag{3.14}$$

This new formulation will allow us to easily construct a numerical scheme for the Poisson equation preserving the key energy estimate (3.10), where in this framework, the linearized free energy $\mathcal{E}$ reads

$$\mathcal{E}(t) \;=\; \frac{1}{2}\left(\|D(t)-D_\infty\|_{L^2}^2 \;+\; \left\|\frac{\mathcal{A}\,v}{\sqrt{\rho_\infty}}\right\|_{L^2}^2\right)\,, \tag{3.15}$$

where $\|\cdot\|_{L^2}$ stands for the overall L^2-norm **with no weight**

$$\|D\|_{L^2}^2 \;=\; \sum_{k\in\mathbb{N}}\|D_k\|_{L^2(\mathbb{T})}^2\,.$$

Indeed, we prove

Proposition 3.1. *Consider the solution $D=(D_k)_{k\in\mathbb{N}}$ to the linearized Vlasov-Poisson-Fokker-Planck system* (3.8)-(3.9) *formulated in the Hermite basis as*

$$\left\{\begin{array}{l}
\partial_t D_k \;+\; \dfrac{1}{\varepsilon}\,\left(\sqrt{k}\,\mathcal{A}\,D_{k-1} \;-\; \sqrt{k+1}\,\mathcal{A}^\star D_{k+1} \;+\; \mathcal{A}\,v\,\delta_{k,1}\right) \;=\; -\dfrac{k}{\varepsilon\,\tau_0}\,D_k\,,\\[2mm]
\left(\mathcal{A}^\star\,\rho_\infty^{-1}\,\mathcal{A}\right)\,v \;=\; D_0-\sqrt{\rho_\infty}\,,\\[2mm]
D_k(t=0)=D_k^{0,\varepsilon}\,,
\end{array}\right. \tag{3.16}$$

where $\delta_{k,1}$ is the Kronecker symbol ($\delta_{1,1}=1$ and $\delta_{k,1}=0$ when $k\neq 1$). Then the following free energy estimate holds for all $t\geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \;+\; \frac{1}{\varepsilon\,\tau_0}\sum_{k\in\mathbb{N}^\star} k\,\|D_k(t)\|_{L^2(\mathbb{T})}^2 \;=\; 0\,. \tag{3.17}$$

Proof. We multiply equation (3.16) by $D_k - D_{\infty,k}$, sum over all $k\in\mathbb{N}$ and integrate in $x\in\mathbb{T}$, this yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D-D_\infty\|_{L^2}^2 \;+\; \frac{1}{\varepsilon}\int_{\mathbb{T}}\mathcal{A}\,v\,D_1\mathrm{d}x \;=\; -\frac{1}{\varepsilon\,\tau_0}\sum_{k\in\mathbb{N}^*}k\|D_k\|_{L^2}^2,$$

We rewrite the integral term in the latter estimate using the equation on D_0 and the duality estimate

$$\int_{\mathbb{T}}\mathcal{A}\,v\,D_1\mathrm{d}x \;=\; \int_{\mathbb{T}}v\,\mathcal{A}^\star D_1\mathrm{d}x \;=\; \varepsilon\,\int_{\mathbb{T}}v\,\partial_t(D_0-\sqrt{\rho_\infty})\,\mathrm{d}x.$$

Using the reformulated Poisson equation in (3.16) and the duality estimate, we deduce

$$\varepsilon \int_{\mathbb{T}} v\, \partial_t D_0 \,\mathrm{d}x \;=\; \varepsilon \int_{\mathbb{T}} v\, \partial_t \left(\mathcal{A}^\star \rho_\infty^{-1} \mathcal{A}\right) v \,\mathrm{d}x \;=\; \frac{\varepsilon}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \frac{\mathcal{A}\, v(t)}{\sqrt{\rho_\infty}} \right\|^2_{L^2(\mathbb{T})} .$$

It finally yields the free energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \;=\; -\frac{1}{\varepsilon\, \tau_0} \sum_{k\in\mathbb{N}^\star} k\, \|D_k(t)\|^2_{L^2(\mathbb{T})} .$$

□

We now write the Hermite reformulation of the Vlasov-Poisson-Fokker-Planck system according to previous steps

$$\begin{cases} \varepsilon\, \partial_t D_k \,+\, \sqrt{k}\, \mathcal{A}\, D_{k-1} \,-\, \sqrt{k+1}\, \mathcal{A}^\star D_{k+1} \,+\, \sqrt{\dfrac{k}{\rho_\infty}}\, \mathcal{A} v\, D_{k-1} \;=\; -\dfrac{k}{\tau_0}\, D_k\,, \\ \left(\mathcal{A}^\star \rho_\infty^{-1} \mathcal{A}\right)\, v \;=\; D_0 - \sqrt{\rho_\infty}\,, \\ D_k(t=0) = D_k^{0,\varepsilon}\,. \end{cases} \tag{3.18}$$

Unfortunately, the latter estimate given in Proposition 3.1 is not sufficient to prove convergence of the solution to linearized system (3.16) the stationary state (3.13) because of the lack of coercivity. To bypass this difficulty, we define a modified relative energy $\mathcal{H}$ as

$$\mathcal{H}(t) \;=\; \mathcal{E}(t) \,+\, \alpha_0\, \langle \mathcal{A}^\star D_1(t),\, u(t)\rangle\,, \tag{3.19}$$

where $\alpha_0 > 0$ is a small free parameter and u is solution to

$$\begin{cases} \mathcal{A}^\star \mathcal{A}\, u \;=\; D_0 \,-\, \sqrt{\rho_\infty}\,, \\ \displaystyle\int_{\mathbb{T}} u\, \sqrt{\rho_\infty}\, \mathrm{d}x \;=\; 0\,. \end{cases}$$

To get the convergence to the solution to the linearized system (3.16) to the stationary state, the strategy consists in proving that $\mathcal{H}$ and $\mathcal{E}$ are equivalent and that there exists $\kappa > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) \le -\frac{\kappa}{\varepsilon}\, \min\left(\tau_0, \tau_0^{-1}\right)\, \mathcal{H}(t).$$

In the following subsection, we design a well-balanced finite volume discretization to the nonlinear system (3.18) such that the discrete approximation of the linearized system (3.16) satisfies such a hypocoercive estimate (Section 3.3).

3.2.3 Finite volume discretization for the space variable

We now turn to the space and time discretization to the system (3.18) and propose a well-balanced time splitting scheme. Consider an interval (a,b) of $\mathbb{R}$ and for $N_x \in \mathbb{N}^\star$, we introduce the set $\mathcal{J} = \{1,\dots,N_x\}$ and a family of control volumes $(K_j)_{j\in\mathcal{J}}$ such that $K_j = \left]x_{j-1/2}, x_{j+1/2}\right[$ with x_j the middle of the intervall K_j and

$$a = x_{1/2} < x_1 < x_{3/2} < \dots < x_{j-1/2} < x_j < x_{j+1/2} < \dots < x_{N_x} < x_{N_x+1/2} = b\,.$$

Let us set

$$\begin{cases} \Delta x_j = x_{j+1/2} - x_{j-1/2}, \text{ for } j \in \mathcal{J}\,, \\[2mm] \Delta x_{i+1/2} = x_{j+1} - x_j, \text{ for } 1 \le j \le N_x - 1\,. \end{cases}$$

We also introduce the parameter h such that

$$h \,=\, \max_{j\in\mathcal{J}} \Delta x_j\,.$$

Let Δt be the time step. We set $t^n = n\Delta t$ with $n \in \mathbb{N}$. A time discretization of $\mathbb{R}^+$ is then given by the increasing sequence of $(t^n)_{n\in\mathbb{N}}$. In the sequel, we will denote by D_k^n the approximation of $D_k(t^n)$, where the index k represents the k-th mode of the Hermite decomposition, whereas $\mathcal{D}_{k,j}^n$ is an approximation of the mean value of D_k over the cell K_j at time t^n.

First of all, the initial condition is discretized on each cell K_j by :

$$\mathcal{D}_{k,j}^0 = \frac{1}{\Delta x_j}\int_{K_j} D_k(t=0,x)\,dx, \quad j \in \mathcal{J}\,.$$

On the one hand, we apply a finite volume scheme to discretize operators $\mathcal{A}$ and $\mathcal{A}^\star$ and a fully implicit scheme for the time discretization, it gives $(D^{n+1/2})_{k\ge 0}$ and $v_h^{n+1/2}$ solution to

$$\begin{cases} \varepsilon\,\dfrac{D_1^{n+1/2} - D_1^n}{\Delta t} \,+\, \mathcal{A}_h\,D_0^{n+1/2} \,-\, \sqrt{2}\,\mathcal{A}_h^\star\,D_2^{n+1/2} \,+\, \mathcal{A}_h\,v_h^{n+1/2} \;=\; -\,\dfrac{1}{\tau_0}D_1^{n+1/2}\,, \\[3mm] \varepsilon\,\dfrac{D_k^{n+1/2} - D_k^n}{\Delta t} \,+\, \sqrt{k}\,\mathcal{A}_h\,D_{k-1}^{n+1/2} \,-\, \sqrt{k+1}\,\mathcal{A}_h^\star\,D_{k+1}^{n+1/2} \;=\; -\,\dfrac{k}{\tau_0}\,D_k^{n+1/2}\,, \text{ for } k \neq 1\,, \\[3mm] (\mathcal{A}_h^\star\,\rho_\infty^{-1}\,\mathcal{A}_h)\,v_h^{n+1/2} \;=\; D_0^{n+1/2} - \sqrt{\rho_\infty}\,, \\[3mm] \displaystyle\sum_{j\in\mathcal{J}} \Delta x_j\,v_j^{n+1/2}\,\sqrt{\rho}_{\infty,j}^{-1} \;=\; 0\,, \end{cases} \tag{3.20}$$

where $\mathcal{A}_h$ (resp. $\mathcal{A}_h^\star$) is an approximation of the operator $\mathcal{A}$ (resp. $\mathcal{A}^\star$) given by

$$\mathcal{A}_h = (\mathcal{A}_j)_{j\in\mathcal{J}} \quad \text{and} \quad \mathcal{A}_h^\star = (\mathcal{A}_j^\star)_{j\in\mathcal{J}} \tag{3.21}$$

and where for $D = (\mathcal{D}_j)_{j\in\mathcal{J}}$ it holds

$$\begin{cases} \mathcal{A}_j D \;=\; \sqrt{T_0}\,\dfrac{\mathcal{D}_{j+1}-\mathcal{D}_{j-1}}{2\Delta x_j} \;-\; \dfrac{E_{\infty,j}}{2\sqrt{T_0}}\,\mathcal{D}_j\,, \quad j\in\mathcal{J}\,, \\[2ex] \mathcal{A}_j^\star D \;=\; -\sqrt{T_0}\,\dfrac{\mathcal{D}_{j+1}-\mathcal{D}_{j-1}}{2\Delta x_j} \;-\; \dfrac{E_{\infty,j}}{2\sqrt{T_0}}\,\mathcal{D}_j\,, \quad j\in\mathcal{J}\,, \end{cases} \tag{3.22}$$

whereas the discrete electric field $E_{\infty,j}$ is given by

$$E_{\infty,j} \;=\; -\frac{\phi_{\infty,j+1}-\phi_{\infty,j-1}}{2\Delta x_j} \;=\; \frac{2\,T_0}{\sqrt{\rho}_{\infty,j}}\,\frac{\sqrt{\rho}_{\infty,j+1}-\sqrt{\rho}_{\infty,j-1}}{2\,\Delta x_j}\,, \tag{3.23}$$

where $\rho_{\infty,j}$ is an approximation of the stationary density ρ_∞ on the cell K_j. This latter formula is consistent with the definition of $\sqrt{\rho}_\infty = e^{-\phi_\infty/(2T_0)}$ and the fact that

$$\frac{1}{2\,T_0}\,\partial_x\phi_\infty \;=\; -\frac{1}{\sqrt{\rho}_\infty}\,\partial_x\sqrt{\rho}_\infty\,.$$

This first step requires the numerical resolution of linear system which does not depend on the time step n. Hence a direct solver based on LU factorization is applied to get the solution $(D^{n+1/2}, v_h^{n+1/2})$.

On the other hand, we solve the nonlinear part using again a fully implicit Euler scheme

$$\begin{cases} D_0^{n+1} \;=\; D_0^{n+1/2}\,, \\[1ex] v_h^{n+1} \;=\; v_h^{n+1/2}\,, \\[1ex] \varepsilon\,\dfrac{D_k^{n+1}-D_k^{n+1/2}}{\Delta t} \;+\; \sqrt{\dfrac{k}{\rho_\infty}}\,\mathcal{A}_h\,v_h^{n+1}\left(D_{k-1}^{n+1} \;-\; D_{\infty,k-1}\right) \;=\; 0\,, \quad \text{if } k\geq 1\,. \end{cases} \tag{3.24}$$

Observe that since $v_h^{n+1/2}$ and $D_0^{n+1/2}$ do not change during this second step, applying an implicit scheme does not require any linear solver since the latter system is trivially invertible.

3.3 Trend to equilibrium for the linearized system

In this section, we only consider the numerical scheme applied to the linearized system corresponding to the first step of the splitting scheme, that is,

$$\begin{cases} \varepsilon\,\dfrac{D_1^{n+1}-D_1^n}{\Delta t} \;+\; \mathcal{A}_h\,D_0^{n+1} \;-\; \sqrt{2}\,\mathcal{A}_h^\star\,D_2^{n+1} \;+\; \mathcal{A}_h\,v_h^{n+1} \;=\; -\dfrac{1}{\tau_0}D_1^{n+1}\,, \\[2ex] \varepsilon\,\dfrac{D_k^{n+1}-D_k^n}{\Delta t} \;+\; \sqrt{k}\,\mathcal{A}_h\,D_{k-1}^{n+1} \;-\; \sqrt{k+1}\,\mathcal{A}_h^\star\,D_{k+1}^{n+1} \;=\; -\dfrac{k}{\tau_0}\,D_k^{n+1}\,, \text{ for } k\neq 1\,, \\[2ex] (\mathcal{A}_h^\star\,\rho_\infty^{-1}\,\mathcal{A}_h)\,v_h^{n+1} \;=\; D_0^{n+1} - \sqrt{\rho}_\infty\,, \\[1ex] \displaystyle\sum_{j\in\mathcal{J}}\Delta x_j\,v_j^{n+1}\,\sqrt{\rho}_{\infty,j}^{-1} \;=\; 0\,, \end{cases} \tag{3.25}$$

and we define the discrete free energy of the solution $D_k^n = (\mathcal{D}_{k,j}^n)_{j\in\mathcal{J}}$ to (3.25) as follows

$$\mathcal{E}^n \,=\, \frac{1}{2}\left(\|D^n - D_\infty\|_{L^2}^2 \,+\, \left\|\frac{\mathcal{A}_h\, v_h^n}{\sqrt{\rho}_\infty}\right\|_{L^2}^2\right). \tag{3.26}$$

We prove the following theorem

Theorem 3.2. *Consider the solution $D_k^n = (\mathcal{D}_{k,j}^n)_{j\in\mathcal{J}}$ to (3.25). Then the following discrete energy estimate holds for all $n \geq 0$*

$$\mathcal{E}^n \leq 3\left(1 + \frac{\kappa}{\varepsilon}\,\min\left(\tau_0, \tau_0^{-1}\right)\,\Delta t\right)^{-n}\,\mathcal{E}^0\,,$$

where $\kappa > 0$ depends only on ρ_i, T_0 and $|b-a|$.

This result is obtained as the discrete version of the free energy estimate proved in Proposition 3.1 coupled with hypocoercive estimates proved on the discrete version of the modified free energy defined in (3.19).

3.3.1 *A priori* estimates

First, we prove a discrete energy estimate as follows

Proposition 3.3. *Consider the solution $D_k^n = (\mathcal{D}_{k,j}^n)_{j\in\mathcal{J}}$ to (3.25). The following discrete energy estimate holds for all $n \geq 0$*

$$\frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} \,+\, \Delta t\,\mathcal{R}_h^n \,=\, -\,\frac{1}{\varepsilon\,\tau_0}\sum_{k\in\mathbb{N}^\star} k\,\left\|D_k^{n+1}\right\|_{L^2}^2,$$

where $\mathcal{R}_h^n$ is the following positive remainder due to numeric dissipation

$$\mathcal{R}_h^n \,=\, \frac{1}{2}\left(\left\|\frac{\mathcal{A}_h\left(v_h^{n+1} - v_h^n\right)}{\Delta t\,\sqrt{\rho}_\infty}\right\|_{L^2}^2 + \left\|\frac{D^{n+1} - D^n}{\Delta t}\right\|_{L^2}^2\right).$$

Proof. To compute the variations of $\|D^n - D_\infty\|_{L^2}^2$ between time step n and $n+1/2$, we multiply equation (3.25) by $D_k^{n+1} - D_{\infty,k}$ and sum over all $k \in \mathbb{N}$, this yields

$$\frac{1}{2\,\Delta t}\left(\left\|D^{n+1} - D_\infty\right\|_{L^2}^2 - \|D^n - D_\infty\|_{L^2}^2 + \left\|D^{n+1} - D^n\right\|_{L^2}^2\right) \,=\, \mathcal{I}_1 \,+\, \mathcal{I}_2\,,$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are given by

$$\begin{cases} \mathcal{I}_1 \,:=\, -\,\dfrac{1}{\varepsilon\,\tau_0}\displaystyle\sum_{k\in\mathbb{N}^\star} k\,\left\|D_k^{n+1}\right\|_{L^2}^2 \,-\, \dfrac{1}{\varepsilon}\displaystyle\sum_{k\in\mathbb{N}}\left\langle\sqrt{k}\,\mathcal{A}_h\,D_{k-1}^{n+1} \,-\, \sqrt{k+1}\,\mathcal{A}_h^\star\,D_{k+1}^{n+1}\,,\,D_k^{n+1} - D_{\infty,k}\right\rangle, \\ \mathcal{I}_2 \,:=\, -\,\dfrac{1}{\varepsilon}\left\langle\mathcal{A}_h\,v_h^{n+1}\,,\,D_1^{n+1}\right\rangle, \end{cases} \tag{3.27}$$

where $\mathcal{I}_2$ stands for the contribution of the electric field and $\mathcal{I}_1$ gathers all the other terms.

First, using that $\mathcal{A}\, D_{\infty,0} = 0$, splitting and re-indexing the second sum in the definition of $\mathcal{I}_1$, we see that it is in fact zero. Therefore, $\mathcal{I}_1$ rewrites as follows

$$\mathcal{I}_1 \;=\; -\frac{1}{\varepsilon\,\tau_0}\sum_{k\in\mathbb{N}^\star} k\,\left\|D_k^{n+1}\right\|_{L^2}^2 .$$

Furthermore, considering the case $k = 0$ in the second equation of system (3.25), we deduce that $\mathcal{I}_2$ rewrites as follows

$$\mathcal{I}_2 \;=\; -\left\langle v_h^{n+1}\,,\,\frac{D_0^{n+1}-D_0^n}{\Delta t}\right\rangle .$$

Then, we replace $D_0^{n+1} - D_0^n$ in the latter relation using the third line in (3.25), it yields

$$\mathcal{I}_2 \;=\; -\left\langle v_h^{n+1}\,,\,(\mathcal{A}_h^\star\,\rho_\infty^{-1}\,\mathcal{A}_h)\,\frac{v_h^{n+1}-v_h^n}{\Delta t}\right\rangle .$$

Therefore, we deduce the following relation

$$\mathcal{I}_2 \;=\; -\frac{1}{2\,\Delta t}\left(\left\|\frac{\mathcal{A}_h\,v_h^{n+1}}{\sqrt{\rho_\infty}}\right\|_{L^2}^2 - \left\|\frac{\mathcal{A}_h\,v_h^{n}}{\sqrt{\rho_\infty}}\right\|_{L^2}^2 + \left\|\frac{\mathcal{A}_h\left(v_h^{n+1}-v_h^n\right)}{\sqrt{\rho_\infty}}\right\|_{L^2}^2\right),$$

which concludes the proof. □

We now define the following modified relative entropy functional

$$\mathcal{H}^n \;=\; \mathcal{E}^n \,+\, \alpha_0\,\langle \mathcal{A}_h^\star\,D_1^n, u_h^n\rangle\,, \tag{3.28}$$

where α_0 is a positive constant and where u_h^n is solution the solution to

$$\begin{cases} \mathcal{A}_h^\star\,\mathcal{A}_h\,u_h^n \;=\; D_0^n \;-\; \sqrt{\rho_\infty}\,, \\[2mm] \displaystyle\sum_{j\in\mathcal{J}} \Delta x_j\,u_j\,\sqrt{\rho_{\infty,j}} \;=\; 0\,. \end{cases} \tag{3.29}$$

The following Lemma ensures that the modified relative entropy functional $\mathcal{H}^n$ is in fact equivalent to the energy for α_0 small enough. The proof of this result may be found in [25] (see the proof of Lemma 3.6)

Lemma 3.4. *For all ρ_i, there exists a positive constant $\overline{\alpha}_0$ such that for all $\alpha_0 \in (0, \overline{\alpha}_0)$ and all $D^n = (\mathcal{D}_{k,j}^n)_{j\in\mathcal{J}\,,\,k\in\mathbb{N}}$, one has*

$$\frac{1}{4}\,\mathcal{E}^n \le \mathcal{H}^n \;\le\; \frac{3}{4}\,\mathcal{E}^n\,, \tag{3.30}$$

where $\mathcal{E}^n$ and $\mathcal{H}^n$ are given by (3.26) *and* (3.28).

Building on the latter lemma, we now prove that $\mathcal{H}^n$ verifies a dissipation relation

Proposition 3.5. *Consider the solution $D^n = (D^n_k)_{k\in\mathbb{N}}$ to (3.25). The modified relative entropy functional defined by (3.28) verifies for all $n \geq 0$*

$$\frac{\mathcal{H}^{n+1} - \mathcal{H}^n}{\Delta t} \;\leq\; -\frac{\kappa}{\varepsilon}\,\min\left(\tau_0, \tau_0^{-1}\right)\mathcal{H}^{n+1}\,,$$

for some positive constant κ depending only on ρ_i and T_0 and $|b-a|$.

Proof. To estimate the variations of the additional term $\langle \mathcal{A}_h^\star D_1^n, u_h^n\rangle$ in the definition of $\mathcal{H}^n$ between time steps n and $n+1$, we start from the following relation, obtained after applying $\mathcal{A}_h^\star$ to the first line of system (3.25) and multiplying the obtained equation by u_h^{n+1}

$$\frac{1}{\Delta t}\left(\left\langle \mathcal{A}_h^\star D_1^{n+1}, u_h^{n+1}\right\rangle \,-\, \langle \mathcal{A}_h^\star D_1^n, u_h^n\rangle\right) \;=\; \mathcal{J}_1 \,+\, \mathcal{J}_2\,,$$

where $\mathcal{J}_1$, $\mathcal{J}_2$ are given by

$$\left\{\begin{aligned}
\mathcal{J}_1 \;&:=\; -\frac{1}{\varepsilon}\left\langle \mathcal{A}_h^\star\mathcal{A}_h\left(D_0^{n+1} - \sqrt{\rho_\infty}\right) \,-\, \sqrt{2}\,(\mathcal{A}_h^\star)^2\, D_2^{n+1} \,+\, \frac{1}{\tau_0}\,\mathcal{A}_h^\star\, D_1^{n+1},\, u_h^{n+1}\right\rangle\\
&\quad + \left\langle \mathcal{A}_h^\star\, D_1^{n+1},\, \frac{u_h^{n+1} - u_h^n}{\Delta t}\right\rangle \,-\, \left\langle \mathcal{A}_h^\star\left(D_1^{n+1} - D_1^n\right),\, \frac{u_h^{n+1} - u_h^n}{\Delta t}\right\rangle\,,\\
\mathcal{J}_2 \;&:=\; -\frac{1}{\varepsilon}\left\langle \mathcal{A}_h^\star\, \mathcal{A}_h\, v_h^{n+1},\, u_h^{n+1}\right\rangle\,.
\end{aligned}\right.$$

The term $\mathcal{J}_2$ takes into account the contribution of the electric field in the equation on D_1^n of (3.25), whereas $\mathcal{J}_1$ gathers all the other terms.

Relying on computations detailed in [25] (see the proof of Theorem 3.1, in Section 3.4), $\mathcal{J}_1$ meets the following estimate

$$\begin{aligned}
\mathcal{J}_1 \;\leq\; &-\frac{1}{\varepsilon}\left(1 - \frac{C}{\bar\tau_0}\,\eta\right)\|D_0^{n+1} - D_{\infty,0}\|^2_{L^2} \,+\, \frac{C}{\eta\,\varepsilon}\left(\|D_2^{n+1}\|^2_{L^2} \,+\, \frac{1}{\tau_0}\,\|D_1^{n+1}\|^2_{L^2}\right)\\
&+ \frac{C}{\Delta t}\,\|D^{n+1} - D^n\|^2_{L^2}\,,
\end{aligned}$$

for any positive η, for some positive constant C depending only on ρ_i and where $\bar\tau_0$ stands for $\bar\tau_0 = \min(1, \tau_0)$.

To estimate $\mathcal{J}_2$, rely on the following relation, which holds for all $n \geq 0$ according to the third line in (3.25) and equation (3.29)

$$(\mathcal{A}_h^\star\, \rho_\infty^{-1}\, \mathcal{A}_h)\, v_h^n \;=\; \mathcal{A}_h^\star\, \mathcal{A}_h\, u_h^n\,.$$

Multiplying the latter relation by v^n, we deduce

$$\mathcal{J}_2 \;=\; -\frac{1}{\varepsilon}\left\|\frac{\mathcal{A}_h\, v_h^n}{\sqrt{\rho_\infty}}\right\|^2_{L^2}\,.$$

Gathering the latter computations, we deduce

$$\frac{\mathcal{H}^{n+1}-\mathcal{H}^n}{\Delta t} + \Delta t\,(1 - C\,\alpha_0)\,\mathcal{R}_h^n \leq$$
$$-\frac{1}{\varepsilon\,\tau_0}\left[\left(1-\frac{C}{\eta\,\bar\tau_0}\,\alpha_0\,\tau_0\right)\sum_{k\in\mathbb{N}^\star}\left\|D_k^{n+1}\right\|_{L^2}^2 + \alpha_0\,\tau_0\left(1-\frac{C}{\bar\tau_0}\,\eta\right)\|D_0^{n+1}-D_{\infty,0}\|_{L^2}^2\right.$$
$$\left. + \alpha_0\,\tau_0\left\|\frac{\mathcal{A}_h\,v_h^{n+1}}{\sqrt{\rho}_\infty}\right\|_{L^2}^2\right],$$

Taking $\eta = \bar\tau_0/(2C)$ and $\alpha_0\tau_0 = 2\bar\tau_0^2/(\bar\tau_0^2 + 4\,C^2)$, the latter inequality becomes

$$\frac{\mathcal{H}^{n+1}-\mathcal{H}^n}{\Delta t} + \Delta t\left(1 - C\,\frac{2\bar\tau_0^2}{\tau_0(\bar\tau_0^2+4C^2)}\right)\mathcal{R}_h^n \leq -\frac{1}{\varepsilon\tau_0}\,\frac{2\bar\tau_0^2}{4\,C^2+\bar\tau_0^2}\,\mathcal{E}^{n+1}.$$

Taking $C \geq 0$ great enough in order for Lemma 3.4 to apply and performing simple computations, we deduce the result

$$\frac{\mathcal{H}^{n+1}-\mathcal{H}^n}{\Delta t} \leq -\frac{\kappa}{\varepsilon}\,\min\left(\tau_0,\tau_0^{-1}\right)\mathcal{H}^{n+1},$$

for some $\kappa > 0$ depending only on ρ_i and T_0 and $|b-a|$. □

3.3.2 Proof of Theorem 3.2

We now proceed to proof of Theorem 3.2. First, from Proposition 3.5, it is straightforward to obtain

$$\mathcal{H}^n \leq \left(1 + \frac{\kappa}{\varepsilon}\,\min\left(\tau_0,\tau_0^{-1}\right)\Delta t\right)^{-n}\mathcal{H}^0.$$

Then, we apply Lemma 3.4 on each side of the latter inequality and obtain the result.

3.4 Numerical simulations

For numerical experiments, we apply a slight modification of the scheme (3.20)-(3.24) since a Strang splitting scheme with a second order Runge-Kutta scheme is used to get second order accuracy in time.

In the numerical simulations presented in this section, we fix $T_0 = 1$ and $\varepsilon = 1$ and take different values for the collisional frequency τ_0 describing either weakly collisional regime when $\tau_0 \gg 1$ or strongly collisional plasmas when $\tau_0 \simeq 1$. Then, we choose $N_x = 128$ and $\Delta t = 0.1$ and adapt the number of Hermite modes according to the collisional regime.

3.4.1 Perturbation of non uniform density

For this first numerical test, we consider the following spatially inhomogeneous equilibrium

$$\rho_\infty(x) \,=\, c_\infty\, e^{-\phi_\infty(x)}, \quad x \in (-L, L)\,,$$

where the potential ϕ_∞ is given by

$$\phi_\infty(x) \,=\, 0.2\,\sin(k\,x)\,, \quad x \in (-L, L),$$

with $k = \pi/L$ and $L = 6$ whereas the constant c_∞ ensures that

$$\frac{1}{2L}\int_{-L}^{L} \rho_\infty(x)\,\mathrm{d}x \,=\, 1.$$

Thus, we take the initial distribution function as a perturbation of this steady state, that is,

$$f(t=0,x,v) \,=\, \frac{1}{\sqrt{2\pi}}\,\left(\rho_\infty(x) \,+\, \delta\,\cos(k\,x)\right)\, e^{-v^2/2}\,, \quad (x,v) \in (-L,L)\times\mathbb{R},$$

where $\delta = 0.01$.

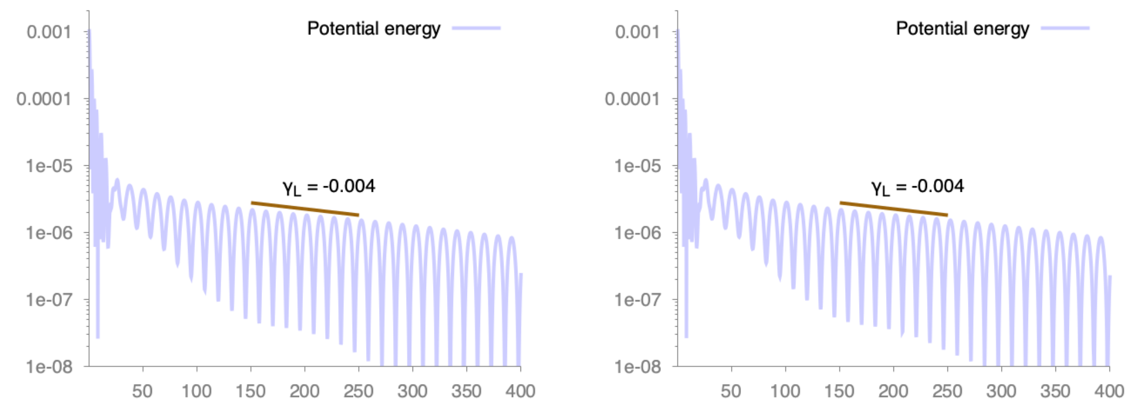
To obtain our results, we have chosen a large number of Hermite modes $N_H = 400$ when the plasma is weakly collisional, that is when $\tau_0 \gg 10^2$, since filamentation may occur in phase space whereas $N_H = 50$ is enough when collisions dominate.

On the one hand, we take $\tau_0 = 10^4$ corresponding to the weakly collisional regime and compare two solutions, one is obtained using (3.25) corresponding to the linearized Vlasov-Poisson-Fokker-Planck system (3.8) and the second one is given by (3.20)-(3.24) corresponding to the nonlinear Vlasov-Poisson-Fokker-Planck system (3.1). Our results show that both solutions have the same behavior, which means that, for such a small perturbation, the linear regime governs the dynamics. To illustrate this observation, we report in Figure 3.1 the time evolution of the potential energy

$$\mathcal{E}_p(t) := \int_{\mathbb{T}} |\partial_x\psi(t,x)|^2\,\mathrm{d}x$$

obtained using (3.25) in (a) and (3.20)-(3.24) in (b). Both solutions first produce fast damped oscillations up to time $t \leq 20$ and then oscillate with a lower frequency while converging exponentially fast to zero with the same convergence rate $\gamma_L \simeq 0.004$.

In Figure 3.2, we show different snapshots of the difference between the distribution function f and its equilibrium f_∞ for $t \in [4, 70]$. As expected, thin filaments propagate in phase space for large velocities but surprisingly we also observe that a vortex is generated in the region where $|v| \leq 1$. For large time, this vortex remains and continues to rotate around the point $(x_C, v_C) = (-3, 0)$. For such a regime, where collisions are almost negligible, the amplitude of the perturbation does not vanish even when $t \simeq 70$ and transport phenomena dominate.



(a) (b)

FIGURE 3.1 – *Perturbation of non uniform density for* $\tau_0 = 10^4$ *(weakly collisional regime) : time development of the potential energy in log scale (for (a) the linearized Vlasov-Poisson-Fokker-Planck system and (b) the nonlinear Vlasov-Poisson-Fokker-Planck system.*

On the other hand, we study the influence of the collision frequency τ_0 and perform several numerical simulations for the nonlinear Vlasov-Poisson-Fokker-Planck system (3.1) with the same initial data for $\tau_0 = 10^k$, with $k = 0, \ldots, 4$ (see Figure 3.3). For a weakly collision regime, that is $k \geq 3$, we again observe oscillations of the potential energy and an exponential decay. However, when collisions dominate, fast oscillations only occur for short time, then the potential energy decays rapidly to zero without any oscillations. This trend to equilibrium can be also observed on the distribution function as shown on Figure 3.3-(b), where we present the time evolution of the quantity

$$\mathcal{L}_2(t) := \|f(t) - f_\infty\|_{L^2(f_\infty^{-1})}.$$

Finally in Figure 3.4, we again present snapshots of $f - f_\infty$ at different time when $\tau_0 = 10^2$. In this situation, collisions dissipate thin filaments generated by the transport term and the amplitude of the perturbation vanishes. As a consequence, we do not observe anymore the vortex structure on the difference $f - f_\infty$ and the solution converges exponentially fast to the equilibrium as it has been shown in Theorem 3.2 for the linearized system.

3.4.2 Plasma echo

We now investigate a much more intricate problem where non-linearity plays the main role. Following the work [164, 110] or more recently [10, 128], we will consider a perturba-

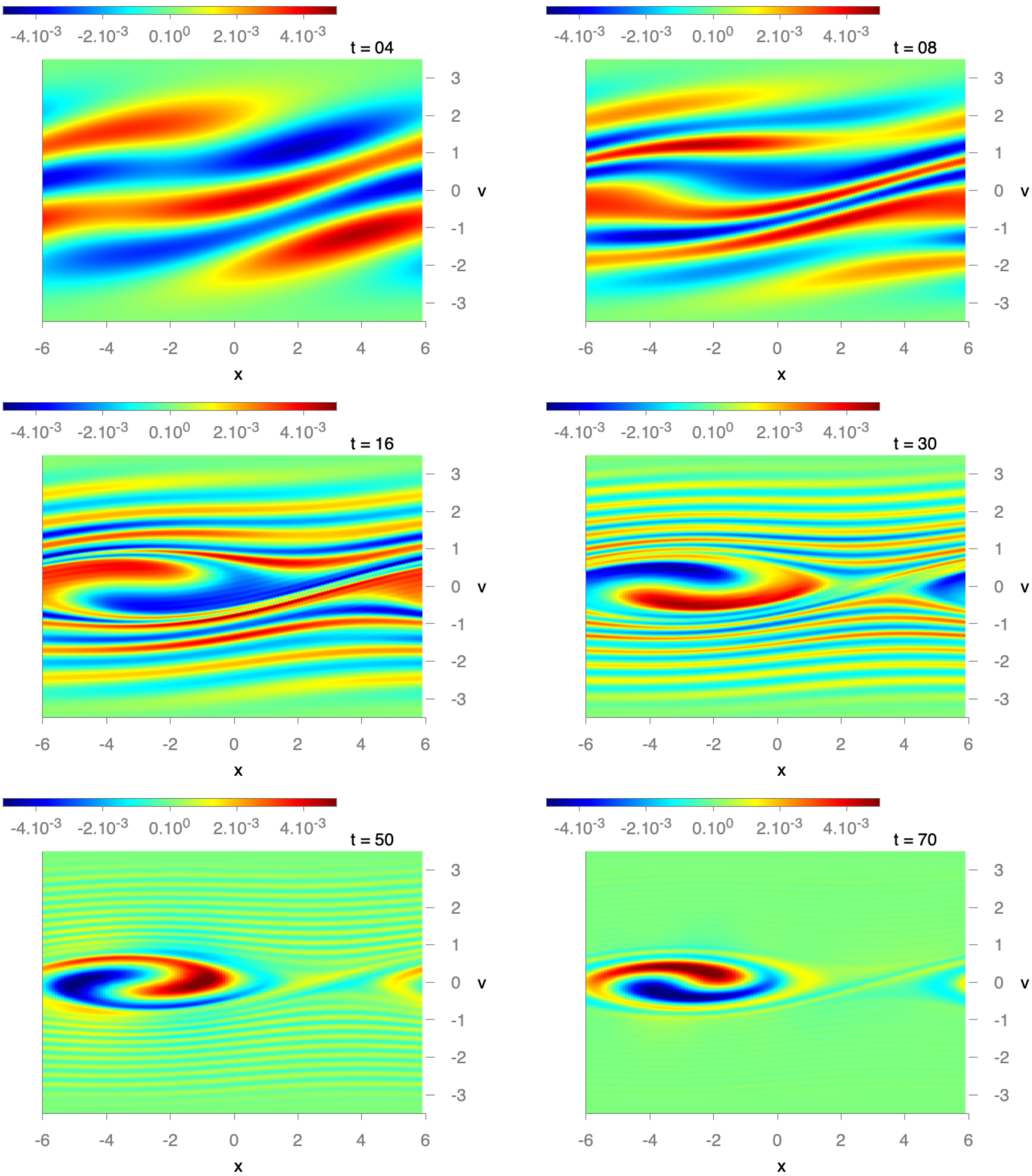


FIGURE 3.2 – *Perturbation of non uniform density (weakly collisional regime, $\tau_0 = 10^4$) : snapshots of the difference between the solution f and the equilibrium f_∞ at time $t = 4$, 8, 16, 30, 40 and 70.*

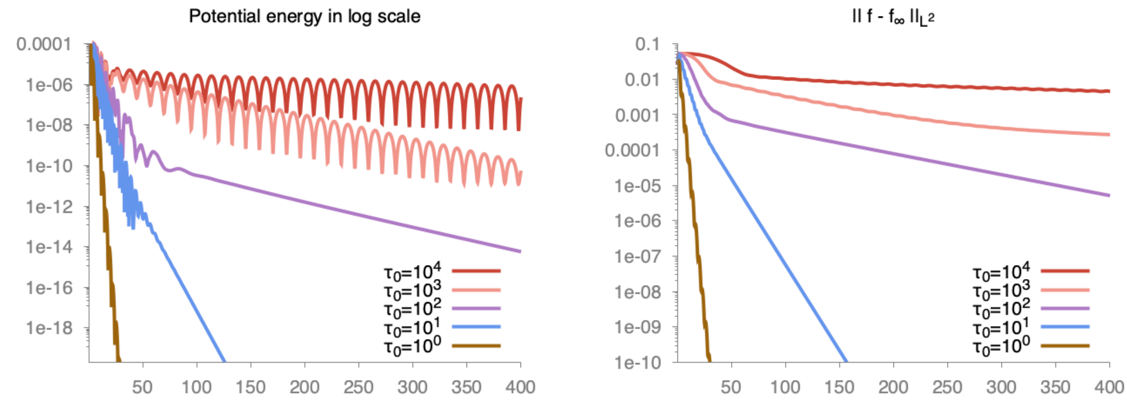


FIGURE 3.3 – *Perturbation of non uniform density : time development of (a) the potential energy (b)* $\|f - f_\infty\|_{L^2(f_\infty^{-1})}$ *for various* $\tau_0 = 1, \ldots, 10^4$ *(in log scale).*

tion of an homogeneous Maxwellian distribution $f_\infty(x, v) := \mathcal{M}(v)$ where

$$\mathcal{M}(v) = \frac{1}{\sqrt{2\pi}} \exp(-v^2/2), \quad v \in \mathbb{R},$$

on the space interval $[-L, L]$, with $L = 6$. Of course, this homogeneous Maxwellian is stable and for a small perturbation, we expect to observe the well known Landau damping on the macroscopic quantities. However, our aim is to investigate a transient regime where a plasma echo occurs for a well chosen perturbation. This echo appears when two waves with distinct frequencies interact. For a large time period, the effect is very small but at certain particular times, the interaction becomes strong : this is known in plasma physics as the plasma echo, and can be thought of as a kind of resonance [127, 163]. Such situation, where usual Landau damping is not respected also occur in magntized plasmas [67]

To build this initial data, we proceed in two steps. We first solve numerically the Vlasov-Poisson system **without any collision** on the time interval $[-30, 0]$ with an initial data at time $t_0 = -30$ given by

$$\tilde{f}_{in}(x, v) \;=\; (1 + \alpha \cos(k_1 x))\; \mathcal{M}(v),$$

where $\alpha = 0.01$ and $k_1 = \pi/L$. This choice induces an exponential decay of the potential energy by Landau damping at the rate associated to the perturbed mode k_1, hence it gives a distribution function which is almost space homogeneous with small and fast oscillations in velocity. Then, at $t = 0$, we consider this solution, denoted by $\tilde{f}(0)$, and choose it as initial data with a perturbation of the mode $k_2 := 2k_1$. More precisely, we take

$$f_0(x, v) = (1 + \alpha \cos(k_2 x))\, \tilde{f}(0, x, v).$$

This initial data is represented in Figure 3.7. Then, starting from f_0, we solve the Vlasov-Poisson-Fokker-Planck system on the time interval $[0, 120]$.

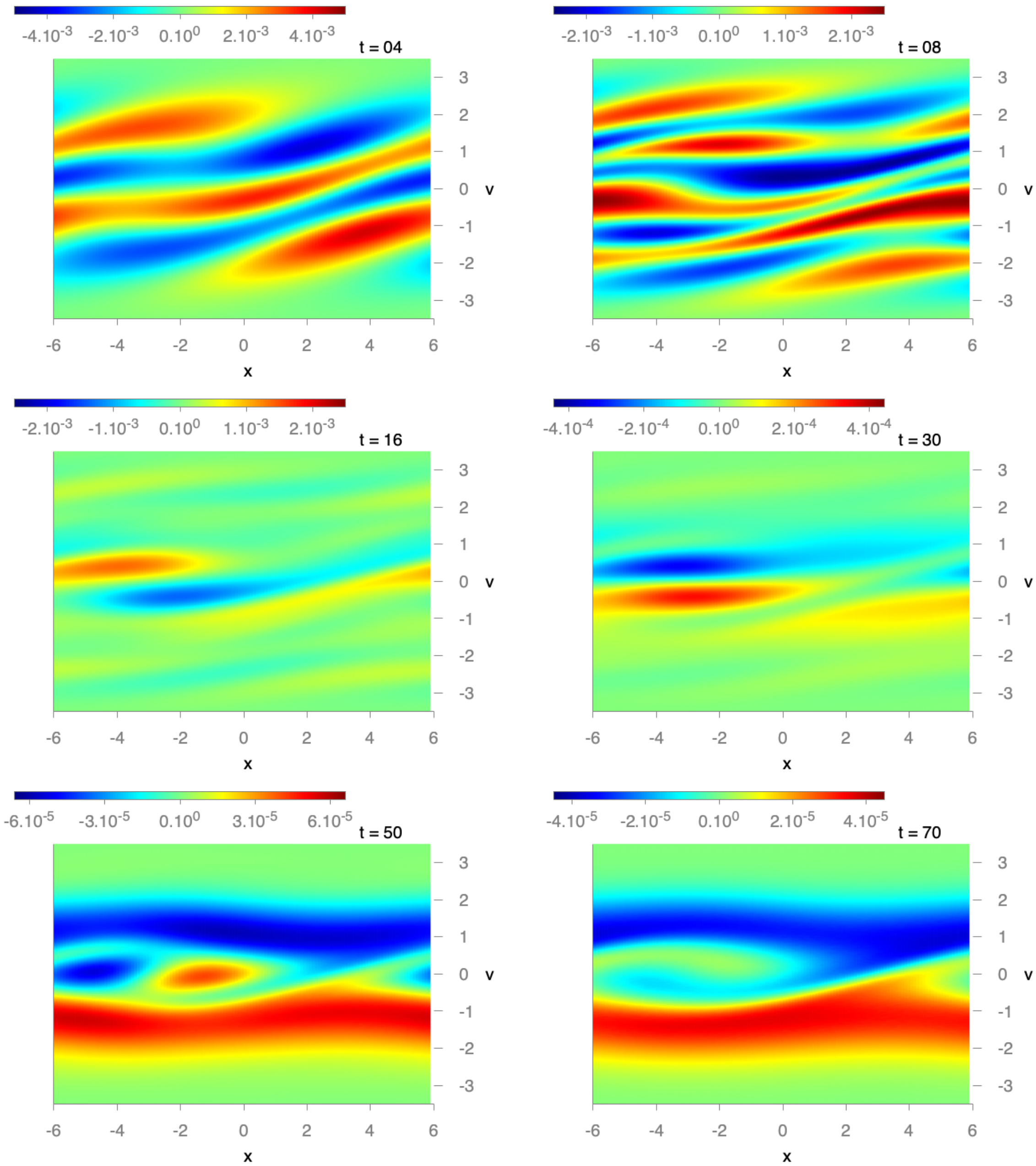


Figure 3.4 – *Perturbation of non uniform density (moderate collisional regime,* $\tau_0 = 10^2$*) : snapshots of the difference between the solution* f *and the equilibrium* f_∞ *at time* $t = 4, 8, 16, 30, 40$ *and* 70.

On the one hand, we take $\tau_0 = 10^6$, which corresponds to a weakly collisional regime generating an oscillatory solution in velocity. For this reason, we choose a large number of Hermite modes $N_H = 8000$ in such a way that our numerical results are not anymore sensitive to the numerical parameters. Let us emphasize that we compare our numerical results with those obtained using a semi-Lagrangian method [110, 208, 107], which also requires such a large number of points in order to reach convergence. The phenomena is so intricate that we want to be sure that numerical parameters do not produce any artefact...

Now let us comment our numerical results. In this weakly collisional regime, we expect that this initial data will induce a first Landau damping phenomena due to the perturbation of the second mode k_2. However, after a time much longer than the inverse Landau damping rate, a new wave, called "echo" will appear as a modulation of the density at the wave number $k_{echo} = k_2 - k_1 = k_1$. This echo is due to the interaction between modes and is essentially a phenomenon of beating between two waves. First, we will see that the nonlinearity is crucial here. Indeed, in Figure 3.5, we show the time evolution of the potential energy and the first modes of the electric field obtained by applying the scheme (3.25) corresponding to the linearized Vlasov-Poisson-Fokker-Planck system and the scheme (3.20)-(3.24) corresponding to the nonlinear system. The numerical solution corresponding to the linearized system exhibits a simple Landau damping, when $t \geq 5$, with a decay rate corresponding to the predicted value $\gamma_L = 0.355$, whereas the numerical solution corresponding the nonlinear system differs completely. It exhibits a first fast decay as for the linearized system, but when $t \geq 15$, it produces an echo on the potential energy and the first mode (see right hand side of Figure 3.5). The echo reaches its maximal amplitude at $t = 30$ for which we report the snapshots of the $f - f_\infty$ in Figure 3.7. The first damping of the perturbed mode k_2 for short time $t \leq 5$ and the subsequent echo are accurately reproduced. From [164], the echo wave number is indeed expected to be $k_{echo} = k_1$ the first echo time is predicted at

$$t_{echo} \,=\, t_0 + \frac{k_2}{k_2 - k_1}(0 - t_0) \,=\, 30\,,$$

which corresponds very well with the numerical value we obtain here. From time $t = 0$ to $t \simeq 20$, the second wave corresponding to the mode k_2 has no effect on the first mode k_1 of the electric field, but at time $t = 30$, it is strongly perturbed by the echo effect. Actually, our numerical results illustrate the complexity of these phenomena.

On the one hand, we observe that **echoes are repeated through time**. On the potential energy (see the top right chart of Figure 3.5), a Landau damping is observed when $30 \leq t \leq 80$, then we again observe a new echo around time $t = 90$ followed by a new damping. We also make the observation that this second damping ($t \geq 90$) unfolds with a smaller decay rate than the first one ($30 \leq t \leq 80$). This repetition of echoes can be also observed on the modes of the electric field.

On the other hand, we report the time evolution of the first modes of the electric field (see the bottom right chart of Figure 3.5) and observe that **other modes are also subjected to echoes** but at times which differ from the "macroscopic" echo time $t_{echo} =$

30. These echoes are not visible on the potential energy since their amplitude is smaller to the one of the potential energy by several order of magnitude. However, we point out that the third mode is subjected to a dramatic echo whose maximal amplitude, reached at time $t = 15$, is greater than the initial amplitude by a factor almost 10^5. A careful inspection allows to perceive the effect of this echo on the overall amplitude of the potential energy (compare bottom and top right charts of Figure 3.5). It is worth to mention that all modes corresponding to the linearized system are subjected to the classical Landau daming without any echo. This is a nice example where nonlinear effects, even small, induce intricate oscillatory behaviours.

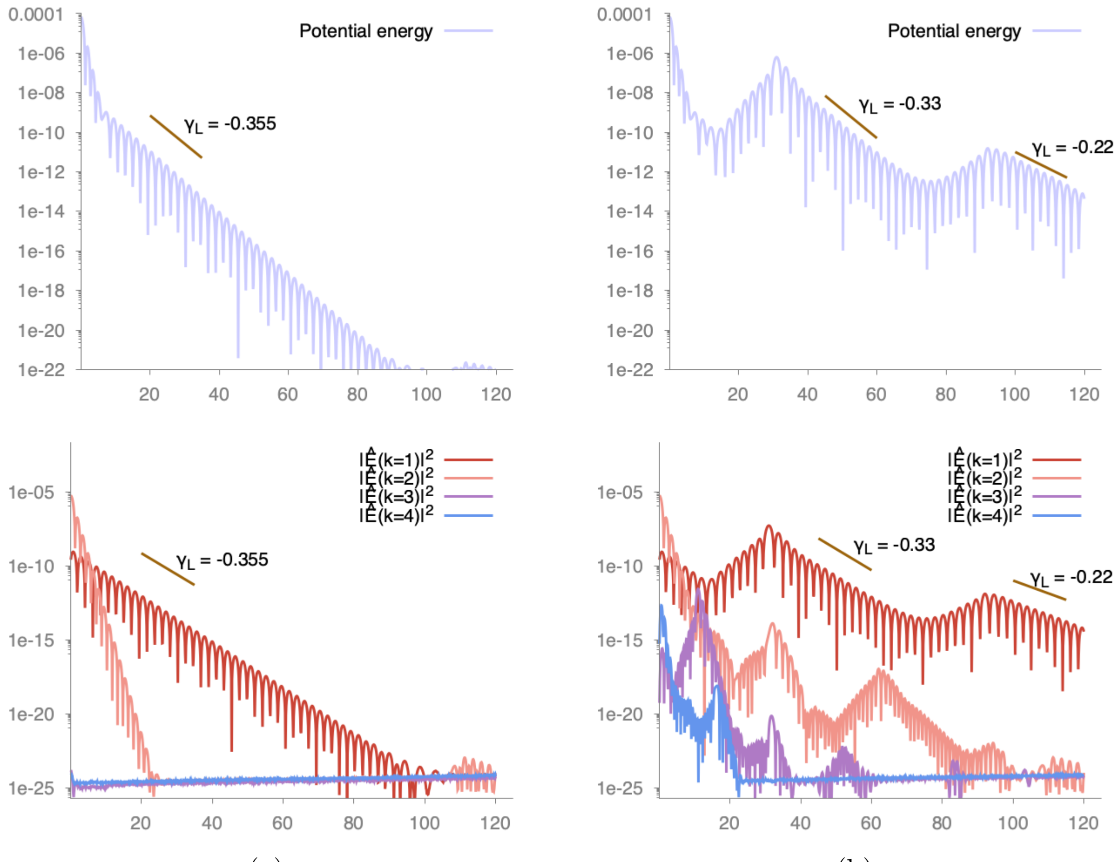
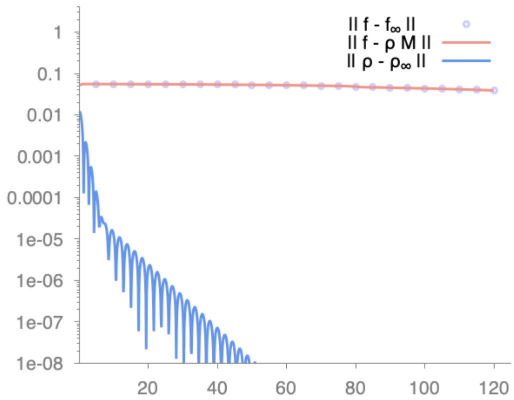


Figure 3.5 – *Plasma echo for $\tau_0 = 10^6$ (weakly collisional regime) : time development of the potential energy (top) and square of the k-th mode of the electric field for $k = 1, ..., 4$ in log scale (bottom) for (a) the linearized Vlasov-Poisson-Fokker-Planck system and (b) the nonlinear Vlasov-Poisson-Fokker-Planck system.*

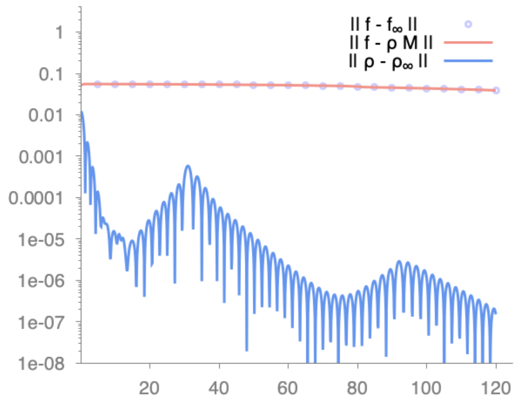
For this weakly collisional regime ($\tau_0 = 10^6$), we also report (on Figure 3.6) the time evolution of the quantity

$$\mathcal{L}_2(t) := \|f - f_\infty\|_{L^2(f_\infty^{-1})}.$$

First, we point out that unlike for potential energies, we do not observe any difference between the behavior of the linearized (3.25) and the nonlinear (3.20)-(3.24) solutions at the level of distribution functions on these charts. Second, we see that on this time interval, collisions are negligible and we clearly see that the distribution function f does not yet converge to f_∞. Figure 3.6 also shows that unlike in the case of strong landau damping, variations of the spatial distribution occurs at an amplitude which is way smaller than the error between kinetic distributions.



(a)



(b)

FIGURE 3.6 – *Plasma echo for* $\tau_0 = 10^6$ *(weakly collisional regime) : time development of* $\|f - f_\infty\|_{L^2(f_\infty^{-1})}$, $\|f - \rho\,\mathcal{M}\|_{L^2(f_\infty^{-1})}$ *and* $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ *in log scale for (a) the linearized Vlasov-Poisson-Fokker-Planck system and (b) the nonlinear Vlasov-Poisson-Fokker-Planck system.*

This can be also observed in Figure 3.7, where we report the snapshots of the difference between the distribution function f solution to the non-linear system (3.20)-(3.24) and the equilibrium f_∞. We first observe the projection of the initial data which exhibits oscillations in velocity and a smooth perturbation in x with a small amplitude of order 10^{-3}. At time $t = 30$ when the echo occurs, we clearly see that the first mode $k_1 = 1$ dominates, then the solution continues to oscillate due to the transport operator in a periodic domain in space.

A natural question in physics is "how to cancel plasma echo ?" for which a natural answer is that collisions may play a role, as shown in several recent articles [10, 68]. To illustrate this phenomenon, we perform new numerical simulations passing from weakly to strongly collisional regime and again compare the two solutions corresponding to the the linearized system (3.25) and the nonlinear one (3.20)-(3.24). The results are now reported

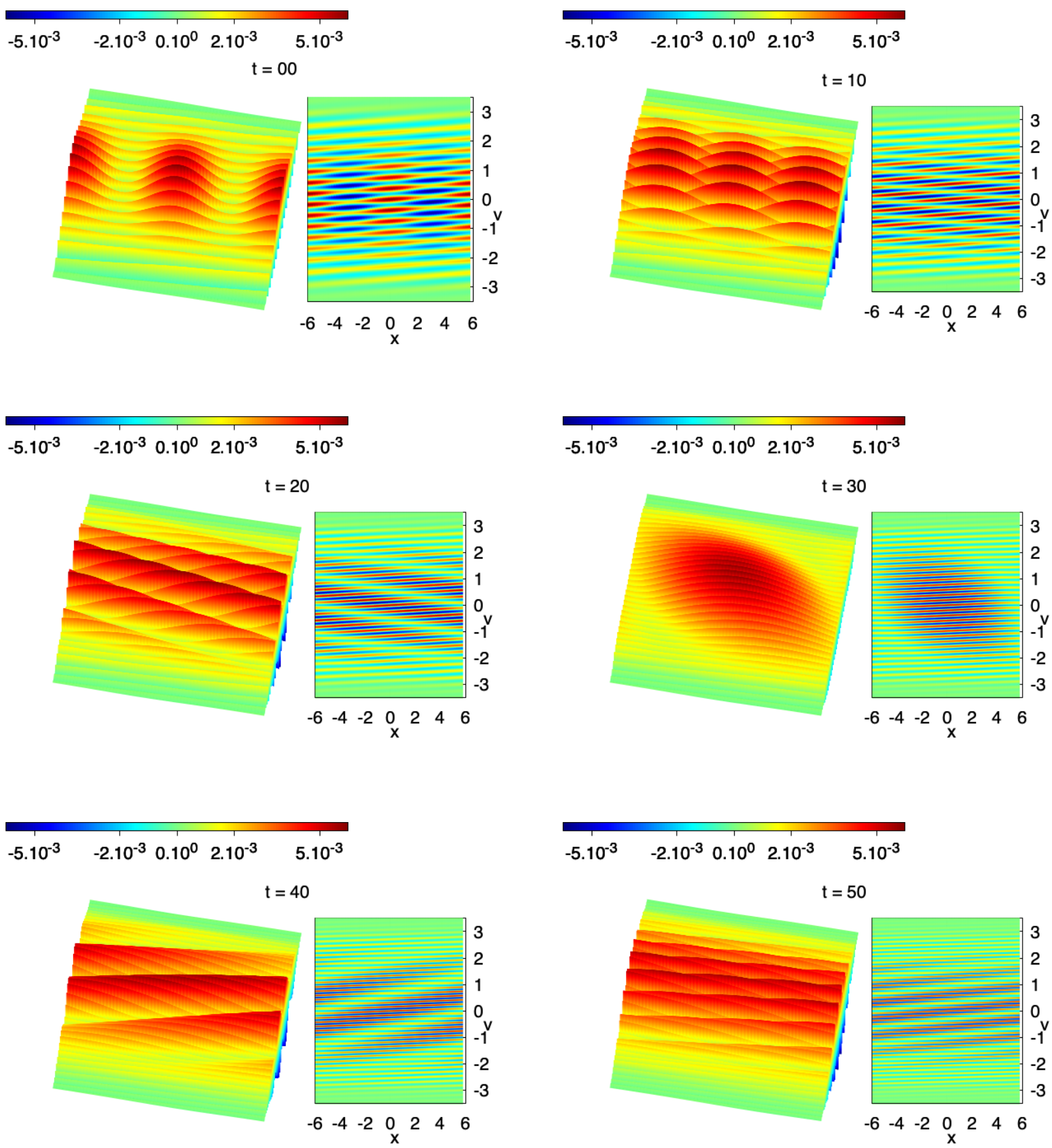
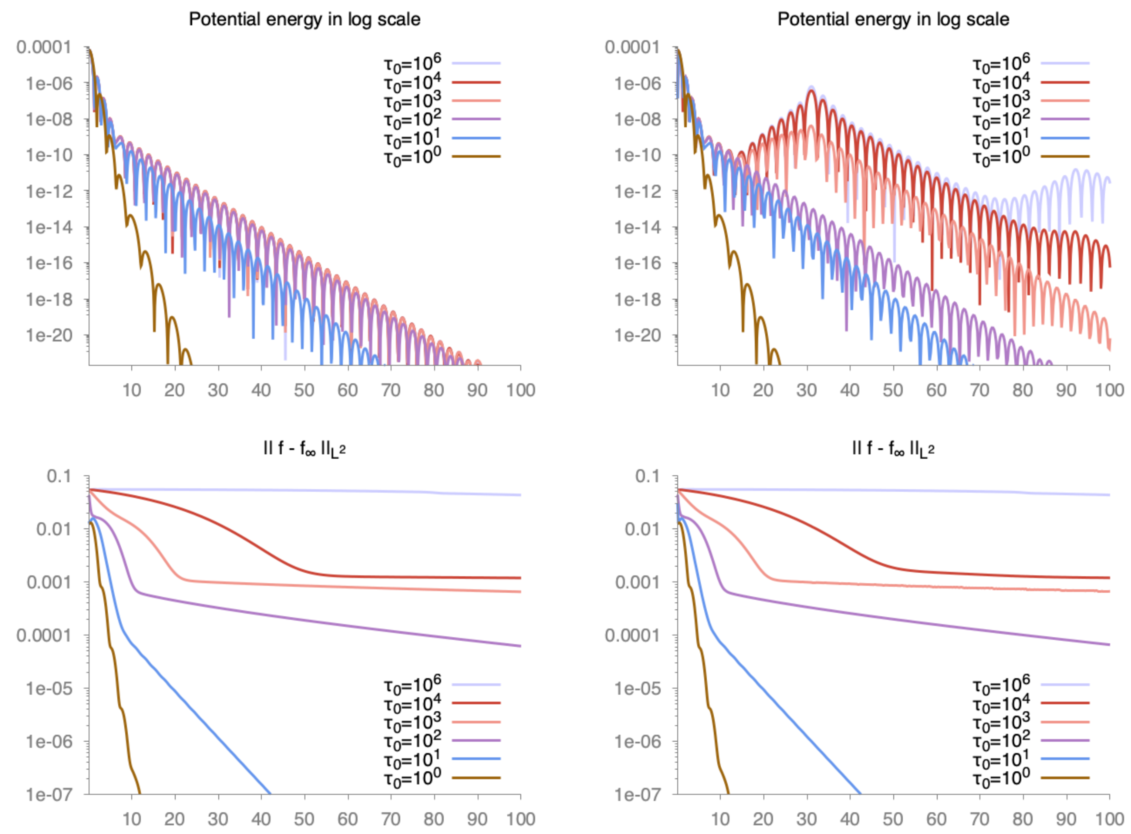


FIGURE 3.7 – *Plasma echo for* $\tau_0 = 10^6$ *(weakly collisional regime) : snapshots of the difference between the solution* f *and the equilibrium* f_∞ *at time* $t = 0$, 20, 30, 40 *and* 50.

in Figure 3.8. Roughly speaking, when $\tau_0 > 10^2$, the nonlinear system exhibits a plasma echo whereas when collisions dominate, the electric field is rapidly damped and the solution converges to its equilibrium f_∞ exponentially fast as predicted by our analysis. It is worth to mention again that at the level of the distribution function, the $\mathcal{L}_2$ time evolution of linearized (3.25) and nonlinear (3.20)-(3.24) solutions are globally the same for various regimes independently of the plasma echo.



(a) (b)

FIGURE 3.8 – *Plasma echo for* $\tau_0 = 1, \ldots, 10^6$ *(various regimes) : time development of the potential energy (top) and* $\|f - f_\infty\|_{L^2(f_\infty^{-1})}$ *(bottom) in log scale for (a) the linearized Vlasov-Poisson-Fokker-Planck system and (b) the nonlinear Vlasov-Poisson-Fokker-Planck system.*

3.4.3 Two streams

In this last experiment, we consider the equilibrium

$$\phi_\infty(x) = 0.1\ (1 - \cos(k\,x))\,, \quad x \in (-L, L),$$

with $k = \pi/L$ with $L = 6$. The equilibrium is therefore given by

$$f_\infty(x,v) \;=\; c_0 \,\exp\left(-\,\phi_\infty(x) \;-\; \frac{|v|^2}{2}\right)\,,$$

where c_0 is renormalizing constant. Furthermore, we consider the initial distribution function as

$$f(t=0,x,v) \;=\; \frac{1}{6\,\sqrt{2\pi}}\;(1+\delta\cos(k\,x))\;(1+5v^2)\,e^{-v^2/2}\,,\quad (x,v)\in(-L,L)\times\mathbb{R},$$

where $\delta = 10^{-2}$. These conditions can be viewed as a perturbation of data leading to the well-known two stream instability when $\phi_\infty \equiv 0$. For this case, we fix the number of Hermite modes at $N_H = 800$ and consider the solution f to the non-linear scheme (3.20)-(3.24). The purpose of this experiment is to compare our result with the classical two stream instability which is usually performed with an homogeneous background distribution and to study the influence of collisions on our results.

Our first comment is that unlike in the classical two streams instability [18], it is not clear that the electric field develops an exponential growth in this case. This may be observed on the left plot of Figure 3.9 considering the curves of the $\|E - E_\infty\|_{L^2(\mathbb{T})}$ in weak and intermediate collisional regimes $\tau_0 = 10^k$, with $2 \le k \le 6$, and also on the bottom charts of Figure 3.10 considering the blue curves which represent the time development of $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ when $\tau_0 = 10^k$, with $k = 2\,,3$.

However, similarly to classical two stream instabilities, we observe that in weakly collisional regimes ($\tau_0 = 10^k$, with $3 \le k \le 6$), the electric field is not damped over time since collisions are negligible on the timescale of our experiments (see Figure 3.9). When collisions are extremely weak $k = 4\,,6$, we even observe oscillations of the electric field (see Figure 3.9).

This similarity with the classical two stream instability may also be observed at the level of kinetic distributions, as we may see on the left hand side of Figure 3.11 where are represented snapshots of $f - f_\infty$ at different times when $\tau_0 = 10^4$. Indeed, we witness the formation of a vortex persisting over time.

When collisions are intense enough, that is $\tau_0 = 10^k$, with $k \le 2$, we observe exponential relaxation towards equilibrium at the level of the electric field (see left chart of Figure 3.9), kinetic distribution (see right chart of Figure 3.9), spatial density and higher Hermite modes (see Figure 3.10). This relaxation may also be observed on Figure 3.11, columns (b) and (c), where the vortex structure is affected by collisions and even canceled completely when $\tau_0 = 10^2$.

Our last comment on this experiment concerns the strongly collisional regime $\tau_0 = 10^k$ when $k = 0\,,1$. A somehow surprising phenomena unfolds since new oscillations appear on all the quantities of interest : electric field (see left chart of Figure 3.9), kinetic distribution (see right chart of Figure 3.9), spatial density and higher Hermite modes (see Figure 3.10, plots (a) and (b)). We have already observed oscillations in a similar setting [25, Section

4.1] where a non-constant stationary force field was applied in strongly collisional settings. However we deal here with a self induced force field whereas [25] focuses on the linear case. These oscillations seem robust enough to persist in the present situation.

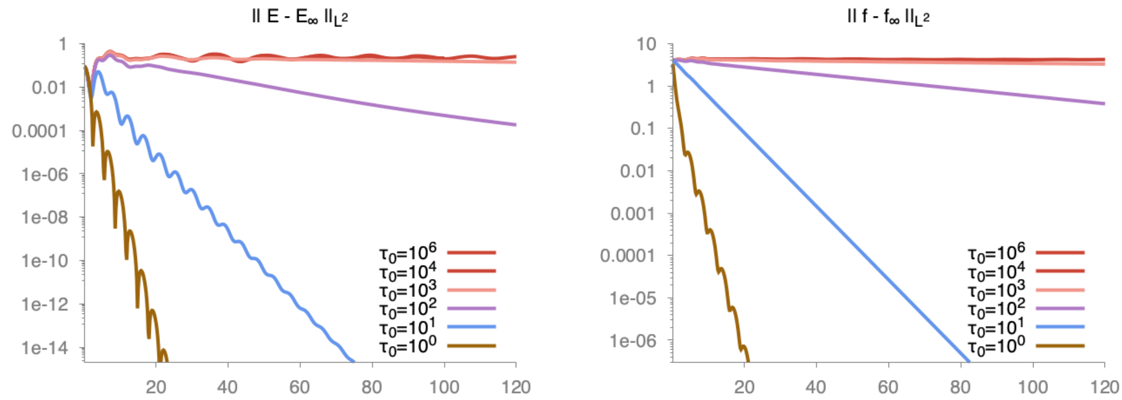
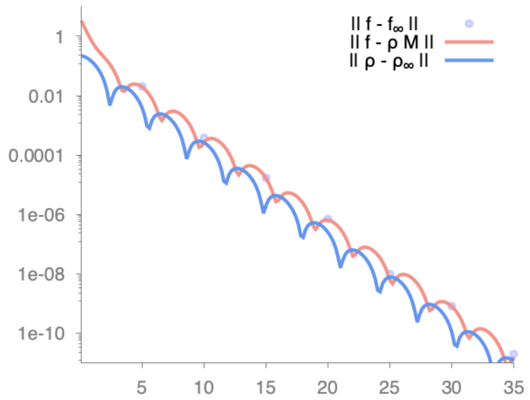


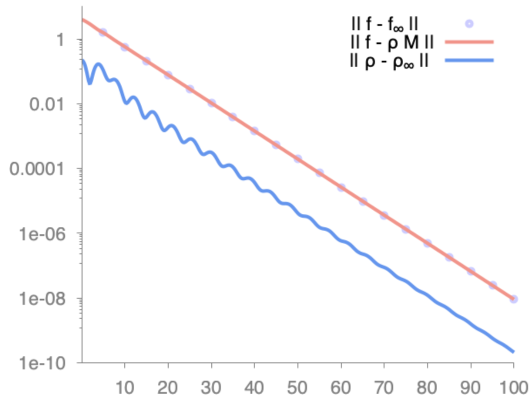
FIGURE 3.9 – *Two streams : time development of (a)* $\|E - E_\infty\|_{L^2(\mathbb{T})}$ *(b) and* $\|f - f_\infty\|_{L^2(f_\infty^{-1})}$ *for various* $\tau_0 = 10^2, \ldots, 10^6$ *(in log scale).*

3.5 Conclusion and perspectives

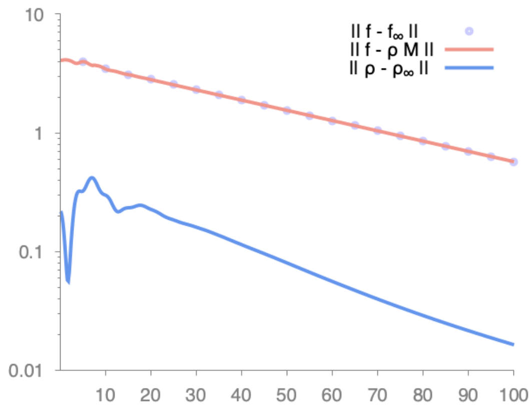
In this work, we proposed a numerical scheme for the Vlasov-Poisson-Fokker-Planck model. On the one hand, we proved that our method is asymptotic preserving in the long time regime for the linearized model. To do so, we derived the exponential relaxation of the numerical solution towards its equilibrium with rates independent of scaling and discretization parameters. On the other hand, we tested the robustness of the method in various numerical experiments. These experiments show the accuracy of our method in both weakly collisional regime where small scales of the system are captured, allowing to reproduce filamentation, vortex formation as well as fine non-linear phenomena such as plasma echoes but also in strong collisional regime, where we witness exponential trend to equilibrium, as predicted by our analysis of the linearized model.
Many interesting perspectives arise from this work. On the theoretical view point, an important continuation consists in extending our results, which apply for a linear coupling with the Poisson equation, to the non-linear scheme (3.20)-(3.24) by proving its asymptotic preserving properties and exponential trend towards equilibrium of discrete solutions. This might be doable in a pertubative setting by controlling the non-linear contribution using discrete Sobolev inequalities. Carrying such proof in higher dimensions $d = 2, 3$ would be a great challenge and would probably require new theoretical tools. Indeed, equivalent



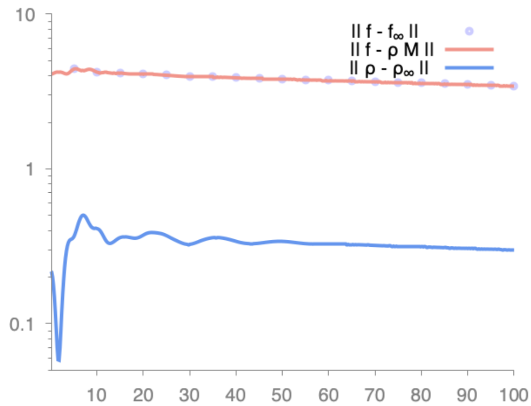
(a) $\tau_0 = 1$



(b) $\tau_0 = 10$

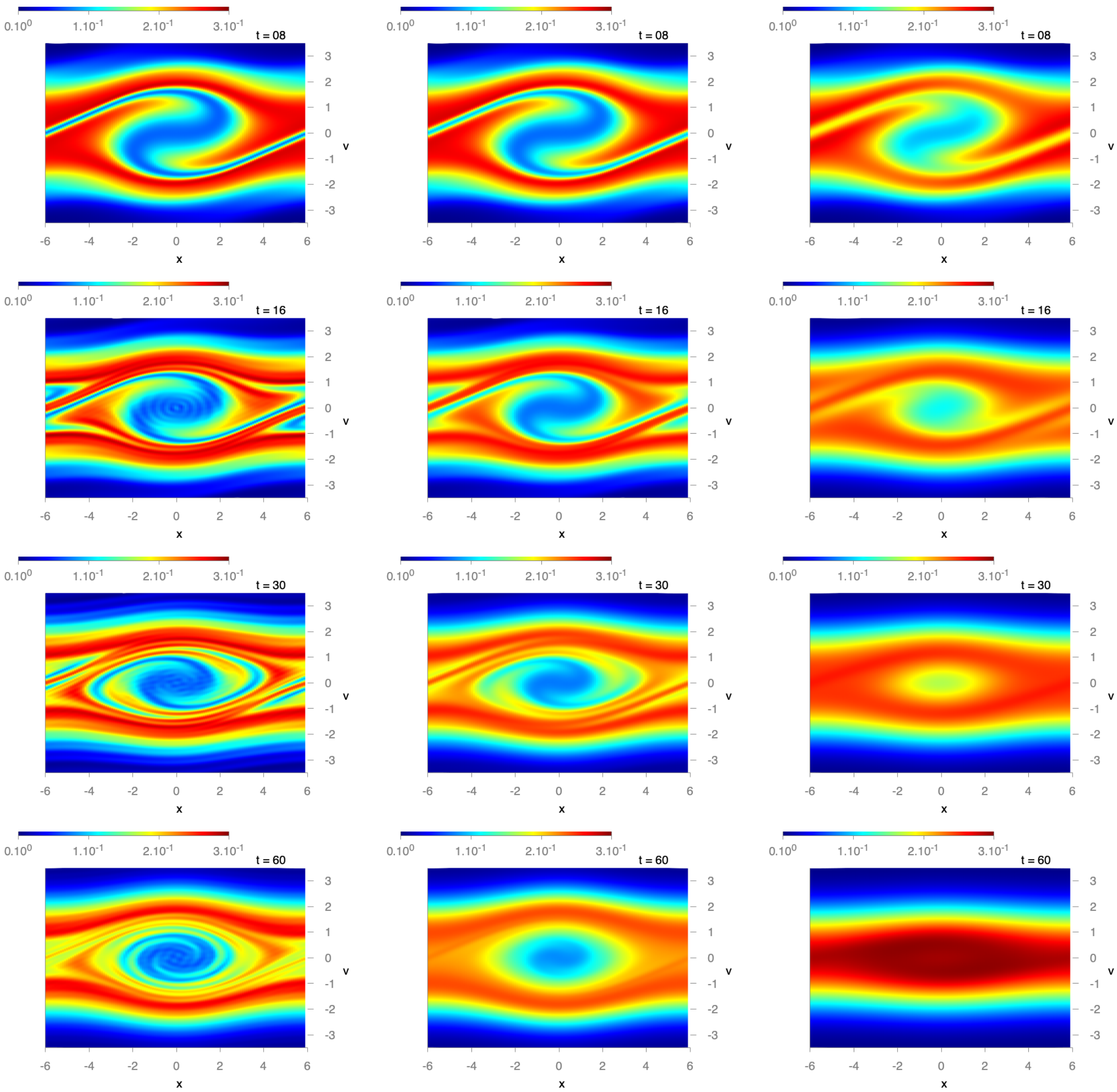


(c) $\tau_0 = 100$



(d) $\tau_0 = 1000$

FIGURE 3.10 – *Two streams : time development of* $\|f - f_\infty\|_{L^2(f_\infty^{-1})}$, $\|f - \rho\,\mathcal{M}\|_{L^2(f_\infty^{-1})}$ *and* $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ *for various* τ_0 *(in log scale).*



(a) $\tau_0 = 10^4$ (b) $\tau_0 = 10^3$ (c) $\tau_0 = 10^2$

FIGURE 3.11 – *Two streams : snapshots of the distribution function f at time $t = 8$, 16, 30 and 60 for various τ_0.*

studies on the continuous model in the literature rely on propagation of regularity [136, 143, 138]. In our case it would require to propagate regularity at the discrete level. The groundwork towards such result has been laid in [25], where we propagated discrete H^1 norms in the linear setting.
Regarding simulations, the studies of echoes also raises interesting perspectives. In [11, 128] were constructed theoretical solutions to the Vlasov-Poisson equation which display infinite cascades of echoes and for which Landau damping is therefore not verified. Furthermore, sharp joint conditions on the collision frequency and the size of the initial data were obtained in order to ensure suppression of these echoes in [10, 68]. Constructing such numerical solutions and illustrating the threshold obtained in these analysis would be of great interest. Another possible continuation would consist in finding a non-homogeneous background configuration where damping phenomena occur as in the homogeneous case, and then construct an experiment where non-linear effect play the main role, even for small perturbation as for plasma echoes.

Acknowledgement

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Annexe A

Annexe of Chapter 1

A.1 Proof of Proposition 1.3

Let us start with the case where exponent p is strictly less than $+\infty$. We compute the time derivative of $\|\rho\|_{L^p}^p$ by multiplying equation (1.5) by ρ^{p-1} and integrating with respect to $\boldsymbol{x}$. After an integration by part, we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\rho\,\|_{L^p}^p\,+\,\Delta_p\,[\,\rho\,]\,=\,\frac{p-1}{p}\,\mathcal{D}\,,$$

where $\mathcal{D}$ is given by

$$\mathcal{D}\,=\,\int_{\mathbb{K}^d}\mathbf{E}\cdot\nabla_{\boldsymbol{x}}\rho^p\,\mathrm{d}\boldsymbol{x}\,.$$

Integrating by part and according to equation (1.5), we rewrite $\mathcal{D}$ as follows

$$\mathcal{D}\,=\,\int_{\mathbb{K}^d}(\rho_i\,-\,\rho)\,\rho^p\,\mathrm{d}\boldsymbol{x}\,=\,\int_{\mathbb{K}^d}\rho_i\,\rho^p\,\mathrm{d}\boldsymbol{x}\,-\,\|\rho\|_{L^{p+1}}^{p+1}\,.$$

To estimate the $\mathcal{D}$, we apply Hölder's inequality

$$\mathcal{D}\,\leq\,\|\rho_i\|_{L^{p+1}}\,\|\rho\|_{L^{p+1}}^{p}\,-\,\|\rho\|_{L^{p+1}}^{p+1}\,,$$

and therefore deduce

$$\mathcal{D}\,\leq\,\|\rho_i\|_{L^{p+1}}^{p+1}\,\mathbb{1}_{\|\rho\|_{L^{p+1}}\,\leq\,\|\rho_i\|_{L^{p+1}}}\,.$$

Furthermore, we use that ρ is a probability measure to apply Jensen's inequality and we deduce

$$\|\rho\|_{L^p}\,\leq\,\|\rho\|_{L^{p+1}}^{(p^2-1)/p^2}\,.$$

Injecting this inequality in the latter estimate on $\mathcal{D}$, we obtain

$$\mathcal{D}\,\leq\,\|\rho_i\|_{L^{p+1}}^{p+1}\,\mathbb{1}_{\|\rho\|_{L^p}\,\leq\,\|\rho_i\|_{L^{p+1}}^{(p^2-1)/p^2}}\,.$$

Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \| \rho \|_{L^p}^p \leq (p-1) \|\rho_i\|_{L^{p+1}}^{p+1} \mathbb{1}_{\|\rho\|_{L^p}^p \leq \|\rho_i\|_{L^{p+1}}^{(p^2-1)/p}} .$$

One can check that for any positive η, the constant

$$C_\eta = \max\left(\|\rho_0\|_{L^p}^p , \|\rho_i\|_{L^{p+1}}^{(p^2-1)/p} \right) + \eta$$

is a super solution to the latter differential inequality. Therefore, it holds for all $\eta > 0$

$$\| \rho \|_{L^p}^p \leq C_\eta .$$

Hence, taking $\eta \to 0$, we obtain the expected result

$$\| \rho \|_{L^p} \leq \max\left(\|\rho_0\|_{L^p} , \|\rho_i\|_{L^{p+1}}^{1-1/p^2} \right),$$

for all time $t \geq 0$. The case $p = +\infty$ is obtained taking the limit $p \to +\infty$ in the latter estimate.

Deuxième partie

Neurosciences

Chapitre 4

Concentration phenomena in FitzHugh-Nagumo's equations : a mesoscopic approach

We consider a spatially extended mesoscopic FitzHugh-Nagumo model with strong local interactions and prove that its asymptotic limit converges towards the classical nonlocal reaction-diffusion FitzHugh-Nagumo system. As the local interactions strongly dominate, the weak solution to the mesoscopic equation under consideration converges to the local equilibrium, which has the form of Dirac distribution concentrated to an averaged membrane potential.
Our approach is based on techniques widely developed in kinetic theory (Wasserstein distance, relative entropy method), where macroscopic quantities of the mesoscopic model are compared with the solution to the nonlocal reaction-diffusion system. This approach allows to make the rigorous link between microscopic and reaction-diffusion models.

This work has been published in *SIAM J. on Mathematical Analysis, 55 (1), pages 367-404, (2023)*, with Francis Filbet.

Contents

4.1 Introduction

4.1.1 Physical model and motivations

Neuron models often focus on the dynamics of the electrical potential through the membrane of a nerve cell. These dynamics are driven by ionic exchanges between the neuron and its environment through its cellular membranes. A very precise modeling of these ion exchanges led to the well-known Hodgkin-Huxley model [140]. In this paper, we shall focus on a simplified version, called the FitzHugh-Nagumo model [114] [174], which keeps its most valuable aspects and remains relatively simple mathematically. More precisely, the FitzHugh-Nagumo model accounts for the variations of the membrane potential $v \in \mathbb{R}$ of a neuron coupled to an auxiliary variable $w \in \mathbb{R}$ called the adaptation variable. It is usually written as follows :

$$\left\{ \begin{array}{l} \mathrm{d}v_t \,=\, (N(v_t) \,-\, w_t \,+\, I_{ext}) \,\mathrm{d}t \,+\, \sqrt{2}\,\mathrm{d}B_t\,, \\ \mathrm{d}w_t \,=\, A\,(v_t, w_t)\,\mathrm{d}t\,, \end{array} \right.$$

where the drift N is a confining non-linearity with the following typical form

$$N(v) \,=\, v \,-\, v^3\,,$$

even though a broader class of drifts N is considered here. On the other hand, the drift A is an affine mapping that has the following form

$$A(v,w) \,=\, a\,v \,-\, b\,w \,+\, c\,,$$

where a, $c \in \mathbb{R}$ and $b > 0$, which means that A also has some confining properties. Here, the Brownian motion B_t has been added in order to take into account random fluctuations in the dynamics of the membrane potential v_t. Another mathematical reason for looking at this system is that it is a prototypical model of excitable kinetics. Interest in such systems stems from the fact that although the kinetics are relatively simple, couplings between neurons can produce complex dynamics, where well-known examples are the propagation of excitatory pulses, spiral waves in two-dimensions, and spatio-temporal chaos. Here, we introduce coupling through the input current I_{ext}. More specifically, we consider that neurons interact with one another following Ohm's law and that the conductance between two neurons depends on there spatial location $\boldsymbol{x} \in K$, where K is a compact set of $\mathbb{R}^d$. The conductance between two neurons is given by a connectivity kernel $\Phi : K \times K \to \mathbb{R}$. Hence, in the case of a network composed with n interacting neurons described by the triplet voltage-adaptation-position $(v_i, w_i, \boldsymbol{x}_i)_{1\leq i\leq m}$, the current received by neuron i from the other neurons is given by

$$I_{ext} \;=\; -\frac{1}{m}\sum_{j=1}^{m} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)\,(v_t^i - v_t^j)\,,$$

where the scaling parameter m is introduced here to re-normalize the contribution of each neuron. According to the former discussion, a neural network of size m is described by the system of equations

$$\begin{cases} \mathrm{d}v_t^i \;=\; \left(N(v_t^i) \;-\; w_t^i \;-\; \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)\,(v_t^i - v_t^j)\right)\,\mathrm{d}t \;+\; \sqrt{2}\,\mathrm{d}B_t^i\,, \\[2ex] \mathrm{d}w_t^i \;=\; A\left(v_t^i, w_t^i\right)\,\mathrm{d}t\,, \end{cases}$$

where $i \in \{1, ..., m\}$. In the formal limit $m \to +\infty$, the behavior of the latter system may be described by the evolution of a distribution function $f := f(t, \boldsymbol{x}, \boldsymbol{u})$, with $\boldsymbol{u} = (v, w) \in \mathbb{R}^2$, representing the density of neurons at time t, position $\boldsymbol{x} \in K$ with a membrane potential v and adaptation variable $w \in \mathbb{R}$. It turns out that the distribution function f solves the following mean-field equation

$$\partial_t f \;+\; \partial_v\left((N(v) \;-\; w \;-\; \mathcal{K}_\Phi[f])\; f\right) \;+\; \partial_w\left(A(v,w) f\right) \;-\; \partial_v^2 f \;=\; 0\,,$$

where the operator $\mathcal{K}_\Phi[f]$ takes into account spatial interactions and is given by

$$\mathcal{K}_\Phi[f](t, \boldsymbol{x}, v) \;=\; \int_{K\times\mathbb{R}^2} \Phi(\boldsymbol{x}, \boldsymbol{x}')\,(v - v')\, f(t, \boldsymbol{x}', \boldsymbol{u}')\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{u}'\,.$$

See for instance [3, 30, 168, 161] for more details on the mean field limit for the FitzHugh-Nagumo system and [26] for a related model in collective dynamics.

Various other types of kinetic models have been derived during the past decades depending on the hypotheses assumed for the dynamics of the emission of an action potential. They include for example integrate-and-fire neural networks [41, 44, 56] and time-elapsed neuronal models [181, 71, 70, 72].

Let us be more specific on the modeling of interactions between neurons. A common assumption consists in considering that there are two types of interactions : strong short range interactions and weak long range interactions (see [39], [196] & [161]). Here we consider a connectivity kernel of the following type

$$\Phi^\varepsilon(\boldsymbol{x}, \boldsymbol{x}') \,=\, \frac{1}{\varepsilon}\delta_0(\boldsymbol{x}-\boldsymbol{x}') \,+\, \Psi(\boldsymbol{x}, \boldsymbol{x}')\,, \tag{4.1}$$

where the Dirac mass δ_0 accounts for strong short range interactions with strength $\varepsilon > 0$, whereas the connectivity kernel $\Psi : K \times K \to \mathbb{R}$ is more regular and represents weak long range interactions.

The purpose of this article is to go through the mathematical analysis of the neural network in the regime of strong interactions, that is when $\varepsilon \ll 1$. More precisely, we prove that the voltage distribution concentrates to a Dirac mass by providing a comprehensive description of this concentration phenomenon.

4.1.2 Formal derivation

Our problem is multiscale due to interactions between neurons, which induce macroscopic effects at the mesoscopic level. Consequently, we introduce integrated quantities. First, we consider the spatial distribution of neurons throughout the network

$$\rho_0^\varepsilon(\boldsymbol{x}) = \int_{\mathbb{R}^2} f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,.$$

It is straightforward to check that ρ_0^ε is indeed time-homogeneous, integrating the mean field equation with respect to $\boldsymbol{u} \in \mathbb{R}^2$. Second, we introduce the averaged voltage $\mathcal{V}^\varepsilon$ and adaptation variable $\mathcal{W}^\varepsilon$ at a spatial location $\boldsymbol{x}$

$$\begin{cases} \rho_0^\varepsilon\,(\boldsymbol{x})\; \mathcal{V}^\varepsilon(t,\, \boldsymbol{x}) & = \displaystyle\int_{\mathbb{R}^2} v\; f^\varepsilon(t,\, \boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,, \\[2ex] \rho_0^\varepsilon\,(\boldsymbol{x})\; \mathcal{W}^\varepsilon(t,\, \boldsymbol{x}) & = \displaystyle\int_{\mathbb{R}^2} w\; f^\varepsilon(t,\, \boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,. \end{cases} \tag{4.2}$$

In the sequel, we use the vector notation $\mathcal{U}^\varepsilon \,=\, (\,\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon\,)$. At the mesoscopic level, we compare probability density functions using the Wasserstein distances. Hence, we renormalize f^ε as

$$\rho_0^\varepsilon\, \mu^\varepsilon \,=\, f^\varepsilon\,,$$

where μ^ε is a non-negative function which lies in $\mathscr{C}^0\left(\mathbb{R}^+ \times K\,,\, L^1\left(\mathbb{R}^2\right)\right)$ and verifies

$$\int_{\mathbb{R}^2} \mu^\varepsilon(t,\, \boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u} \,=\, 1, \quad \forall\,(t,\, \boldsymbol{x}) \,\in\, \mathbb{R}^+ \times K\,.$$

Consequently, we denote $\mu^\varepsilon_{t,\,\boldsymbol{x}}$ the probability density function defined as $\mu^\varepsilon_{t,\,\boldsymbol{x}} \,=\, \mu^\varepsilon\,(t,\, \boldsymbol{x}, \cdot)$. With these notations and our modeling assumptions on the connectivity kernel Φ^ε defined by (4.1), the mean-field equation rewrites

$$\partial_t\, \mu^\varepsilon \,+\, \mathrm{div}_{\boldsymbol{u}}\,[\,\mathbf{b}^\varepsilon \mu^\varepsilon\,] \,-\, \partial_v^2\, \mu^\varepsilon \,=\, \frac{1}{\varepsilon}\, \rho_0^\varepsilon\, \partial_v\,[\,(v - \mathcal{V}^\varepsilon)\, \mu^\varepsilon\,]\,, \tag{4.3}$$

where $\mathbf{b}^\varepsilon$ is defined for all $(t,\, \boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$ as

$$\mathbf{b}^\varepsilon(t,\, \boldsymbol{x}, \boldsymbol{u}) \,=\, \begin{pmatrix} N(v) \,-\, w \,-\, \mathcal{K}_\Psi[\rho_0^\varepsilon\, \mu^\varepsilon]\,(t,\, \boldsymbol{x}, v) \\ A(\boldsymbol{u}) \end{pmatrix}.$$

Furthermore, one can notice that the non local term $\mathcal{K}_\Psi[\rho_0^\varepsilon\, \mu^\varepsilon]$ can be expressed in terms of the macroscopic quantities

$$\mathcal{K}_\Psi[\rho_0^\varepsilon\, \mu^\varepsilon](t,\, \boldsymbol{x}, v) \,=\, \Psi *_r \rho_0^\varepsilon(\boldsymbol{x})\, v \,-\, \Psi *_r (\rho_0^\varepsilon\, \mathcal{V}^\varepsilon)(t,\, \boldsymbol{x})\,,$$

where $*_r$ is a shorthand for the convolution on the right side of any function g with Ψ

$$\Psi *_r g(\boldsymbol{x}) \,=\, \int_K \Psi(\boldsymbol{x}, \boldsymbol{x}')\, g(\boldsymbol{x}')\, \mathrm{d}\boldsymbol{x}'\,.$$

Coming back to the analysis of the strong interaction regime, we look for the leading order in (4.3). In our case, it is induced by strong short range interactions between neurons, and as $\varepsilon \to 0$, we expect

$$(v \,-\, \mathcal{V}^\varepsilon)\, \mu^\varepsilon \,=\, 0\,,$$

which means that μ^ε concentrates around its mean value with respect to the voltage variable at each spatial location $\boldsymbol{x}$ in K, that is, μ^ε converges to a Dirac mass centred in $\mathcal{V}^\varepsilon$. Thus, to quantify the asymptotic behavior of μ^ε when $\varepsilon \ll 1$, we denote by $\mathscr{P}_2(\mathbb{R}^2)$ the set of probability laws with finite second order moments

$$\mathscr{P}_2\left(\mathbb{R}^2\right) \,=\, \left\{\mu \in \mathscr{P}(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} |\boldsymbol{u}|^2 \mathrm{d}\mu(\boldsymbol{u}) \,<\, +\infty\right\}$$

and the Wasserstein distance of order two W_2 defined as follows : for any μ and ν probability measures in $\mathscr{P}_2(\mathbb{R}^2)$,

$$W_2^2\,(\mu, \nu) \,=\, \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^4} |\boldsymbol{u} - \boldsymbol{u}'|^2 \mathrm{d}\pi(\boldsymbol{u}, \boldsymbol{u}')\,,$$

where $\Pi(\mu, \nu)$ stands for the set of distributions π over $\mathbb{R}^4$ with marginals μ with respect to $\boldsymbol{u}$ and ν with respect to $\boldsymbol{u}'$, that is, for $\pi \in \Pi(\mu, \nu)$

$$\pi\left(A,\, \mathbb{R}^2\right) = \mu(A) \quad \text{and} \quad \pi\left(\mathbb{R}^2,\, A\right) = \nu(A)\,,$$

for any Borel set $A \subset \mathbb{R}^2$. The choice of the W_2 metric in our analysis is somehow arbitrary as it might be possible to adapt our approach to other distances metrizing probability spaces. However, it is easy to quantify the distance between μ^ε and a Dirac mass in this framework since we have

$$W_2^2\left(\mu^\varepsilon_{t,\boldsymbol{x}},\, \delta_{\mathcal{V}^\varepsilon} \otimes \bar{\mu}^\varepsilon_{t,\boldsymbol{x}}\right) \,=\, \int_{\mathbb{R}^2} |v - \mathcal{V}^\varepsilon|^2\, \mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u}) \mathrm{d}\boldsymbol{u}\,,$$

where $\bar{\mu}^\varepsilon$ is defined as the marginal of μ^ε with respect to the voltage variable

$$\bar{\mu}^\varepsilon_{t,\boldsymbol{x}}(w) \,=\, \int_{\mathbb{R}} \mu^\varepsilon_{t,\boldsymbol{x}}(v,w)\,\mathrm{d}v\,.$$

Hence, we multiply equation (4.3) by $|v-\mathcal{V}^\varepsilon|^2$ and integrate with respect to $\boldsymbol{u}\in\mathbb{R}^2$. We obtain that at each spatial location $\boldsymbol{x}$ in K, we have (see Proposition 4.12 below for a more precise estimate)

$$W_2\left(\mu^\varepsilon_{t,\boldsymbol{x}},\, \delta_{\mathcal{V}^\varepsilon(t,\boldsymbol{x})}\otimes\,\bar{\mu}^\varepsilon_{t,\boldsymbol{x}}\right) \underset{\varepsilon\to 0}{\sim} \sqrt{\varepsilon}\,.$$

Considering this estimate, we infer that the dynamics of the network when $\varepsilon\ll 1$ are driven by the couple $(\mathcal{V}^\varepsilon,\bar{\mu}^\varepsilon)$, which displays both the macroscopic & the mesoscopic scale. We complete this step of our analysis by deriving the limit of $(\mathcal{V}^\varepsilon,\bar{\mu}^\varepsilon)$. Multiplying equation (4.3) by v (resp. 1) and then integrating over $\boldsymbol{u}\in\mathbb{R}^2$ (resp. $v\in\mathbb{R}$), we obtain that the couple $(\mathcal{V}^\varepsilon,\bar{\mu}^\varepsilon)$ solves the following system

$$\left\{\begin{array}{l}\partial_t\mathcal{V}^\varepsilon \,=\, N(\mathcal{V}^\varepsilon) \,-\, \mathcal{W}^\varepsilon \,-\, \mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon] \,+\, \mathcal{E}(\mu^\varepsilon)\,,\\[2mm] \partial_t\bar{\mu}^\varepsilon \,+\, \partial_w\left(a\displaystyle\int_{\mathbb{R}} v\mu^\varepsilon\mathrm{d}v \,-\, b\,w\,\bar{\mu}^\varepsilon \,+\, c\,\bar{\mu}^\varepsilon\right) = 0\,,\end{array}\right. \tag{4.4}$$

with

$$\mathcal{W}^\varepsilon \,=\, \int_{\mathbb{R}} w\,\bar{\mu}^\varepsilon\,\mathrm{d}w\,.$$

In equation (4.4), $\mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon]$ is a non local operator given by

$$\mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon] = \mathcal{V}^\varepsilon\,\Psi *_r \rho_0^\varepsilon \,-\, \Psi *_r (\rho_0^\varepsilon\mathcal{V}^\varepsilon)\,,$$

and the error term $\mathcal{E}(\mu^\varepsilon)$ is given by

$$\mathcal{E}(\mu^\varepsilon) \,=\, \int_{\mathbb{R}^2} N(v)\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \,-\, N(\mathcal{V}^\varepsilon)\,. \tag{4.5}$$

Before computing the limit for $(\mathcal{V}^\varepsilon,\bar{\mu}^\varepsilon)$, we emphasize that multiplying the second equation in (4.4) by w and integrating with respect to $w\in\mathbb{R}$, we get a closed equation for $\mathcal{W}^\varepsilon$ since A is affine,

$$\partial_t\mathcal{W}^\varepsilon \,=\, A\left(\mathcal{V}^\varepsilon,\mathcal{W}^\varepsilon\right)\,.$$

Both equations on $\mathcal{V}^\varepsilon$ and $\bar{\mu}^\varepsilon$ in (4.4) depend on the distribution function μ^ε. However since our interest here lies in the regime of strong interactions, we replace μ^ε in (4.4) by the *ansatz*

$$\mu^\varepsilon \underset{\varepsilon\to 0}{=} \delta_{\mathcal{V}^\varepsilon}\otimes\bar{\mu}^\varepsilon \,+\, O(\sqrt{\varepsilon})\,.$$

This removes the dependence with respect to μ^ε from the system (4.4). Indeed, we obtain on the one hand (see Proposition 4.14 for more details)

$$(\mathcal{V}^\varepsilon,\,\mathcal{W}^\varepsilon) \underset{\varepsilon\to 0}{=} (\mathcal{V},\,\mathcal{W}) \,+\, O\left(\varepsilon\right)\,,\quad \text{as}\quad \varepsilon\to 0\,,$$

and on the other hand

$$\bar{\mu}^\varepsilon \underset{\varepsilon\to 0}{=} \bar{\mu} \;+\; O\left(\sqrt{\varepsilon}\right), \quad \text{as} \quad \varepsilon \to 0\,,$$

where the couple $(\mathcal{V}, \bar{\mu})$ solves the following system

$$\begin{cases} \partial_t \mathcal{V} \;=\; N(\mathcal{V}) \;-\; \mathcal{W} \;-\; \mathcal{L}_{\rho_0}[\mathcal{V}]\,, \\[2mm] \partial_t \bar{\mu} \;+\; \partial_w \left(A(\mathcal{V}\,,\, w)\, \bar{\mu}\right) \;=\; 0\,, \end{cases} \tag{4.6}$$

with

$$\mathcal{W} \;=\; \int_{\mathbb{R}} w \, \mathrm{d}\bar{\mu}_{t,\,\boldsymbol{x}}(w)\,.$$

Similar results have already been obtained in a deterministic setting in [79] using relative entropy methods. In the end, it can be proven that μ^ε converges to a mono-kinetic distribution in v with mean $\mathcal{V}$ and we get the following rate of convergence (see Corollary 4.9 for more details)

$$W_2\left(\mu^\varepsilon_{t,\,\boldsymbol{x}},\, \delta_{\mathcal{V}(t,\,\boldsymbol{x})} \otimes \bar{\mu}_{t,\,\boldsymbol{x}}\right) \underset{\varepsilon\to 0}{=} O(\sqrt{\varepsilon})\,.$$

Actually, this latter convergence estimate corresponds to the expansion at order 0 of f^ε in the regime of strong interactions.

4.1.3 Introduction of rescaled variables

In this paper, we introduce some rescaled variables, in order to study more precisely the asymptotic behavior of the solution and to improve the order of convergence. The strategy consists in finding the concentration's profile with respect to the potential variable v. This leads to considering the following re-scaled version ν^ε of μ^ε

$$\mu^\varepsilon(t,\, \boldsymbol{x}, v, w) \;=\; \frac{1}{\varepsilon^\alpha}\, \nu^\varepsilon\left(t,\, \boldsymbol{x},\, \frac{v - \mathcal{V}^\varepsilon}{\varepsilon^\alpha},\, w - \mathcal{W}^\varepsilon\right),$$

where ε^α is the concentration rate of μ^ε around its mean value $\mathcal{V}^\varepsilon$ and α needs to be determined. For a proper choice of α, we expect ν^ε to converge to some limit as ε vanishes. This limit will be interpreted as the concentration profile of the voltage's distribution throughout the network in the regime of strong interaction.

In order to determine the proper concentration exponent α, we derive the equation solved by ν^ε. To this aim, we perform the following change of variable

$$(v,\, w) \mapsto \left(\frac{v - \mathcal{V}^\varepsilon}{\varepsilon^\alpha},\, w - \mathcal{W}^\varepsilon\right) \tag{4.7}$$

in equation (4.3) and use the first equation of (4.4) on $\mathcal{V}^\varepsilon$. It yields

$$\partial_t\, \nu^\varepsilon \;+\; \mathrm{div}_{\boldsymbol{u}}\left[\, \mathbf{b}_0^\varepsilon\, \nu^\varepsilon\right] \;=\; \frac{1}{\varepsilon^{2\alpha}} \partial_v \left[\varepsilon^{2\alpha-1}\, \rho_0^\varepsilon\, v\, \nu^\varepsilon + \partial_v\, \nu^\varepsilon\right],$$

where $\mathbf{b}_0^\varepsilon$ is a centered version of $\mathbf{b}^\varepsilon$ and is given by

$$\mathbf{b}_0^\varepsilon\,(t,\,\boldsymbol{x},\,\boldsymbol{u})\;=\;\begin{pmatrix}\varepsilon^{-\alpha}\,(\,N(\mathcal{V}^\varepsilon\,+\,\varepsilon^\alpha v)\,-\,N(\mathcal{V}^\varepsilon)\,-\,w\,-\,\varepsilon^\alpha v\,\Psi *_r \rho_0^\varepsilon(\boldsymbol{x})\,-\,\mathcal{E}\,(\mu^\varepsilon))\\ A_0(\varepsilon^\alpha v,\,w)\end{pmatrix},$$

and where A_0 is the linear version of A

$$A_0(\boldsymbol{u}) = A(\boldsymbol{u}) - A(\mathbf{0})\,.$$

It turns out that the only suitable value for α is 1/2. Indeed, when α is less than 1/2, we check that ν^ε converges towards a Dirac mass, which means that the scaling is not precise enough. On the contrary, when $\alpha > 1/2$, ν^ε converges to 0, which means that we "zoom in" too much. Hence we obtain the following equation

$$\partial_t\,\nu^\varepsilon\;+\;\mathrm{div}_{\boldsymbol{u}}\,[\,\mathbf{b}_0^\varepsilon\,\nu^\varepsilon\,]\;=\;\frac{1}{\varepsilon}\,\partial_v\,[\,\rho_0^\varepsilon\,v\,\nu^\varepsilon+\partial_v\,\nu^\varepsilon\,]\;, \tag{4.8}$$

where we take $\alpha = 1/2$ in the definition of $\mathbf{b}_0^\varepsilon$. Keeping only the leading order, it yields that

$$\nu_{t,\boldsymbol{x}}^\varepsilon(v,w)\underset{\varepsilon\to 0}{=}\mathcal{M}_{\rho_0^\varepsilon(\boldsymbol{x})}(v)\otimes\bar{\nu}_{t,\boldsymbol{x}}^\varepsilon(w)\;+\;O(\sqrt{\varepsilon})\,,$$

where the Maxwellian $\mathcal{M}_{\rho_0^\varepsilon}$ is defined as

$$\mathcal{M}_{\rho_0^\varepsilon}(v) = \sqrt{\frac{\rho_0^\varepsilon}{2\pi}}\exp\left(-\frac{\rho_0^\varepsilon\,|v|^2}{2}\right)\,,$$

whereas $\bar{\nu}^\varepsilon$ is the marginal of ν^ε with respect to the re-scaled adaptation variable

$$\bar{\nu}_{t,\boldsymbol{x}}^\varepsilon(w)\;=\;\int_{\mathbb{R}}\nu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\,\mathrm{d}v\,.$$

At this point, it is possible to answer our initial concern : μ^ε concentrates with Gaussian profile as $\varepsilon \to 0$. Then we complete the analysis by deriving the limit of $\bar{\nu}^\varepsilon$: integrating equation (4.8) with respect to the re-scaled voltage variable v, we obtain the equations solved by $\bar{\nu}^\varepsilon$,

$$\partial_t\bar{\nu}^\varepsilon\;-\;b\,\partial_w\,(w\,\bar{\nu}^\varepsilon)\;=\;-a\,\sqrt{\varepsilon}\,\partial_w\int_{\mathbb{R}}v\,\nu_{t,\boldsymbol{x}}^\varepsilon(\mathrm{d}v,w)\,.$$

Once again, the equation still depends on ν^ε through the source term in the right-hand side. However we obtain that it is in fact of order ε when we replace ν^ε with the following *ansatz*

$$\nu^\varepsilon\underset{\varepsilon\to 0}{=}\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon\;+\;O(\sqrt{\varepsilon})\,.$$

Consequently, we expect the following convergence

$$\bar{\nu}^\varepsilon\underset{\varepsilon\to 0}{=}\bar{\nu}\;+O(\varepsilon)\,,$$

where $\bar{\nu}$ solves the following linear transport equation

$$\partial_t \bar{\nu} \,-\, b\, \partial_w \left(w\, \bar{\nu} \right) \,=\, 0\,,$$

which corresponds to the same equation as (4.6) for $\bar{\mu}$ after inverting the change of variable (4.7).

We come back to our initial problem, which consisted in building a precise model for the dynamics of μ^ε in the regime of strong interactions. We invert the change of variable (4.7) and obtain in the end

$$W_2 \left(\mu^\varepsilon_{t,\boldsymbol{x}},\, \mathcal{M}_{\frac{1}{\varepsilon}\rho_0(\boldsymbol{x})\,,\mathcal{V}(t,\boldsymbol{x})} \otimes \bar{\mu}_{t,\boldsymbol{x}} \right) \underset{\varepsilon \to 0}{=} O(\varepsilon)\,,$$

where $\mathcal{M}_{\rho_0/\varepsilon,\mathcal{V}}$ is given by $\mathcal{M}_{\rho_0/\varepsilon,\mathcal{V}}\,(v) \,=\, \mathcal{M}_{\rho_0/\varepsilon}\,(v - \mathcal{V})$. This result should be regarded as the expansion of μ^ε at order 1 in the regime of strong interactions. It may be compared with the expansion of μ^ε at order 0. Furthermore, it enables us to characterize the blow up profile of the distribution function μ^ε and to improve the order of convergence.

Our result is in line with a broader collection of publications, which focus on the mathematical analysis of the dynamics in a FitzHugh-Nagumo neural networks with strongly interacting neurons. First, we mention [196], in which similar results are obtained following a Hamilton Jacobi approach. The authors study the so-called Hopf-Cole transform ϕ^ε of μ^ε defined by the following *ansatz*

$$\mu^\varepsilon \,=\, \exp\left(\phi^\varepsilon/\varepsilon\right).$$

However, due to this *ansatz*, authors deal with well prepared initial condition in the sense that it is already concentrated at time $t = 0$. In our case, we lift this assumption and deal with non-concentrated initial data. Furthermore, the results are stated in a spatially homogeneous setting and the limiting distribution $\bar{\mu}$ of the adaptation variable is not identified in [196]. Secondly, our work follows on from [79], which focuses on the expansion of μ^ε at order 0 in a deterministic setting. On top of that, we cite [26] which deals with mean-field limit in the context of collective dynamics. This article locates itself in a probability framework and the authors develop mathematical methods based on the Wasserstein distance and similar to the ones in the present article. However, the authors adopt a stochastic point of view whereas we focus on the analytic point of view all along this paper. To end with, we mention [117], where methods related to Wasserstein distances are reviewed for a broad class of models.

The rest of the paper consists in making the asymptotic expansion rigorous. In the next section, we state our assumptions on the parameters of our problem : N, Ψ and $(\mu_0^\varepsilon)_{\varepsilon>0}$. Then we state the main result, Theorem 4.7. Section 4.3 is devoted to *a priori* estimates on the solutions $(\mu^\varepsilon)_{\varepsilon>0}$, whereas Section 4.4 contains the proof of Theorem 4.7. Finally, in the Appendix, we give the main ingredients to prove existence and uniqueness of a solution μ^ε to equation (4.3) for any $\varepsilon > 0$. We mention that even though this well posedness result is not our main concern here, we develop interesting arguments using a modified relative entropy which might have other applications.

4.2 Mathematical setting & main results

In this section, we give the precise mathematical setting of our analysis and present our main results on the profile of the distribution function f^ε when $\varepsilon \ll 1$.

4.2.1 Mathematical setting

We suppose the drift N to be of class $\mathscr{C}^2$ over $\mathbb{R}$. Then we set $\omega(v) = N(v)/v$ and suppose that the following coupled pair of confining assumptions are met

$$\begin{cases} \limsup\limits_{|v| \to +\infty} \omega(v) \,=\, -\infty\,, & (4.9a)\\[2ex] \sup\limits_{|v| \geq 1} \dfrac{|\omega(v)|}{|v|^{p-1}} \,<\, +\infty\,, & (4.9b)\end{cases}$$

for some $p \geq 2$. Assumption (4.9a) ensures that N is super-linearly confining at infinity. It allows us to obtain uniform estimates in time. However, it would have been replaced by

$$\sup_{|v| \geq 1} \omega(v) \,<\, +\infty\,,$$

had we worked on finite time intervals. Assumption (4.9b) ensures that the confinement property of N is controlled by a polynomial. This assumption is merely technical. Indeed, it may be possible to consider exponential control instead, as long as we also suppose exponential moments for the initial condition μ_0^ε. In the end, the assumption only dictates the localization we need on the initial condition : in our case, we choose polynomial control on the drift N and polynomial moments on the initial condition. A typical choice for N would be any polynomial of the form

$$P(v) \,=\, Q(v) \,-\, Cv|v|^{p-1}\,,$$

for some positive constant $C \,>\, 0$ and where Q has degree less then p.

We turn to the connectivity kernel Ψ. We suppose $\Psi \in \mathscr{C}^0\left(K_{\boldsymbol{x}},\, L^1\left(K_{\boldsymbol{x}'}\right)\right)$ and assume the following bound to hold

$$\sup_{\boldsymbol{x}' \in K} \int_K \left|\Psi(\boldsymbol{x}, \boldsymbol{x}')\right| \,\mathrm{d}\boldsymbol{x} \,<\, +\infty\,. \tag{4.10}$$

Moreover, we consider r in $]1, +\infty]$, define its conjugate $r' \geq 1$ as $1/r + 1/r' = 1$ and suppose

$$\sup_{\boldsymbol{x} \in K} \int_K \left|\Psi(\boldsymbol{x}, \boldsymbol{x}')\right|^r \,\mathrm{d}\boldsymbol{x}' \,<\, +\infty\,. \tag{4.11}$$

Our set of assumptions on the connectivity kernel is quite general since we consider non-symmetric interactions between neurons and also authorize the connectivity kernel to follow a negative power law, a case which is considered in the physical literature (see [161]).

Remark 4.1. *It may be possible to adapt our analysis to the case of a discrete spatial variable. This could be done by replacing the Lebesgue measure* $\mathrm{d}\boldsymbol{x}$ *in the definition of* Ψ *by a positive Borel measure* λ_K *with finite mass over* K *(typically a sum of Dirac mass if* K *is discrete). In this case, we should also suppose that* ρ_0^ε *lies in* $L^1(\lambda_K)$.

We now state our assumptions on the sequence of initial data $\left(\mu_{0,\boldsymbol{x}}^\varepsilon\right)_{\varepsilon>0}$. We suppose that for each $\varepsilon > 0$ we have

$$\left(\boldsymbol{x} \mapsto \mu_{0,\boldsymbol{x}}^\varepsilon\right) \in \mathscr{C}^0\left(K\,,\, \mathscr{P}_2\left(\mathbb{R}^2\right)\right).$$

We also suppose the spatial distribution of the network ρ_0^ε to be continuous over K and uniformly bounded from above and below, that is, there exists a constant $m_* > 0$ such that for all $\varepsilon > 0$

$$m_* \leq \rho_0^\varepsilon \leq 1/m_*\,, \quad \rho_0^\varepsilon \in \mathscr{C}^0(K)\,. \tag{4.12}$$

On top of that, we assume the following condition : there exist two positive constants m_p and $\overline{m}_p$, independent of ε, such that

$$\sup_{\boldsymbol{x}\in K} \int_{\mathbb{R}^2} |\boldsymbol{u}|^{2p} \mu_{0,\boldsymbol{x}}^\varepsilon(\mathrm{d}\boldsymbol{u}) \,\leq\, m_p\,, \tag{4.13}$$

and such that

$$\int_{K\times\mathbb{R}^2} |\boldsymbol{u}|^{2pr'}\, \rho_0^\varepsilon(\boldsymbol{x})\, \mu_{0,\boldsymbol{x}}^\varepsilon(\mathrm{d}\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \,\leq\, \overline{m}_p\,, \tag{4.14}$$

where p and r' are given in (4.9b) and (4.11).

Now let us define the notion of solution we will consider for equation (4.3).

Definition 4.2. *For all* $\varepsilon > 0$ *we say that* μ^ε *solves* (4.3) *with initial condition* μ_0^ε *if we have*

1. μ^ε *lies in*
$$\mathscr{C}^0\left(\mathbb{R}^+ \times K\,,\, L^1\left(\mathbb{R}^2\right)\right) \cap L^\infty_{loc}\left(\mathbb{R}^+ \times K\,,\, \mathscr{P}_2\left(\mathbb{R}^2\right)\right),$$

2. *for all* $\boldsymbol{x} \in K$, $t \geq 0$, *and* $\varphi \in \mathscr{C}_c^\infty\left(\mathbb{R}^2\right)$, *it holds*

$$\int_{\mathbb{R}^2} \varphi(\boldsymbol{u})\,\left(\mu_{t,\boldsymbol{x}}^\varepsilon - \mu_{0,\boldsymbol{x}}^\varepsilon\right)(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} = -\frac{\rho_0^\varepsilon(\boldsymbol{x})}{\varepsilon}\int_0^t \int_{\mathbb{R}^2} \partial_v \varphi(\boldsymbol{u})\,\left(v - \mathcal{V}^\varepsilon(s,\boldsymbol{x})\right) \mu_{s,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}s$$
$$\int_0^t \int_{\mathbb{R}^2} \left(\nabla_{\boldsymbol{u}}\varphi(\boldsymbol{u}) \cdot \mathbf{b}^\varepsilon(s,\boldsymbol{x},\boldsymbol{u}) + \partial_v^2 \varphi(\boldsymbol{u})\right) \mu_{s,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\mathrm{d}\boldsymbol{u}\,\mathrm{d}s\,,$$

where $\mathcal{V}^\varepsilon$ *and* $\mathbf{b}^\varepsilon$ *are given by* (4.2) *and* (4.3).

With this notion of solution, equation (4.3) is well-posed. Indeed, we have

Theorem 4.3. *For any $\varepsilon > 0$, suppose that assumptions (4.9a)-(4.9b) and (4.10)-(4.12) are fulfilled and that the initial condition μ_0^ε also verifies*

$$\begin{cases} \sup_{\boldsymbol{x}\in K} \int_{\mathbb{R}^2} e^{|\boldsymbol{u}|^2/2}\, \mu_{0,\boldsymbol{x}}^\varepsilon(\mathrm{d}\boldsymbol{u}) \,\leq\, M^\varepsilon\,, \\ \sup_{\boldsymbol{x}\in K} \int_{\mathbb{R}^2} \ln(\mu_{0,\boldsymbol{x}}^\varepsilon)\, \mu_{0,\boldsymbol{x}}^\varepsilon(\mathrm{d}\boldsymbol{u}) \,\leq\, m^\varepsilon\,, \end{cases} \tag{4.15}$$

and

$$\sup_{\boldsymbol{x}\in K} \left\| \nabla_{\boldsymbol{u}} \sqrt{\mu_{0,\boldsymbol{x}}^\varepsilon} \right\|_{L^2(\mathbb{R}^2)}^2 \,\leq\, m^\varepsilon\,, \tag{4.16}$$

where M^ε and m^ε are two positive constant. Then there exists a unique solution μ^ε to equation (4.3) with initial condition μ_0^ε, in the sense of Definition 4.2 which verifies

$$\sup_{(t\,,\,\boldsymbol{x})\in[0,T]\times K} \int_{\mathbb{R}^2} e^{|\boldsymbol{u}|^2/2}\, \mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} < +\infty\,,$$

for all $T \,\geq\, 0$.
Furthermore, the macroscopic quantities $\mathcal{V}^\varepsilon$ and $\mathcal{W}^\varepsilon$ given in (4.2) lie in $\mathscr{C}^0\left(\mathbb{R}^+ \times K\right)$.

We postpone the proof of this result to the Appendix B.1, which relies on relative entropy estimates. We take advantage of assumption (4.15) to derive continuity estimates for both the time and the spatial variable. More precisely, we apply an abstract result from [27] which ensures that if we suppose some exponential moments such as in (4.15), then the Wasserstein metric is controlled by the relative entropy. We make use of assumption (4.16) in order to obtain strong continuity with respect to the time variable. We emphasize that since we are not able to close the estimates using directly the L^1 norm, we introduce some modified relative entropy, which turns out to be, in some sense, equivalent to the L^1 distance. Our approach is original to our knowledge.

Remark 4.4. *We do not make use of assumptions (4.15) & (4.16) in the analysis of the asymptotic $\varepsilon \to 0$. Therefore, both constants M^ε and m^ε may blow up as ε vanishes.*

We also define solutions for the limiting system (4.6)

Definition 4.5. *We say that $(\mathcal{V}\,,\,\bar{\mu})$ solves (4.6) with initial condition $(\mathcal{V}_0\,,\,\bar{\mu}_0)$ if we have*

1. *$\mathcal{V} \in \mathscr{C}^0\left(\mathbb{R}^+ \times K\right)$ and $\bar{\mu} \in \mathscr{C}^0\left(\mathbb{R}^+ \times K\,,\, L^1(\mathbb{R})\right)$,*
2. *$\mathcal{V}$ is a mild solution to (4.6),*
3. *for all $t \,\geq\, 0$, all $\boldsymbol{x} \in K$ and all $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} \phi(w)\,\bar{\mu}_{t,\boldsymbol{x}}(w)\,\mathrm{d}w \,=\, \int_{\mathbb{R}} \phi(w)\,\bar{\mu}_{0,\boldsymbol{x}}(w)\,\mathrm{d}w \,+\, \int_0^t \int_{\mathbb{R}} A(\mathcal{V}, w)\,\partial_w\,\phi(w)\,\bar{\mu}_{s,\boldsymbol{x}}(w)\,\mathrm{d}w\,\mathrm{d}s\,.$$

Theorem 4.6. *Under assumptions* (4.9a)*,* (4.11)-(4.12)*, and for any initial condition*

$$(\mathcal{V}_0\,,\,\bar{\mu}_0) \in \mathscr{C}^0\,(K) \times \mathscr{C}^0\left(K\,,\,L^1\,(\mathbb{R})\right),$$

there exists a unique solution to (4.6) *in the sense of Definition 4.5 with initial condition* $(\mathcal{V}_0\,,\,\bar{\mu}_0)$*.*
Furthermore, $\mathcal{V}$ *is uniformly bounded over* $\mathbb{R}^+ \times K$*.*

Proof. The key argument is that the system (4.6) may be decoupled through the change of variable (4.7). Indeed, we consider the following system

$$\begin{cases} \partial_t \mathcal{V} = N(\mathcal{V}) - \mathcal{W} - \mathcal{L}_{\rho_0}[\mathcal{V}]\,, \\ \partial_t \mathcal{W} \,=\, A\,(\mathcal{V}\,,\,\mathcal{W})\,, \\ \partial_t \bar{\nu} \,-\, b\,\partial_w\,(w\,\bar{\nu}) \,=\, 0\,, \end{cases}$$

which turns out to be equivalent to (4.6) in the sense that it is solved by $(\mathcal{V}\,,\,\mathcal{W}\,,\,\bar{\nu})$ if and only if $(\mathcal{V}\,,\,\bar{\mu})$ solves (4.6), where $\bar{\mu}$ is defined as

$$\bar{\mu}\,(t\,,\,\boldsymbol{x}\,,\,w) \,=\, \bar{\nu}\,(t\,,\,\boldsymbol{x}\,,\,w \,-\, \mathcal{W}(t\,,\,\boldsymbol{x}))\,,$$

for all $(t,\,\boldsymbol{x},\,w)$ in $\mathbb{R}^+ \times K \times \mathbb{R}$. Existence and uniqueness for $\bar{\nu}$ relies on classical arguments and we refer to [79], where one can find the proof of existence and uniqueness for the system $(\mathcal{V},\,\mathcal{W})$. □

4.2.2 Main results

The following theorem is the main result of this article. It states that in the regime of strong interactions, the distribution of the voltage variable concentrates with rate $\sqrt{\varepsilon}$ around $\mathcal{V}$ with Gaussian profile. The distribution of the adaptation variable converges towards $\bar{\mu}$. Hence, the couple $(\mathcal{V},\,\bar{\mu})$, which solves (4.6), encodes the behavior of the system when $\varepsilon \ll 1$. The result provides an explicit convergence rate which is global in time and uniform in $\boldsymbol{x} \in K$.

Theorem 4.7. *Under assumptions* (4.9a)-(4.9b) *on the drift* N*,* (4.10)-(4.11) *on* Ψ*,* (4.12)-(4.14) *on the initial conditions* μ_0^ε *and under the additional assumptions of Theorem 4.3, consider the solutions* μ^ε *&* $(\mathcal{V},\,\bar{\mu})$ *provided by Theorem 4.3 & 4.6 respectively. Furthermore, define the initial macroscopic and mesoscopic errors as*

$$\mathcal{E}_{\mathrm{mac}} \,=\, \|\,\mathcal{U}_0 \,-\, \mathcal{U}_0^\varepsilon\,\|_{L^\infty(K)} \,+\, \|\,\rho_0 - \rho_0^\varepsilon\,\|_{L^\infty(K)}\,,$$

and

$$\mathcal{E}_{\mathrm{mes}} \,=\, \sup_{\boldsymbol{x}\in K} W_2\left(\bar{\nu}^\varepsilon_{0,\boldsymbol{x}},\bar{\nu}_{0,\boldsymbol{x}}\right).$$

Then there exists $(C, \varepsilon_0) \in \left(\mathbb{R}_*^+\right)^2$ *such that the following expansion holds for all* $\varepsilon \leq \varepsilon_0$,

$$W_2\left(\mu^\varepsilon_{t,\boldsymbol{x}}\,,\,\mathcal{M}_{\frac{1}{\varepsilon}\rho_0(\boldsymbol{x}),\,\mathcal{V}(t,\boldsymbol{x})}\otimes\bar{\mu}_{t,\boldsymbol{x}}\right)\leq C\left(\min\left(e^{Ct}\left(\mathcal{E}_{\text{mac}}+\varepsilon\right)\,,\,1\right)+\mathcal{E}_{\text{mes}}\,e^{-b\,t}+e^{-\rho_0^\varepsilon(\boldsymbol{x})\,t/\varepsilon}\right),$$

for all $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$.
Moreover, suppose the initial errors to be of order ε, *that is*

$$\mathcal{E}_{\text{mac}}\,+\,\mathcal{E}_{\text{mes}}\underset{\varepsilon\to 0}{=}O\left(\varepsilon\right)\,,$$

then the following estimate holds,

$$W_2\left(\mu^\varepsilon_{t,\,\boldsymbol{x}}\,,\,\mathcal{M}_{\frac{1}{\varepsilon}\,\rho_0(\boldsymbol{x})\,,\,\mathcal{V}(t,\,\boldsymbol{x})}\otimes\bar{\mu}_{t,\,\boldsymbol{x}}\right)\,\leq\,C\,\left(\min\left(e^{C\,t}\,\varepsilon\,,\,1\right)\,+\,e^{-\rho_0^\varepsilon(\boldsymbol{x})\,t/\varepsilon}\right)\,,$$

for all $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$.

An important feature in our work is that we **do not suppose** the initial condition μ_0^ε to be concentrated, nor the initial profile ν_0^ε to be close to its limit. In particular, this means that the problem on ν^ε is **ill-prepared**. Indeed, performing the change of variable (4.7) in the following *ansatz*

$$\inf_{\boldsymbol{x}\in K}\int_{\mathbb{R}^2}|v\,-\,\mathcal{V}_0^\varepsilon(\boldsymbol{x})|^2\,\mu^\varepsilon_{0,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\geq\,1\,,$$

we obtain that ν_0^ε blows up in $\mathscr{P}_2$ as ε vanishes

$$\inf_{\boldsymbol{x}\in K}\int_{\mathbb{R}^2}|v|^2\,\nu^\varepsilon_{0,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\geq\,\varepsilon^{-1}\,.$$

This is why we prove in Proposition 4.12 some exponential localizing effects with respect to the variable t/ε, which also appears in Theorem 4.7.

Let us outline the main steps of the proof. First, we obtain in Proposition 4.10 some uniform moment estimates for μ^ε. Second, in Proposition 4.12, we estimate a relative energy, which should be interpreted as the moments of the re-scaled quantity ν^ε after a proper re-normalization, as mentioned in Remark 4.13. The last step consists in the convergence estimate. We focus on the re-scaled quantity ν^ε and develop an analytical coupling method in order to estimate its Wasserstein distance with the limiting profile. The key idea is to consider a coupled equation which is solved by couplings between ν^ε and its limit and then to estimate some energy for the solutions to the coupled equation. The improvement with respect to [79] is twofold. On the one hand, we prove error estimate not only on the macroscopic quantities but also on the distribution functions by using the Wasserstein distance. On the other hand, by considering diffusive effect in v and rescaled variables, we can characterize the asymptotic profile of the distribution function and then get a better convergence rate.

Let us now mention some interpretations and consequences of our result. The first consequence of the latter result is the convergence of the averaged quantities $\mathcal{V}^\varepsilon$ and $\mathcal{W}^\varepsilon$. In fact, we prove a finer result since we obtain that the couple $(\mathcal{V}^\varepsilon, \bar{\mu}^\varepsilon)$ converges towards $(\mathcal{V}, \bar{\mu})$ with rate ε.

Corollary 4.8. *Under the assumptions of Theorem 4.7 and supposing the initial errors to be of order ε, that is*

$$\mathcal{E}_{\text{mac}} + \mathcal{E}_{\text{mes}} \underset{\varepsilon\to 0}{=} O(\varepsilon) ,$$

there exists $(C,\varepsilon_0) \in \left(\mathbb{R}_*^+\right)^2$ such that for all $\varepsilon \leq \varepsilon_0$,

$$\sup_{\boldsymbol{x}\in K}\left(|\,\mathcal{V}^\varepsilon(t,\boldsymbol{x}) - \mathcal{V}(t,\boldsymbol{x})\,| + W_2\left(\bar{\mu}^\varepsilon_{t,\boldsymbol{x}},\, \bar{\mu}_{t,\boldsymbol{x}}\right)\right) \leq C\left(e^{-m_*\,t/\varepsilon} + \min\left(e^{C\,t}\varepsilon,\, 1\right)\right) ,$$

for all $t \geq 0$, where m_* is given by (4.12).

Proof. Using Jensen's inequality, we check that

$$|\mathcal{V}^\varepsilon(t,\boldsymbol{x}) - \mathcal{V}(t,\boldsymbol{x})| \leq W_2\left(\mu^\varepsilon_{t,\boldsymbol{x}},\, \mathcal{M}_{\frac{\rho_0(\boldsymbol{x})}{\varepsilon},\mathcal{V}(t,\boldsymbol{x})} \otimes \bar{\mu}_{t,\boldsymbol{x}}\right) .$$

Furthermore, we use the definition of the Wasserstein distance to obtain

$$W_2\left(\bar{\mu}^\varepsilon_{t,\boldsymbol{x}},\, \bar{\mu}_{t,\boldsymbol{x}}\right) \leq W_2\left(\mu^\varepsilon_{t,\boldsymbol{x}},\, \mathcal{M}_{\frac{\rho_0(\boldsymbol{x})}{\varepsilon},\mathcal{V}(t,\boldsymbol{x})} \otimes \bar{\mu}_{t,\boldsymbol{x}}\right) .$$

Then we apply Theorem 4.7 and replace ρ_0^ε with its lower bound m_* given in assumption (4.12). □

Another interesting consequence of Theorem 4.7, which may be interpreted as the expansion of f^ε at order 1 when $\varepsilon \ll 1$, consists in recovering order 0. In fact, we prove a stronger result. Indeed, let us make the analogy with other types of expansions as Taylor expansions. The expansion at order 1 yields an equivalence result at order 0. This is exactly what we obtain in our case : we prove that the distance between f^ε and $\delta_\mathcal{V} \otimes \bar{\mu}$ is exactly of order $\sqrt{\varepsilon}$. This justifies our approach for two reasons. First, it means that it is not possible to achieve convergence at order ε if we restrict the analysis to the convergence towards a Dirac mass. Second, the choice of the Wasserstein metric in our analysis instead of another stronger norm enables to compare easily our result with the convergence of f^ε towards a Dirac mass.

Corollary 4.9. *Under the assumptions of Theorem 4.7 and supposing the initial errors to be of order $\sqrt{\varepsilon}$, that is*

$$\mathcal{E}_{\text{mac}} + \mathcal{E}_{\text{mes}} \underset{\varepsilon\to 0}{=} O\left(\sqrt{\varepsilon}\right) ,$$

there exists $(C,\varepsilon_0) \in \left(\mathbb{R}_*^+\right)^2$, such that for all $\varepsilon \leq \varepsilon_0$

$$\sup_{\boldsymbol{x}\in K}\left(W_2\left(\mu^\varepsilon_{t,\boldsymbol{x}},\, \delta_{\mathcal{V}(t,\boldsymbol{x})} \otimes \bar{\mu}_{t,\boldsymbol{x}}\right)\right) \leq C\left(e^{-m_*\,t/\varepsilon} + \min\left(e^{Ct}\sqrt{\varepsilon},\, 1\right)\right) ,$$

for all $t \geq 0$, where m_* is given by (4.12).
Moreover, supposing the initial errors to be of order ε, that is

$$\mathcal{E}_{\text{mac}} + \mathcal{E}_{\text{mes}} \underset{\varepsilon\to 0}{=} O(\varepsilon) ,$$

and considering two positive times t_0 and T such that $t_0 < T$, there exists $(C, \varepsilon_0) \in (\mathbb{R}_*^+)^2$ such that for all $\varepsilon \le \varepsilon_0$

$$C^{-1}\sqrt{\varepsilon} \le W_2\left(\mu^\varepsilon_{t,\boldsymbol{x}},\, \delta_{\mathcal{V}(t,\boldsymbol{x})} \otimes \bar{\mu}_{t,\boldsymbol{x}}\right) \le C\sqrt{\varepsilon}\,, \quad \forall (t,\boldsymbol{x}) \in [\,t_0\,,\,T\,] \times K\,.$$

Proof. The proof is a direct consequence of Theorem 4.7 and the triangular inequality for W_2. □

4.3 A priori estimates

In this section, we provide uniform estimates with respect to ε for the moments of μ^ε and for the relative energy given by

$$\begin{cases} M_q\,[\,\mu^\varepsilon\,]\,(t,\boldsymbol{x}) \,:=\, \displaystyle\int_{\mathbb{R}^2} |\boldsymbol{u}|^q \,\mathrm{d}\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,, \\ D_q\,[\,\mu^\varepsilon\,]\,(t,\boldsymbol{x}) \,:=\, \displaystyle\int_{\mathbb{R}^2} |v - \mathcal{V}^\varepsilon(t,\boldsymbol{x})|^q \,\mathrm{d}\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,, \end{cases}$$

where $q \ge 2$.

The key point here is to obtain uniform estimates with respect to time for both M_q^ε and D_q^ε using confining properties of N and A. It is actually the only place where we use the super-linear confinement of the drift N (4.9a).

Proposition 4.10 (Propagation of moment)**.** *Under assumptions* (4.9a) *on the drift N and* (4.10)-(4.11) *on the interaction kernel Ψ, consider a sequence of solutions $(\mu^\varepsilon)_{\varepsilon>0}$ to* (4.3) *with initial conditions satisfying assumptions* (4.12)-(4.14)*. Then, for all positive ε and all q lying in $[\,2,\,2p\,]$ holds the following estimate*

$$M_q[\,\mu^\varepsilon\,](t,\boldsymbol{x}) \;\le\; M_q[\,\mu^\varepsilon\,](0,\boldsymbol{x})\,\exp\left(-t/C\right) \,+\, C, \quad \forall\,(t,\boldsymbol{x}) \,\in\, \mathbb{R}^+ \times K\,,$$

where $C > 0$ is a positive constant which only depends on m_*, $\overline{m}_p$ and the data of the problem : Ψ, A_0 and N.

Proof. Let us choose an exponent $\theta \ge 2$, multiply equation (4.3) by $|\boldsymbol{u}|^\theta/\theta$ and integrate with respect to $\boldsymbol{u} \in \mathbb{R}^2$. Integrating by part, this leads us to the following relation

$$\frac{1}{\theta}\,\frac{\mathrm{d}}{\mathrm{d}t} M_\theta[\mu^\varepsilon](t,\boldsymbol{x}) \,=\, \mathcal{I}\,,$$

where $\mathcal{I}$ splits into $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4$ with

$$\begin{cases} \mathcal{I}_1 \,=\, -\dfrac{1}{\varepsilon}\,\rho_0^\varepsilon(\boldsymbol{x}) \displaystyle\int_{\mathbb{R}^2} v\,|v|^{\theta-2}\,\left(v \,-\, \mathcal{V}^\varepsilon(t)\right)\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,, \\ \mathcal{I}_2 \,=\, \displaystyle\int_{\mathbb{R}^2} v\,|v|^{\theta-2}\,N(v)\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,, \\ \mathcal{I}_3 \,=\, \displaystyle\int_{\mathbb{R}^2} \left(w\,|w|^{\theta-2}\,A(\boldsymbol{u}) - v|v|^{\theta-2}\,w \,+\, (\theta-1)\,|v|^{\theta-2}\right)\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,, \\ \mathcal{I}_4 \,=\, -\displaystyle\int_{\mathbb{R}^2} v\,|v|^{\theta-2}\,\mathcal{K}_\Psi[\rho_0^\varepsilon\,\mu^\varepsilon]\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,. \end{cases}$$

We first handle the stiff term $\mathcal{I}_1$, which can be simply re-written as

$$\mathcal{I}_1 \;=\; -\frac{\rho_0^\varepsilon(\boldsymbol{x})}{\varepsilon}\int_{\mathbb{R}^2}\left(v\,|v|^{\theta-2}\;-\;\mathcal{V}^\varepsilon\,|\mathcal{V}^\varepsilon|^{\theta-2}\right)\,(v\,-\,\mathcal{V}^\varepsilon)\;\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\;\leq\;0\,.$$

Then we evaluate $\mathcal{I}_2$, which may involve higher order moments due to the non-linearity N. To overcome this difficulty, we split it in two parts

$$v\,|v|^{\theta-2}\,N(v)\;=\;|v|^\theta\,\frac{N(v)}{v}\,\mathbb{1}_{|v|\geq 1}\;+\;v\,|v|^{\theta-2}\,N(v)\,\mathbb{1}_{|v|<1}\,.$$

According to assumption (4.9a) and since N is continuous, we obtain

$$\mathcal{I}_2\;\leq\;\int_{\mathbb{R}^2}\left(C\,-\,\omega^-(v)\right)\,|v|^\theta\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\;+\;C\,,$$

for some constant $C>0$ only depending on N and where ω^- is the following nonnegative function

$$\omega^-(v)\;=\;\left(\omega(v)\;\mathbb{1}_{|v|\geq 1},\right)^-\,,$$

with $s^-=\max(0,-s)$ and where ω is given by (4.9a).

Now we evaluate $\mathcal{I}_3$, which gathers low order terms. Applying Young's inequality, we obtain the following estimate

$$\mathcal{I}_3\;\leq\;\frac{C}{\eta^\theta}\int_{\mathbb{R}^2}|v|^\theta\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\;+\;\int_{\mathbb{R}^2}(C\,\eta\,-\,b)\,|w|^\theta\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\;+\;\frac{C}{\eta^\theta}\,,$$

for some constant $C>0$ and all $\eta\in]0,1[$.

Finally, to evaluate the non-local term $\mathcal{I}_4$, we estimate $\mathcal{V}^\varepsilon$ by applying Jensen's inequality, which yields

$$|\mathcal{V}^\varepsilon|^\theta\;\leq\;\int_{\mathbb{R}^2}|v|^\theta\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,$$

hence, applying Young's inequality, we obtain that

$$\mathcal{I}_4\leq C\left(\|\Psi *_r \rho_0^\varepsilon\|_{L^\infty(K)}\int_{\mathbb{R}^2}|v|^\theta\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}+\int_{K\times\mathbb{R}^2}\left|\Psi(\boldsymbol{x},\boldsymbol{x}')\right|\,|v|^\theta\rho_0^\varepsilon(\boldsymbol{x}')\,\mu^\varepsilon_{t,\boldsymbol{x}'}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}'\right),$$

where C is a positive constant only depending on θ. Then we use Hölder's inequality and assumption (4.11) & (4.12) (we do not use the constraint $r>1$ here), which yields

$$\|\,\Psi *_r \rho_0^\varepsilon\,\|_{L^\infty(K)}\;\leq\;\|\,\rho_0^\varepsilon\,\|_{L^\infty(K)}\,\sup_{\boldsymbol{x}\in K}\|\,\Psi(\boldsymbol{x},\cdot)\,\|_{L^r(K)}\;\leq\;C\,,$$

for some $C>0$ independent of ε.

In the former computations we choose η such that $b\,-\,C\eta\;>\;0$. With the notation $\alpha\,=\,b\,-\,C\eta$, this yields

$$\begin{aligned}\frac{1}{\theta}\,\frac{\mathrm{d}}{\mathrm{d}t}\,M_\theta[\,\mu^\varepsilon\,](t,\boldsymbol{x})\quad&\leq\;\int_{\mathbb{R}^2}\left[\,(C\,-\,\omega^-(v))\,|v|^\theta\,-\,\alpha\,|w|^\theta\right]\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\;+\;C\\&\quad+C\int_{K\times\mathbb{R}^2}\left|\Psi(\boldsymbol{x},\boldsymbol{x}')\right|\,|v|^\theta\,\rho_0^\varepsilon(\boldsymbol{x}')\,\mu^\varepsilon_{t,\boldsymbol{x}'}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}'\,,\end{aligned}\tag{4.17}$$

for another constant $C > 0$ depending on θ, m_*, A, N and Ψ but not on $(t,\, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ nor on ε.

Now, we fix q in $[2,\, 2p]$ and proceed in two steps. On the one hand, choosing $\theta = qr' \geq 2$ in (4.17), we evaluate the averaged moments $\overline{M}_{qr'}[\rho_0^\varepsilon\, \mu^\varepsilon]$ given by

$$\overline{M}_{qr'}\,[\,\rho_0^\varepsilon\, \mu^\varepsilon\,]\,(t) \;=\; \int_{K\times\mathbb{R}^2} |\boldsymbol{u}|^{qr'}\, \rho_0^\varepsilon(\boldsymbol{x})\, \mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\, \mathrm{d}\boldsymbol{x}\,,$$

where r' is given by (4.11).

On the other hand, choosing $\theta = q$ in (4.17), we use the latter estimate to control the non-local contribution on the right hand side of (4.17) and evaluate $M_q[\mu^\varepsilon](t, \boldsymbol{x})$ at each $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$.

Starting from (4.17) with $\theta = qr' \geq 2$, we multiply it by $\rho_0^\varepsilon(\boldsymbol{x})$ and integrate with respect to $\boldsymbol{x} \in K$, which yields

$$\begin{aligned}\frac{1}{qr'}\,\frac{\mathrm{d}}{\mathrm{d}t}\overline{M}_{qr'}\,[\,\rho_0^\varepsilon\, \mu^\varepsilon\,] \;\leq\; &\int_{K\times\mathbb{R}^2} \Big(\left(C \,-\, \omega^-(v)\right)\, |v|^{qr'} \,-\, \alpha\, |w|^{qr'}\Big)\; \rho_0^\varepsilon(\boldsymbol{x})\, \mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\, \mathrm{d}\boldsymbol{x}\\ &+\, C\int_{K\times K\times\mathbb{R}^2} |\Psi(\boldsymbol{x}, \boldsymbol{x}')|\, |v|^{qr'}\, \rho_0^\varepsilon(\boldsymbol{x})\;\rho_0^\varepsilon(\boldsymbol{x}')\, \mu_{t,\boldsymbol{x}'}^\varepsilon(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\, \mathrm{d}\boldsymbol{x}'\, \mathrm{d}\boldsymbol{x} \,+\, C\,.\end{aligned}$$

According to assumption (4.10) & (4.12), we have

$$\int_{K^2\times\mathbb{R}^2} |\Psi(\boldsymbol{x}, \boldsymbol{x}')|\, |v|^{qr'} \rho_0^\varepsilon(\boldsymbol{x})\rho_0^\varepsilon(\boldsymbol{x}')\mu_{t,\boldsymbol{x}'}^\varepsilon(\boldsymbol{u})\mathrm{d}\boldsymbol{u}\mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{x} \;\leq\; C\int_{K\times\mathbb{R}^2} |v|^{qr'} \rho_0^\varepsilon(\boldsymbol{x})\mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\mathrm{d}\boldsymbol{u}\, \mathrm{d}\boldsymbol{x}\,,$$

for some positive constant C depending on m_* and Ψ. Hence it yields

$$\frac{1}{qr'}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\overline{M}_{qr'}\,[\,\rho_0^\varepsilon\, \mu^\varepsilon\,]\,(t) \leq \int_{K\times\mathbb{R}^2} \Big(\left(C - \omega^-(v)\right)\, |v|^{qr'} - \alpha |w|^{qr'}\Big)\, \rho_0^\varepsilon(\boldsymbol{x})\, \mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\, \mathrm{d}\boldsymbol{x} \,+\, C\,.$$

From assumption (4.9a), $\omega^-(v)$ goes to infinity with $|v|$. Consequently, we are led to

$$\frac{1}{qr'}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\overline{M}_{qr'}\,[\,\rho_0^\varepsilon\, \mu^\varepsilon\,]\,(t) \;\leq\; C \,-\, \frac{1}{C}\,\overline{M}_{qr'}\,[\,\rho_0^\varepsilon\, \mu^\varepsilon\,]\,(t),$$

for $C > 0$ great enough. Using Gronwall's lemma and the assumption (4.14) on the non-local moment of μ_0^ε, we obtain that for all $\varepsilon > 0$,

$$\overline{M}_{qr'}[\rho_0^\varepsilon\, \mu^\varepsilon](t) \;\leq\; C\,,\;\; \forall t \in \mathbb{R}^+\,,$$

where C may depend on $\overline{m}_p$.

Now, we come back to the local estimate on the moment $M_q[\mu^\varepsilon](t, \boldsymbol{x})$. We replace θ with q in (4.17) and estimate the non-local contribution. First, we apply Hölder's inequality and assumptions (4.11) & (4.12) to the non-local contribution in (4.17). This gives

$$\int_{K\times\mathbb{R}^2} |\Psi(\boldsymbol{x}, \boldsymbol{x}')|\, |v|^q\, \rho_0^\varepsilon(\boldsymbol{x}')\, \mu_{t,\boldsymbol{x}'}^\varepsilon(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\, \mathrm{d}\boldsymbol{x}' \;\leq\; C\, \left|\overline{M}_{qr'}^\varepsilon(t)\right|^{1/r'}\,,$$

where we emphasize that we use the constraint $r > 1$ in the former estimate. Then we use the latter bound on $\overline{M}^{\varepsilon}_{qr'}(t)$ and obtain

$$\int_{K\times\mathbb{R}^2} |\Psi(\boldsymbol{x},\boldsymbol{x}')|\, |v|^q\, \rho_0^\varepsilon(\boldsymbol{x}')\, \mu^\varepsilon_{t,\boldsymbol{x}'}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\, \mathrm{d}\boldsymbol{x}' \le C\,.$$

Hence, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t} M_q[\mu^\varepsilon](t,\boldsymbol{x}) \le \int_{\mathbb{R}^2} \left((C - \omega^-(v))\, |v|^q - \alpha |w|^q \right) \mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u} + C\,.$$

Using the same arguments as before, we get some $C > 0$ such that,

$$\frac{\mathrm{d}}{\mathrm{d}t} M_q[\mu^\varepsilon](t,\boldsymbol{x}) \le C - \frac{1}{C} M_q[\mu^\varepsilon](t,\boldsymbol{x})\,,$$

hence we conclude this proof applying Gronwall's lemma. □

As a straightforward consequence of Proposition 4.10, we obtain uniform bounds with respect to time, space and ε for the macroscopic quantities

Corollary 4.11. *Under the assumptions of Proposition 4.10, there exists a constant $C > 0$, independent of ε, such that,*

$$|\mathcal{V}^\varepsilon(t,\boldsymbol{x})| + |\mathcal{W}^\varepsilon(t,\boldsymbol{x})| \le C, \quad \forall\, (t,\boldsymbol{x}) \in \mathbb{R}^+ \times K\,,$$

where $\mathcal{V}^\varepsilon$ and $\mathcal{W}^\varepsilon$ are given by (4.2).

Proof. Applying the Cauchy-Schwarz inequality, we have for any $(t,\boldsymbol{x}) \in \mathbb{R}^+ \times K$,

$$|\mathcal{V}^\varepsilon(t,\boldsymbol{x})| + |\mathcal{W}^\varepsilon(t,\boldsymbol{x})| \le 2\, |M_2[\mu^\varepsilon](t,\boldsymbol{x})|^{1/2}\,,$$

hence the result follows on from Proposition 4.10. □

We turn to the estimates for the relative energy $D_q[\mu^\varepsilon]$, which quantifies the convergence of μ^ε towards a Dirac mass centered on $\mathcal{V}^\varepsilon$.

Proposition 4.12 (Relative energy)**.** *Under assumptions* (4.9a)-(4.9b) *on the drift N and* (4.10)-(4.11) *on the interaction kernel Ψ, consider a sequence of solutions $(\mu^\varepsilon)_{\varepsilon>0}$ to* (4.3) *with initial conditions satisfying assumption* (4.12)-(4.14). *There exists a positive constant $C > 0$, which may depend on m_p and $\overline{m}_p$, such that for all $\varepsilon > 0$ and all q in $[2, 2p]$ holds the following estimate,*

$$D_q[\mu^\varepsilon](t,\boldsymbol{x}) \le C \left[D_q[\mu^\varepsilon](0,\boldsymbol{x}) \exp\left(-\frac{q\rho_0^\varepsilon(\boldsymbol{x})}{\varepsilon}\, t \right) + \left(\frac{\varepsilon}{\rho_0^\varepsilon(\boldsymbol{x})} \right)^{q/2} \right], \quad \forall (t,\boldsymbol{x}) \in \mathbb{R}^+ \times K\,.$$

Proof. We choose some q in $[2,\,2p]$, multiply equation (4.3) by $|v-\mathcal{V}^\varepsilon|^q/q$ and integrate with respect to $\boldsymbol{u}\in\mathbb{R}^2$. After integrating by part and using equation (4.2), it yields

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}D_q[\mu^\varepsilon](t,\boldsymbol{x}) \,=\, \mathcal{J}\,,$$

where $\mathcal{J}$ is split as

$$\mathcal{J} \,=\, \mathcal{J}_1 \,+\, \mathcal{J}_2 \,+\, \mathcal{J}_3 \,-\, \frac{\rho_0^\varepsilon}{\varepsilon}D_q[\mu^\varepsilon](t,\boldsymbol{x}) \,+\, (q-1)\,D^\varepsilon_{q-2}[\mu^\varepsilon](t,\boldsymbol{x})\,,$$

with

$$\begin{cases} \mathcal{J}_1 \,=\, -\displaystyle\int_{\mathbb{R}^2}(v\,-\,\mathcal{V}^\varepsilon)\,|v\,-\,\mathcal{V}^\varepsilon|^{q-2}\,\Big(\mathcal{K}_\Psi[\rho_0^\varepsilon\,\mu^\varepsilon]\,-\,\mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon]\Big)\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,\\ \mathcal{J}_2 \,=\, \displaystyle\int_{\mathbb{R}^2}(v\,-\,\mathcal{V}^\varepsilon)\,|v\,-\,\mathcal{V}^\varepsilon|^{q-2}\,\left(N(v)\,-\,N(\mathcal{V}^\varepsilon)\right)\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,\\ \mathcal{J}_3 \,=\, -\displaystyle\int_{\mathbb{R}^2}(v\,-\,\mathcal{V}^\varepsilon)\,|v\,-\,\mathcal{V}^\varepsilon|^{q-2}\,\Big(w\,-\,\mathcal{W}^\varepsilon\,+\,\mathcal{E}(\mu^\varepsilon_{t,\boldsymbol{x}})\Big)\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,. \end{cases}$$

To estimate the non-local term $\mathcal{J}_1$, we observe that

$$\mathcal{K}_\Psi[\rho_0^\varepsilon\,\mu^\varepsilon]\,-\,\mathcal{L}_{\rho_0^\varepsilon}(\mathcal{V}^\varepsilon)\,=\,(v\,-\,\mathcal{V}^\varepsilon)\,\Psi *_r \rho_0^\varepsilon(\boldsymbol{x})\,,$$

hence, using assumptions (4.11) & (4.12), we obtain

$$\mathcal{J}_1 \,=\, -\,\Psi *_r \rho_0^\varepsilon(\boldsymbol{x})\;D_q[\mu^\varepsilon](t,\boldsymbol{x})\,\leq\, C\,D_q[\mu^\varepsilon](t,\boldsymbol{x})\,.$$

We turn to $\mathcal{J}_2$, which involves higher order moments since it displays the non-linearity N. On the one hand, Corollary 4.11 ensures that $\mathcal{V}^\varepsilon$ is uniformly bounded. On the other hand N lies in $\mathscr{C}^1(\mathbb{R})$ and meets the confining assumption (4.9a). Hence, there exists a constant $C>0$ independent of ε such that

$$(v-\mathcal{V}^\varepsilon)\;(N(v)\,-\,N(\mathcal{V}^\varepsilon))\,\leq\, C\,|v-\mathcal{V}^\varepsilon|^2\,,\quad \forall(t,\,\boldsymbol{x},\,v)\in\mathbb{R}^+\times K\times\mathbb{R}\,.$$

Thus, it yields

$$\mathcal{J}_2\,\leq\, C\,D_q[\mu^\varepsilon](t,\boldsymbol{x})\,.$$

Finally we estimate $\mathcal{J}_3$, which gathers the low order terms. According to assumption (4.9b) on N and applying Proposition 4.10, we have for all positive ε,

$$\int_{\mathbb{R}^2}|w\,-\,\mathcal{W}^\varepsilon|^q\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,+\,|\mathcal{E}(\mu^\varepsilon_{t,\boldsymbol{x}})|\,\leq\, C\,,\quad\forall\,(t,\,\boldsymbol{x})\in\mathbb{R}^+\times K\,,$$

for some constant C that may depend on m_p and $\overline{m}_p$. Hence, applying Hölder's inequality, it yields

$$\mathcal{J}_3\,\leq\, C\,D_q[\mu^\varepsilon]^{(q-1)/q}(t,\boldsymbol{x})\,\leq\, C\,\Big(D_q[\mu^\varepsilon](t,\boldsymbol{x})\,+\,D_q[\mu^\varepsilon]^{(q-2)/q}(t,\boldsymbol{x})\Big)\,.$$

Gathering these computations and applying Hölder's inequality to D^ε_{q-2}, we obtain

$$\frac{1}{q}\,\frac{\mathrm{d}}{\mathrm{d}t}D_q[\mu^\varepsilon](t,\boldsymbol{x}) \,+\, \frac{\rho_0^\varepsilon}{\varepsilon}D_q[\mu^\varepsilon](t,\boldsymbol{x}) \,\leq\, C\,\left(D_q[\mu^\varepsilon](t,\boldsymbol{x}) \,+\, D_q[\mu^\varepsilon]^{(q-2)/q}(t,\boldsymbol{x})\right)\,.$$

To estimate $D_q[\mu^\varepsilon]$, we introduce the function $u \,=\, (D_q[\mu^\varepsilon])^{2/q}$, which satisfies the following differential inequality

$$\frac{1}{2}\,\frac{\mathrm{d}u}{\mathrm{d}t} \,+\, \frac{\rho_0^\varepsilon}{\varepsilon}\,u \,\leq\, C\,(u\,+\,1)\,.$$

Applying Proposition 4.10, we get a first bound on u since

$$D_q[\mu^\varepsilon](t,\boldsymbol{x}) \,\leq\, C\,M_q[\mu^\varepsilon](t,\boldsymbol{x}) \,\leq\, C\,,$$

hence, we substitute this estimate on the right hand side of the former differential inequality and obtain

$$\frac{\mathrm{d}u}{\mathrm{d}t} \,+\, \frac{2\,\rho_0^\varepsilon}{\varepsilon}\,u \,\leq\, C\,,$$

which implies that

$$u(t) \,\leq\, u(0)\,\exp\left(-\frac{2\rho_0^\varepsilon\,t}{\varepsilon}\right) \,+\, C\,\frac{\varepsilon}{\rho_0^\varepsilon}\,\left(1\,-\,\exp\left(-\frac{2\,\rho_0^\varepsilon\,t}{\varepsilon}\right)\right)\,.$$

The result follows from replacing u by $(D_q[\mu^\varepsilon])^{2/q}$. □

Both Propositions 4.10 & 4.12 may be interpreted in terms of the re-scaling ν^ε according to the following remark

Remark 4.13. *Performing the change of variable* (4.7) *in the expression of* $M_q[\mu^\varepsilon]$ *and* $D_q[\mu^\varepsilon]$*, we obtain the following relations*

$$\left\{\begin{array}{l}\displaystyle\int_{\mathbb{R}^2} |v|^q\,\nu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \,=\, \varepsilon^{-q/2}D_q[\mu^\varepsilon](t,\boldsymbol{x})\,,\\ \displaystyle\int_{\mathbb{R}^2} |w|^q\,\nu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \,=\, \int_{\mathbb{R}^2} |w-\mathcal{W}^\varepsilon|^q\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \,\leq\, C\,M_q[\mu^\varepsilon](t,\boldsymbol{x})\,.\end{array}\right.$$

To conclude this section, we deduce from Propositions 4.10 & 4.12 the following error estimate

Proposition 4.14. *Under assumptions* (4.9a)-(4.9b) *on the drift* N*,* (4.10)-(4.11) *on the interaction kernel* Ψ *consider a sequence of solutions* $(\mu^\varepsilon)_{\varepsilon>0}$ *to* (4.3) *with initial conditions satisfying assumption* (4.12)-(4.14)*. There exists a constant* $C > 0$ *such that for all* $\varepsilon > 0$ *we have*

$$\left|\mathcal{E}(\mu^\varepsilon_{t,\boldsymbol{x}})\right| \leq C\left(e^{-2\rho_0^\varepsilon(\boldsymbol{x})t/\varepsilon} + \varepsilon\right)\,,\quad \forall (t,\boldsymbol{x}) \in \mathbb{R}^+ \times K\,.$$

Proof. We rewrite the error term $\mathcal{E}(\mu^{\varepsilon}_{t,\boldsymbol{x}})$ given by (4.5) as follows

$$\mathcal{E}(\mu^{\varepsilon}_{t,\boldsymbol{x}}) \,=\, \mathcal{E}_1 \,+\, \mathcal{E}_2 \,+\, \mathcal{E}_3\,,$$

where

$$\begin{cases} \mathcal{E}_1 \,=\, N'\left(\mathcal{V}^{\varepsilon}\right) \displaystyle\int_{\mathbb{R}^2} \left(v - \mathcal{V}^{\varepsilon}\right) \mathbb{1}_{\,|v-\mathcal{V}^{\varepsilon}|\leq 1}\, \mu^{\varepsilon}_{t,\boldsymbol{x}}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,, \\ \mathcal{E}_2 \,=\, \displaystyle\int_{\mathbb{R}^2} \left(N\left(v\right) - N\left(\mathcal{V}^{\varepsilon}\right) - N'\left(\mathcal{V}^{\varepsilon}\right)\left(v - \mathcal{V}^{\varepsilon}\right)\right) \mathbb{1}_{\,|v-\mathcal{V}^{\varepsilon}|\leq 1}\, \mu^{\varepsilon}_{t,\boldsymbol{x}}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,, \\ \mathcal{E}_3 \,=\, \displaystyle\int_{\mathbb{R}^2} \left(N\left(v\right) - N\left(\mathcal{V}^{\varepsilon}\right)\right) \mathbb{1}_{\,|v-\mathcal{V}^{\varepsilon}|> 1}\, \mu^{\varepsilon}_{t,\boldsymbol{x}}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,. \end{cases}$$

We start with $\mathcal{E}_1$. According to the definition of $\mathcal{V}^{\varepsilon}$, we have

$$\mathcal{E}_1 = 0 \,-\, N'\left(\mathcal{V}^{\varepsilon}\right) \int_{\mathbb{R}^2} \left(v - \mathcal{V}^{\varepsilon}\right) \mathbb{1}_{\,|v-\mathcal{V}^{\varepsilon}|> 1}\, \mu^{\varepsilon}_{t,\boldsymbol{x}}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,.$$

Since $N \in \mathscr{C}^1(\mathbb{R})$ and applying Corollary 4.11 & Proposition 4.12, it yields

$$|\mathcal{E}_1| \,\leq\, C\, \left(e^{-2\rho_0^{\varepsilon} t/\varepsilon} \,+\, \varepsilon\right)\,.$$

We turn to $\mathcal{E}_2$. Since N is of class $\mathscr{C}^2$, and using Corollary 4.11, we have

$$\left|N\left(v\right) - N\left(\mathcal{V}^{\varepsilon}\right) - N'\left(\mathcal{V}^{\varepsilon}\right)\left(v - \mathcal{V}^{\varepsilon}\right)\right| \mathbb{1}_{\,|v-\mathcal{V}^{\varepsilon}|\leq 1} \,\leq\, C\left|v - \mathcal{V}^{\varepsilon}\right|^2,$$

for some constant $C > 0$ independent of $(t, \boldsymbol{x}, v) \,\in\, \mathbb{R}^+ \times K \times \mathbb{R}$ and $\varepsilon > 0$. Hence, we obtain the following estimate applying Proposition 4.12

$$|\mathcal{E}_2| \,\leq\, C\, \left(e^{-2\rho_0^{\varepsilon} t/\varepsilon} \,+\, \varepsilon\right)\,.$$

To end with, we estimate $\mathcal{E}_3$ using the assumption (4.9b) on N and Corollary 4.11. Indeed, we have

$$\left|N\left(v\right) - N\left(\mathcal{V}^{\varepsilon}\right)\right| \mathbb{1}_{\,|v-\mathcal{V}^{\varepsilon}|> 1} \,\leq\, C\left|v - \mathcal{V}^{\varepsilon}\right|^p.$$

Hence, applying Proposition 4.12, we obtain

$$|\mathcal{E}_3| \leq C\, \left(e^{-p\rho_0^{\varepsilon} t/\varepsilon} \,+\, \varepsilon^{p/2}\right)$$

and gathering these computations, it yields the expected result. □

In the present section, we derived pointwise estimates for the moments of μ^{ε}, a relative energy which corresponds to the moments of ν^{ε} and the macroscopic error term $\mathcal{E}(\mu^{\varepsilon})$. We build on these results to prove Theorem 4.7.

4.4 Proof of Theorem 4.7

To achieve the proof of Theorem 4.7, we quantify the concentration of $(\mu^\varepsilon)_{\varepsilon>0}$ around its asymptotic profile $\mathcal{M}_{\rho_0/\varepsilon,\mathcal{V}}\otimes\bar{\mu}$ when $\varepsilon\to 0$. Therefore, this section consists in estimating the error term $W_2\left(\mu^\varepsilon,\mathcal{M}_{\rho_0/\varepsilon,\mathcal{V}}\otimes\bar{\mu}\right)$, which we decompose as follows : for any $(t,\boldsymbol{x})\in\mathbb{R}^+\times K$,

$$W_2^2\left(\mu^\varepsilon_{t,\boldsymbol{x}},\,\mathcal{M}_{\rho_0/\varepsilon,\mathcal{V}}\otimes\bar{\mu}_{t,\boldsymbol{x}}\right)\,\leq\,\mathcal{D}_1\,+\,\mathcal{D}_2\,+\,\mathcal{D}_3,$$

where $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are defined as

$$\begin{cases}\mathcal{D}_1\,:=\,W_2^2\left(\mu^\varepsilon_{t,\boldsymbol{x}}\,,\,\mathcal{M}_{\rho_0^\varepsilon/\varepsilon,\mathcal{V}^\varepsilon}\otimes\bar{\nu}_{t,\boldsymbol{x}}\left(\cdot\,-\,\mathcal{W}^\varepsilon\right)\right)\,,\\[1em] \mathcal{D}_2\,:=\,W_2^2\left(\mathcal{M}_{\rho_0^\varepsilon/\varepsilon,\mathcal{V}^\varepsilon}\otimes\bar{\nu}_{t,\boldsymbol{x}}\left(\cdot-\mathcal{W}^\varepsilon\right)\,,\,\mathcal{M}_{\rho_0^\varepsilon/\varepsilon,\mathcal{V}}\otimes\bar{\mu}_{t,\boldsymbol{x}}\right)\,,\\[1em] \mathcal{D}_3\,:=\,W_2^2\left(\mathcal{M}_{\rho_0^\varepsilon/\varepsilon,\mathcal{V}}\otimes\bar{\mu}_{t,\boldsymbol{x}}\,,\,\mathcal{M}_{\rho_0/\varepsilon,\mathcal{V}}\otimes\bar{\mu}_{t,\boldsymbol{x}}\right)\,.\end{cases}$$

The term $\mathcal{D}_1$ quantifies the distance between the concentration profile ν^ε and the Gaussian distribution since from the change of variable (4.7), we have

$$\mathcal{D}_1\,=\,\inf_{\pi^\varepsilon\in\Pi^\varepsilon}\int_{\mathbb{R}^4}\left(\varepsilon\left|v-v'\right|^2+\left|w-w'\right|^2\right)\mathrm{d}\pi^\varepsilon(\boldsymbol{u},\boldsymbol{u}')\,,$$

where Π^ε stands for the set of distributions over $\mathbb{R}^4$ with marginals $\nu^\varepsilon_{t,\boldsymbol{x}}$ and $\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}_{t,\boldsymbol{x}}$, whereas $\mathcal{D}_2$ quantifies the error between the macroscopic quantities. Indeed, with the same change of variable, we obtain

$$\mathcal{D}_2=\left|\mathcal{V}^\varepsilon-\mathcal{V}\right|^2+\left|\mathcal{W}^\varepsilon-\mathcal{W}\right|^2.$$

Finally, the term $\mathcal{D}_3$ is simply the error on the macroscopic density. In the following we give an estimate for each term $(\mathcal{D}_i)_{1\leq i\leq 3}$.

Estimate for $\mathcal{D}_1$

Our strategy to estimate $\mathcal{D}_1$ consists in introducing a coupled equation on $\pi^\varepsilon\in\Pi^\varepsilon$. Let us consider the following problem,

$$\begin{aligned}&\partial_t\pi^\varepsilon\,+\,\mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}_0^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\,\pi^\varepsilon\right]\,+\,\partial_{w'}\left[A_0\left(0,w'\right)\pi^\varepsilon\right]\\ &=\,\frac{1}{\varepsilon}\,\mathrm{div}_{v,v'}\left[\rho_0^\varepsilon(\boldsymbol{x})\left(v,\,v'\right)^\top\pi^\varepsilon\,+\,D\cdot\nabla_{v,v'}\pi^\varepsilon\right],\end{aligned}\tag{4.18}$$

where $\mathbf{b}_0^\varepsilon$ is given in (4.8) and the diffusion matrix D is given by

$$D\,=\,\begin{pmatrix}1&\beta\\ \beta&1\end{pmatrix}\,,$$

for some $\beta \in [-1,1]$. On the one hand, integrating the former equation with respect to $\boldsymbol{u}' = (v', w')$, we obtain equation (4.8) on ν^ε. On the other hand, equation (4.18) integrated with respect to $\boldsymbol{u}$ is given by

$$\partial_t \nu + \partial_{w'} \left[A_0 \left(0, w'\right) \nu \right] = \frac{1}{\varepsilon} \partial_{v'} \left(\rho_0^\varepsilon v' \nu + \partial_{v'} \nu \right) ,$$

which is solved by $\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}$. Consequently, if we take some initial data π_0^ε such that

$$\pi_0^\varepsilon \in \Pi \left(\nu_{0,\boldsymbol{x}}^\varepsilon , \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}_{0,\boldsymbol{x}} \right) ,$$

we obtain that the solution π^ε to (4.18) has marginals $\nu_{t,\boldsymbol{x}}^\varepsilon$ with respect to $\boldsymbol{u}$ and $\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}_{t,\boldsymbol{x}}$ with respect to $\boldsymbol{u}'$ at all time $t \geq 0$. Hence, according to the definition of the Wasserstein metric, the following inequality holds

$$\mathcal{D}_1 \leq \int_{\mathbb{R}^4} \left(\varepsilon \left| v - v' \right|^2 + \left| w - w' \right|^2 \right) \mathrm{d}\pi^\varepsilon(\boldsymbol{u}, \boldsymbol{u}') .$$

Moreover, we say that the equation is coupled because of the diffusion matrix D. Condition $\beta \in [-1,1]$ ensures that the matrix is positive and hence is required in order for equation (4.18) to be well-posed. In the case $\beta = 0$, there is no coupling between variables $\boldsymbol{u}$ and $\boldsymbol{u}'$. However, we can see that there is only one suitable choice for the parameter β. Indeed, considering only the leading order in equation (4.18), the equation rewrites

$$\partial_t \pi^\varepsilon = \frac{1}{\varepsilon} \operatorname{div}_{v,v'} \left[\rho_0^\varepsilon \left(v, v' \right)^\top \pi^\varepsilon + D \cdot \nabla_{v,v'} \pi^\varepsilon \right] .$$

Multiplying the former equation by $|v - v'|^2 / 2$ and integrating by part, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^4} \left| v - v' \right|^2 \mathrm{d}\pi^\varepsilon(\boldsymbol{u}, \boldsymbol{u}') + \frac{\rho_0^\varepsilon}{\varepsilon} \int_{\mathbb{R}^4} \left| v - v' \right|^2 \mathrm{d}\pi^\varepsilon(\boldsymbol{u}, \boldsymbol{u}') = 2 \frac{1 - \beta}{\varepsilon} .$$

Consequently, $\beta = 1$ is the unique suitable choice in order to avoid blow up in the asymptotic $\varepsilon \to 0$. Hence, **we fix parameter** β to 1 in what follows. On a probabilistic point of view, this choice corresponds to taking the same Brownian motion for the processes associated to ν^ε and $\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}$.
Before estimating $\mathcal{D}_1$, we precise the nature of solutions we consider for equation (4.18)

Definition 4.15. *For any $\boldsymbol{x}$ in K and some $\varepsilon > 0$, we say that π^ε solves* (4.18) *with initial condition π_0^ε if we have*

1. *π^ε lies in*

$$\mathscr{C}^0 \left(\mathbb{R}^+ , \mathcal{D}' \left(\mathbb{R}^4 \right) \right) \cap L^\infty_{loc} \left(\mathbb{R}^+ , L \log L \left(\mathbb{R}^4 \right) \right) ,$$

where $\mathcal{D}'(\mathbb{R}^4)$ stands for the set of distributions over $\mathbb{R}^4$ and $L \log L$ stands for the set of L^1 functions with finite entropy over $\mathbb{R}^4$.

2. for all $t \geq 0$*, and* $\varphi \in \mathscr{C}_c^\infty\left(\mathbb{R}^4\right)$*, it holds*

$$\int_{\mathbb{R}^4} \varphi\,(\pi_t^\varepsilon - \pi_0^\varepsilon)\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u'} \;=\; \int_0^t \int_{\mathbb{R}^4} \left(\nabla_{\boldsymbol{u}}\varphi \cdot \mathbf{b}_0^\varepsilon(s,\boldsymbol{x},\boldsymbol{u}) \;+\; \partial_{w'}\varphi\, A_0(0,\,w')\right)\pi_s^\varepsilon\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u'}\,\mathrm{d}s$$

$$-\,\frac{1}{\varepsilon}\int_0^t \int_{\mathbb{R}^4} \left(\rho_0^\varepsilon(\boldsymbol{x})\nabla_{v,v'}\varphi \cdot (v,v')^\top \;-\; \mathrm{div}_{(v,v')}\left(D\cdot\nabla_{v,v'}\varphi\right)\right)\pi_s^\varepsilon\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u'}\,\mathrm{d}s\,,$$

where $\mathcal{V}^\varepsilon$ *and* $\mathbf{b}_0^\varepsilon$ *are given by* (4.2) *and* (4.3)*.*

We prove the following result for equation (4.18)

Proposition 4.16. *Under the assumptions of Theorem 4.3, consider a positive* ε*, some* $\boldsymbol{x}$ *lying in* K *and a coupling*

$$\pi_0^\varepsilon \;\in\; \Pi\left(\nu_{0,\boldsymbol{x}}^\varepsilon\,,\, \mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}_{0,\boldsymbol{x}}\right).$$

There exists a solution π^ε *to equation* (4.18) *with initial condition* π_0^ε *and parameter* $\beta = 1$ *in the sense of Definition 4.15. Furthermore, we have*

$$\pi^\varepsilon(t,\cdot) \in \Pi\left(\nu_{t,\boldsymbol{x}}^\varepsilon\,,\, \mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}_{t,\boldsymbol{x}}\right), \quad \forall t \in \mathbb{R}^+\,.$$

We postpone the proof to Appendix B.2. It is mainly technical since equation (4.18) is linear.

We come back to the main concern of this section which consists in estimating $\mathcal{D}_1$ and prove the following estimate

Proposition 4.17. *Under the assumptions of Theorem 4.7, there exist two positive constants* C *and* ε_0 *such that for all* $\varepsilon \leq \varepsilon_0$*, holds the following estimate*

$$\mathcal{D}_1 \;\leq\; C\,\left(W_2^2\left(\bar{\nu}_{0,\boldsymbol{x}}^\varepsilon\,,\,\bar{\nu}_{0,\boldsymbol{x}}\right)e^{-2\,b\,t} \;+\; e^{-2\,\rho_0^\varepsilon\,t/\varepsilon} \;+\; \varepsilon^2\right)\,, \quad \forall (t,\boldsymbol{x}) \in \mathbb{R}^+\times K\,.$$

Proof. We consider $\pi_0^\varepsilon \in \Pi\left(\nu_{0,\boldsymbol{x}}^\varepsilon\,,\,\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}_{0,\boldsymbol{x}}\right)$ and the associated solution π^ε to equation (4.18) given by Proposition 4.16. When the context is clear, we omit the dependence with respect to t. To evaluate the term $\mathcal{D}_1$, we introduce the following quantities

$$\begin{cases} \mathcal{A}(t) \;:=\; \displaystyle\int_{\mathbb{R}^4} |v-v'|^2\,\mathrm{d}\pi^\varepsilon(\boldsymbol{u},\boldsymbol{u'})\,, \\[2ex] \mathcal{B}(t) \;:=\; \displaystyle\int_{\mathbb{R}^4} |w-w'|^2\,\mathrm{d}\pi^\varepsilon(\boldsymbol{u},\boldsymbol{u'})\,. \end{cases}$$

Observing that $\mathcal{D}_1(t) \leq \varepsilon\,\mathcal{A}(t) \,+\, \mathcal{B}(t)$, this proof consists in showing that $\mathcal{A}(t)$ is of order ε whereas $\mathcal{B}(t)$ is of order ε^2.

We begin with $\mathcal{A}$ and multiply equation (4.18) by $|v-v'|^2/2$ and integrate by part with respect to $(\boldsymbol{u}, \boldsymbol{u}')$, it yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} + \frac{\rho_0^\varepsilon}{\varepsilon}\mathcal{A} = \mathcal{A}_1\,,$$

where

$$\mathcal{A}_1 \,=\, \frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{R}^4}(v-v')\,\left(N(\mathcal{V}^\varepsilon+\sqrt{\varepsilon}\,v)\,-\,w\,-\,\sqrt{\varepsilon}\,v\,\Psi*_r\rho_0^\varepsilon(\boldsymbol{x})\right)\,\mathrm{d}\pi^\varepsilon(\boldsymbol{u},\boldsymbol{u}')\,.$$

We apply Cauchy-Schwarz inequality, assumptions (4.9b), (4.11) & (4.12) and Corollary 4.11 to estimate $\mathcal{A}_1$, it yields

$$\mathcal{A}_1 \,\leq\, \frac{C}{\sqrt{\varepsilon}}\,\mathcal{A}(t)^{1/2}\left(\int_{\mathbb{R}^2}\left(1\,+\,|w|^2\,+\,|\sqrt{\varepsilon}\,v|^{2p}\right)\,\mathrm{d}\nu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\right)^{1/2}\,.$$

Then we perform the change of variable (4.7) in the latter integral and apply Propositions 4.10 & 4.12 and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t)\,+\,\frac{\rho_0^\varepsilon}{\varepsilon}\,\mathcal{A}(t)\,\leq\,\frac{C}{\sqrt{\varepsilon}}\,\mathcal{A}(t)^{1/2}\,.$$

Dividing the former estimate by $\mathcal{A}(t)^{1/2}$ and applying Gronwall's lemma, this yields

$$\mathcal{A}(t)^{1/2}\,\leq\,\mathcal{A}(0)^{1/2}\,\exp\left(-\rho_0^\varepsilon\,t/\varepsilon\right)\,+\,C\,\varepsilon^{1/2}\,,$$

where the constant $C>0$ may depend on the lower bound m_* of the spatial distribution ρ_0^ε (see assumption (4.12)), m_p and $\bar{m}_p$. We point out that since we do not prepare the initial condition, $\mathcal{A}(0)$ may blow up as ε vanishes. Indeed, we have

$$\mathcal{A}(0)\,\leq\,2\left(\int_{\mathbb{R}^2}|v|^2\,\mathrm{d}\nu_{0,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\,+\,\frac{1}{\rho_0^\varepsilon(\boldsymbol{x})}\right)\,.$$

Hence, operating the change of variable (4.7) in the latter integral and applying assumptions (4.12) & (4.13), the former estimate becomes

$$\mathcal{A}(0)\,\leq\,\frac{C}{\varepsilon}\,.$$

Therefore, for all ε less than 1, we obtain

$$\mathcal{A}(t)\leq\frac{C}{\varepsilon}\left(\exp\left(-2\,\rho_0^\varepsilon(\boldsymbol{x})\,t/\varepsilon\right)\,+\,\varepsilon^2\right)\,.$$

We turn to $\mathcal{B}$ and prove that it is of order ε^2 using the previous estimate on $\mathcal{A}$. Indeed, we compute the derivative of $\mathcal{B}$ multiplying equation (4.18) by $|w-w'|^2/2$ and integrating by part with respect to $(\boldsymbol{u}, \boldsymbol{u}')$, it yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B}(t)\,=\,-\,b\,\mathcal{B}(t)\,+\,a\,\sqrt{\varepsilon}\,\left(\mathcal{B}_1(t)\,+\,\mathcal{B}_2(t)\right)\,,$$

where

$$\begin{cases} \mathcal{B}_1(t) \;=\; \displaystyle\int_{\mathbb{R}^4} (v - v')\,(w - w')\,\mathrm{d}\pi^\varepsilon(\boldsymbol{u}, \boldsymbol{u}')\,, \\[1em] \mathcal{B}_2(t) \;=\; \displaystyle\int_{\mathbb{R}^4} v'\,(w - w')\,\mathrm{d}\pi^\varepsilon(\boldsymbol{u}, \boldsymbol{u}')\,. \end{cases}$$

In order to estimate $\mathcal{B}_1$, we apply Cauchy-Schwarz inequality, use the estimate on $\mathcal{A}(t)$ and assumption (4.12) on ρ_0^ε, which yields

$$a\sqrt{\varepsilon}\,\mathcal{B}_1 \;\leq\; C\,\left(\exp\left(-m_*\,t/\varepsilon\right) \,+\, \varepsilon\right)\,\mathcal{B}(t)^{1/2}\,,$$

for some $C > 0$ and where m_* is given in (4.12). The next step consists in proving that $\mathcal{B}_2(t)$ is of order $\varepsilon^{3/2}$. We compute the derivative of $\mathcal{B}_2(t)$ multiplying equation (4.18) by $v'(w - w')$ and integrating by part with respect to $(\boldsymbol{u}, \boldsymbol{u}')$, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B}_2(t) \;=\; -\left(\frac{\rho_0^\varepsilon}{\varepsilon} \,+\, b\right)\mathcal{B}_2(t) \,+\, a\sqrt{\varepsilon}\int_{\mathbb{R}^4} v\,v'\,\mathrm{d}\pi^\varepsilon(\boldsymbol{u}, \boldsymbol{u}').$$

Then we apply Young's inequality, invert the change of variable (4.7) and apply Proposition 4.12. It yields

$$a\sqrt{\varepsilon}\int_{\mathbb{R}^4} |v\,v'|\;\mathrm{d}\pi^\varepsilon\left(\boldsymbol{u},\,\boldsymbol{u}'\right) \;\leq\; C\,\left(\varepsilon^{-1/2}\,e^{-2\rho_0^\varepsilon t/\varepsilon} \,+\, \varepsilon^{1/2}\right),$$

where the positive constant C may depend on m_*, m_p and $\bar{m}_p$. Then, we multiply the equation on $\mathcal{B}_2$ by its sign and apply Gronwall's lemma. In the end, this leads to

$$|\mathcal{B}_2(t)| \leq |\mathcal{B}_2(0)|\exp\left(-\rho_0^\varepsilon t/\varepsilon\right) + C\left(\varepsilon^{1/2}\exp\left(-\rho_0^\varepsilon t/\varepsilon\right) + \varepsilon^{3/2}\right).$$

for any ε less than $m_*/(2b)$. Furthermore, applying Cauchy-Schwarz inequality in $\mathcal{B}_2(0)$ and using assumption (4.12), we obtain

$$|\mathcal{B}_2(0)| \;\leq\; C\,\mathcal{B}(0)^{1/2}.$$

Gathering the former computations and applying assumption (4.12), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{B}(t), + b\,\mathcal{B}(t) \;\leq\; C\,\left(e^{-m_* t/\varepsilon} + \varepsilon\right)\mathcal{B}(t)^{1/2} + C\left(\varepsilon^{1/2}\,\mathcal{B}(0)^{1/2}\,e^{-m_* t/\varepsilon} + \varepsilon\,e^{-m_* t/\varepsilon} + \varepsilon^2\right).$$

To estimate $\mathcal{B}(t)$, we will construct an upper-bound $\mathcal{B}_+(t)$ by considering the following ODE

$$\begin{cases} \dfrac{1}{2}\,\dfrac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{B}_+(t) \,+\, b\,\mathcal{B}_+(t) \;=\; 2\,C\,\left(e^{(b\,-\,m_*/\varepsilon)\,t} \,+\, \varepsilon\right)\mathcal{B}_+(t)^{1/2}\,, \\[1em] \mathcal{B}_+(0)^{1/2} \;=\; \mathcal{B}(0)^{1/2} \,+\, \dfrac{2\,C\,\varepsilon}{b}\,, \end{cases}$$

whose exact solution is given by

$$\mathcal{B}_+(t)^{1/2} \;=\; \mathcal{B}_+(0)^{1/2}\,e^{-bt} \,+\, \frac{2\,C\,\varepsilon}{b}\left(1 \,-\, e^{-bt}\right) \,+\, \frac{2\,C\,\varepsilon}{m_* \,-\, 2b\varepsilon}\,\left(e^{-bt} \,-\, e^{-(m_*/\varepsilon\,-\,b)\,t}\right).$$

We check that the following condition is fulfilled at all time t,

$$\mathcal{B}_+(t)^{1/2} \geq \varepsilon^{1/2}\,\mathcal{B}(0)^{1/2}\,e^{-b\,t} + \varepsilon\,,$$

as long as $2\,C \geq b$ and $\varepsilon < \min\{\, m_* \,/\, (2b)\,,\, 1\,\}$. Making use of the latter inequality, we get

$$\varepsilon^{1/2}\,\mathcal{B}(0)^{1/2}\,e^{-m_* t/\varepsilon} + \varepsilon\,e^{-m_* t/\varepsilon} + \varepsilon^2 \leq \left(e^{(b\,-\,m_*/\varepsilon)t} + \varepsilon\right)\,\mathcal{B}_+(t)^{1/2}\,.$$

Therefore, defining $u(t) = \mathcal{B}_+(t) - \mathcal{B}(t)$, we check that the following inequality holds

$$\frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}t} + b\,u \geq C\,\frac{e^{-m_* t/\varepsilon} + \varepsilon}{\mathcal{B}(t)^{1/2} + \mathcal{B}_+(t)^{1/2}}\,u\,.$$

Since $\mathcal{B}$ is non negative and $\mathcal{B}_+(t)$ stays lower bounded by ε^2, we apply Gronwall's lemma to the latter estimate and noticing that $u(0) \geq 0$, it yields that $u(t)$ is non-negative. Then, we deduce

$$\mathcal{B}(t) \leq \mathcal{B}_+(t) \leq 2\,\mathcal{B}(0)\,e^{-2bt} + C\,\varepsilon^2\,,$$

for all $t \geq 0$, as long as $2\,C \geq b$ and $\varepsilon < \min\{\, m_* \,/\, (2b)\,,\, 1\,\}$.
Gathering the former computations, we obtain the following estimate for $\mathcal{D}_1$

$$\mathcal{D}_1 \leq C\,\left(\mathcal{B}(0)\,e^{-2bt} + e^{-2\,\rho_0^\varepsilon\,t/\varepsilon} + \varepsilon^2\right)\,.$$

We conclude the proof taking the infimum over all π_0^ε. □

Estimate for $\mathcal{D}_2$

We mentioned that the change of variables (4.7) in $\mathcal{D}_2$ yields

$$\mathcal{D}_2 = |\mathcal{V} - \mathcal{V}^\varepsilon|^2 + |\mathcal{W} - \mathcal{W}^\varepsilon|^2\,.$$

Hence the proof consists in injecting the error estimate obtained in Proposition 4.14 in equations (4.4) & (4.6)

Proposition 4.18. *Under the assumptions of Theorem 4.7, there exist two positive constants C and ε_0 such that for all $\varepsilon \leq \varepsilon_0$, holds the following estimate*

$$\mathcal{D}_2^{1/2} \leq C\,\min\left(e^{C\,t}\,(\mathcal{E}_{\mathrm{mac}} + \varepsilon)\,,1\right), \quad \forall\,(t,\boldsymbol{x}) \in \mathbb{R}^+ \times K\,,$$

where $\mathcal{E}_{\mathrm{mac}}$ is defined in Theorem 4.7.

Proof. We omit the dependence with respect to $(t,\boldsymbol{x})$ when the context is clear and write $\|\cdot\|_\infty$ instead of $\|\cdot\|_{L^\infty(K)}$ in this proof. According to equations (4.4) & (4.6), we have

$$\left\{\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,|\mathcal{V} - \mathcal{V}^\varepsilon| = \operatorname{sgn}\,(\mathcal{V} - \mathcal{V}^\varepsilon)\left[N(\mathcal{V}) - N(\mathcal{V}^\varepsilon) - \left(\mathcal{W} - \mathcal{W}^\varepsilon + \mathcal{L}_{\rho_0}(\mathcal{V}) - \mathcal{L}_{\rho_0^\varepsilon}(\mathcal{V}^\varepsilon)\right) - \mathcal{E}(\mu^\varepsilon)\right],\\ &\frac{\mathrm{d}}{\mathrm{d}t}\,|\mathcal{W} - \mathcal{W}^\varepsilon| = \operatorname{sgn}\,(\mathcal{W} - \mathcal{W}^\varepsilon)\,A_0\,(\,\mathcal{V} - \mathcal{V}^\varepsilon\,,\,\mathcal{W} - \mathcal{W}^\varepsilon\,)\,,\end{aligned}\right.$$

where sgn $(v) = v / |v|$ for all $v \in \mathbb{R}^*$. According to Corollary 4.11 and Theorem 4.6, $(\mathcal{V}, \mathcal{V}^\varepsilon)$ is uniformly bounded by a constant $R > 0$ with respect to ε, time & space. Consequently, we have

$$(\mathcal{V} - \mathcal{V}^\varepsilon)(N(\mathcal{V}) - N(\mathcal{V}^\varepsilon)) \leq C |\mathcal{V} - \mathcal{V}^\varepsilon|^2 ,$$

where C stands for the Lipschitz constant of N over the ball of radius R. We now estimate the contribution of the non-local terms. Using the linearity of $\mathcal{L}$ we split the term as follows

$$\mathcal{L}_{\rho_0}(\mathcal{V}) - \mathcal{L}_{\rho_0^\varepsilon}(\mathcal{V}^\varepsilon) = \mathcal{L}_{\rho_0}(\mathcal{V} - \mathcal{V}^\varepsilon) + \mathcal{L}_{(\rho_0 - \rho_0^\varepsilon)}(\mathcal{V}^\varepsilon) .$$

According to assumption (4.11) (we do not use the constraint $r > 1$ here), and since $\mathcal{V}^\varepsilon$ is uniformly bounded (see Corollary 4.11), we have

$$-\text{sgn }(\mathcal{V} - \mathcal{V}^\varepsilon)\, \mathcal{L}_{(\rho_0 - \rho_0^\varepsilon)}(\mathcal{V}^\varepsilon) \leq C \, \|\rho_0 - \rho_0^\varepsilon\|_\infty .$$

Furthermore, according to assumptions (4.11) & (4.12) (we do not use the constraint $r > 1$ here), we obtain

$$-\text{sgn }(\mathcal{V} - \mathcal{V}^\varepsilon)\, \mathcal{L}_{\rho_0}(\mathcal{V} - \mathcal{V}^\varepsilon) \leq C \, |\mathcal{V} - \mathcal{V}^\varepsilon| + |\Psi| *_r (|\mathcal{V} - \mathcal{V}^\varepsilon| \, \rho_0) .$$

We estimate the non-local term using assumptions (4.11) & (4.12) (we do not use the constraint $r > 1$ here). It yields

$$|\Psi| *_r (|\mathcal{V} - \mathcal{V}^\varepsilon| \, \rho_0) \leq C \, \|\mathcal{V} - \mathcal{V}^\varepsilon\|_\infty ,$$

for some constant C only depending on m_* and Ψ. Then we gather the former computations and replace $\mathcal{E}(\mu^\varepsilon)$ by the bound obtained in Proposition 4.14. It yields

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathcal{U} - \mathcal{U}^\varepsilon| \leq C \left(\|\mathcal{U} - \mathcal{U}^\varepsilon\|_\infty + \varepsilon + e^{-2\, m_*\, t/\varepsilon} + \|\rho_0 - \rho_0^\varepsilon\|_\infty \right) ,$$

for some positive constant C which may depend on m_p, $\overline{m}_p$ and m_* but not on ε and $(t, \boldsymbol{x})$. Integrating the latter inequality between 0 and t and taking the supremum over all $\boldsymbol{x}$ in K, we end up with the following inequality

$$\|\mathcal{U} - \mathcal{U}^\varepsilon\|_\infty (t) \leq \|\mathcal{U}_0 - \mathcal{U}_0^\varepsilon\|_\infty + C \int_0^t \|\mathcal{U} - \mathcal{U}^\varepsilon\|_\infty (s) + \varepsilon + e^{-m_*\, s/\varepsilon} + \|\rho_0 - \rho_0^\varepsilon\|_\infty \, \mathrm{d}s .$$

We conclude the proof applying Gronwall's lemma and using that $\mathcal{U}$ and $\mathcal{U}^\varepsilon$ are uniformly bounded according to Theorem 4.6 and Corollary 4.11. □

Estimate for $\mathcal{D}_3$

We now turn to the last term $\mathcal{D}_3$. This section is only technical and we prove that $\mathcal{D}_3$ is negligible in comparison to $\mathcal{D}_1$ and $\mathcal{D}_2$. Indeed, since $\mathcal{D}_3$ is the distance between two tensors, it is bounded by the sum of the distances appearing in each tensor

$$\mathcal{D}_3 \leq W_2^2 \left(\mathcal{M}_{\rho_0^\varepsilon/\varepsilon, \mathcal{V}} \, , \, \mathcal{M}_{\rho_0/\varepsilon, \mathcal{V}} \right) + W_2^2 \left(\bar{\mu}_{t,\boldsymbol{x}} \, , \, \bar{\mu}_{t,\boldsymbol{x}} \right) = W_2^2 \left(\mathcal{M}_{\rho_0^\varepsilon/\varepsilon, \mathcal{V}} \, , \, \mathcal{M}_{\rho_0/\varepsilon, \mathcal{V}} \right) .$$

Then, changing variables, we obtain

$$\mathcal{D}_3 \leq \frac{2\,\varepsilon}{\rho_0^\varepsilon} W_2^2\left(\mathcal{M}_1, \mathcal{M}_1\right) + 2\,\varepsilon \left(\frac{1}{\sqrt{\rho_0^\varepsilon}} - \frac{1}{\sqrt{\rho_0}}\right)^2 .$$

Hence, according to assumption (4.12), we obtain

$$\mathcal{D}_3 \leq \frac{\varepsilon}{2\,m_*^3} \|\rho_0 - \rho_0^\varepsilon\|_{L^\infty(K)}^2 .$$

Therefore applying Propositions 4.17 and 4.18 together with the latter estimate on $\mathcal{D}_3$, we have proven Theorem 4.7.

4.5 Conclusion & Perspectives

In this paper, we have characterized the blow-up profile (Gaussian distribution) of the voltage distribution in the regime of strong & short-range coupling between neurons and have computed the limiting distribution for the adaptation variable as well. Our result should be interpreted as the first order expansion of the network's distribution as ε vanishes. Indeed, it allows to improve on former convergence result in the sense that we gain an order $\sqrt{\varepsilon}$ in our convergence rate. On top of that, we benefit from the first order expansion to derive an asymptotic equivalent of the distribution at order zero (see Corollary 4.9).

Let us also mention a few natural questions that arise from this work. The natural continuation of this article consists in obtaining a strong convergence result towards the concentration profile (see [21]). It appears that the relative entropy approach we developed in Appendix B.1 may adapt to the analysis of the asymptotic $\varepsilon \ll 1$. We also point out that it should be possible to derive the next corrective terms in the expansion of the macroscopic quantities thanks to this work. Indeed, based on our result with a slight improvement, it may be possible to prove

$$(\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon) \underset{\epsilon\to 0}{=} \left(\overline{\mathcal{V}}^\varepsilon, \overline{\mathcal{W}}^\varepsilon\right) + O\left(\varepsilon^{3/2}\right),$$

where the limiting macroscopic system $\left(\overline{\mathcal{V}}^\varepsilon, \overline{\mathcal{W}}^\varepsilon\right)$ solves the following system

$$\begin{cases} \partial_t \overline{\mathcal{V}}^\varepsilon = N\left(\overline{\mathcal{V}}^\varepsilon\right) - \overline{\mathcal{W}}^\varepsilon - \mathcal{L}_{\rho_0}[\overline{\mathcal{V}}^\varepsilon] + \dfrac{\varepsilon}{2}\rho_0 N''\left(\overline{\mathcal{V}}^\varepsilon\right), \\ \partial_t \overline{\mathcal{W}}^\varepsilon = A\left(\overline{\mathcal{V}}^\varepsilon, \overline{\mathcal{W}}^\varepsilon\right). \end{cases}$$

The corrective term adds a dependence with respect to the spatial distribution of neurons and it might add some complexity to the dynamics of the limiting macroscopic system. Our last comment on this work is that it might be possible to improve it approach in order to obtain uniform in time convergence estimate. This could be done by carrying stability analysis of the limiting macroscopic system (4.6). Indeed, going back to the proof of Theorem 4.7, the only estimate which is not uniform with respect time is the one given for $\mathcal{D}_2$ (see Proposition 4.18), which corresponds to the error between the macroscopic quantities $\mathcal{U}$ and $\mathcal{U}^\varepsilon$. Therefore, it should be possible to obtain some uniform in time convergence results by taking $\mathcal{U}^\varepsilon$ close to an equilibrium state of the limiting macroscopic equation (4.6).

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Chapitre 5

Large coupling in a FitzHugh-Nagumo neural network : quantitative and strong convergence results

We consider a mesoscopic model for a spatially extended FitzHugh-Nagumo neural network and prove that in the regime where short-range interactions dominate, the probability density of the potential throughout the network concentrates into a Dirac distribution whose center of mass solves the classical non-local reaction-diffusion FitzHugh-Nagumo system. In order to refine our comprehension of this regime, we focus on the blow-up profile of this concentration phenomenon. Our main purpose here consists in deriving two quantitative and strong convergence estimates proving that the profile is Gaussian : the first one in a L^1 functional framework and the second in a weighted L^2 functional setting. We develop original relative entropy techniques to prove the first result whereas our second result relies on propagation of regularity.

This work has been submitted and is available on *arXiv :2203.14558*, Large coupling in a FitzHugh-Nagumo neural network : quantitative and strong convergence results.

Contents

5.1 Introduction

5.1.1 Physical model and motivations

Over the last century, mathematical models were built in order to describe biological neural activity, laying the groundwork for computational neuroscience. We mention the pioneer work A. Hodgkin and A. Huxley [140] who derived a precise model for the voltage dynamics of a nerve cell submitted to an external input. However a general and precise description of cerebral activity seems out of reach, due to the number of neurons, the complexity of their behavior and the intricacy of their interactions. Therefore, numerous simplified models arose from neuroscience over the last decade allowing to recover some of the behaviors observed in regimes or situations of interest. They may usually be interpreted as the mean-field limit of stochastic microscopic models. We mention integrate-and-fire neural networks [44, 56, 46], time-elapsed neuronal models [71, 70, 72, 181] and also voltage-conductance firing models [181, 184]. In this article, we study a FitzHugh-Nagumo neural field represented by its distribution $\mu(t, \boldsymbol{x}, \boldsymbol{u})$ depending on time t, position $\boldsymbol{x} \in K$ with K a compact set of $\mathbb{R}^d$, and $\boldsymbol{u} = (v, w) \in \mathbb{R}^2$ where v stands for the membrane potential and w is an adaptation variable. The distribution μ is normalized by the total density $\rho_0(\boldsymbol{x})$ of neurons at position $\boldsymbol{x}$. Therefore μ is a non-negative function taken in $\mathscr{C}^0\left(\mathbb{R}^+ \times K\,,\, L^1\left(\mathbb{R}^2\right)\right)$ which verifies

$$\int_{\mathbb{R}^2} \mu(t, \boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u} \,=\, 1, \quad \forall\, (t, \boldsymbol{x}) \,\in\, \mathbb{R}^+ \times K\,,$$

and which solves the following McKean-Vlasov equation (see [79, 77, 168] for other instances of such model)

$$\partial_t \mu + \partial_v \left(\left(N(v) - w - \mathcal{K}_\Phi[\rho_0 \, \mu] \right) \mu \right) + \partial_w \left(A\,(v,w)\, \mu \right) - \partial_v^2 \mu = 0\,,$$

where the non-linear term $\mathcal{K}_\Phi[\rho_0 \, \mu] \, \mu$ is induced by non-local electrostatic interactions : we suppose that neurons interact through Ohm's law and that the conductance between two neurons is given by an interaction kernel $\Phi : K^2 \to \mathbb{R}$ which depends on their position, this yields

$$\mathcal{K}_\Phi[\rho_0 \, \mu](t, \boldsymbol{x}, v) = \int_{K \times \mathbb{R}^2} \Phi(\boldsymbol{x}, \boldsymbol{x}')\,(v - v')\,\rho_0(\boldsymbol{x}')\mu(t, \boldsymbol{x}', \boldsymbol{u}')\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{u}'\,.$$

The other terms in the McKean-Vlasov equation are associated to the individual behavior of each neuron, driven here by the model of R. FitzHugh and J. Nagumo in [114, 174]. On the one hand, the drift $N \in \mathscr{C}^2(\mathbb{R})$ is a confining non-linearity : setting $\omega(v) = N(v)/v$ we suppose

$$\begin{cases} \limsup\limits_{|v| \to +\infty} \omega(v) = -\infty\,, & (5.1\text{a}) \\[2ex] \sup\limits_{|v| \geq 1} \dfrac{|\omega(v)|}{|v|^{p-1}} < +\infty\,, & (5.1\text{b}) \end{cases}$$

for some $p \geq 2$. For instance, these assumptions are met by the original model proposed by R. FitzHugh and J. Nagumo, where N is a cubic non-linearity

$$N(v) = v - v^3\,.$$

On the other hand, A drives the dynamics of the adaptation variable, it is given by

$$A(v,w) = a\,v - b\,w + c\,,$$

where a, $c \in \mathbb{R}$ and $b > 0$. We also add a diffusion term with respect to v in order to take into account random fluctuations of the voltage. This type of model has been rigorously derived as the mean-field limit of microscopic model in [3, 30, 77, 168, 161]. Well posedeness of the latter equation is well known and will not be discussed here. We refer to [24, Theorem 2.3] for a precise discussion over that matter.

5.1.2 Regime of strong short-range interactions

In this article, we consider a situation where Φ decomposes as follows

$$\Phi(\boldsymbol{x}, \boldsymbol{x}') = \Psi(\boldsymbol{x}, \boldsymbol{x}') + \frac{1}{\varepsilon}\,\delta_0(\boldsymbol{x} - \boldsymbol{x}')\,,$$

where the Dirac mass δ_0 accounts for short-range interactions whereas the interaction kernel Ψ models long-range interactions : it is "smoother" than δ_0 since we suppose

$$\Psi \in \mathscr{C}^0\left(K_{\boldsymbol{x}},\, L^1\left(K_{\boldsymbol{x}'}\right)\right) \quad \text{and} \quad \sup_{\boldsymbol{x}\in K}\int_K |\Psi(\boldsymbol{x}',\boldsymbol{x})| + |\Psi(\boldsymbol{x},\boldsymbol{x}')|^r \,\mathrm{d}\boldsymbol{x}' < +\infty\,, \tag{5.2}$$

for some $r > 1$ (we denote r' its conjugate : $r' = (r-1)/r$). We point out that our assumptions on Ψ are quite general, this is in line with other works which put a lot of effort into considering general interactions [144].
The scaling parameter $\varepsilon > 0$ represents the magnitude of short-range interactions ; we focus on the regime where they dominate, that is, when $\varepsilon \ll 1$. From these assumptions, the equation on μ can be rewritten as

$$\partial_t \mu^\varepsilon + \partial_v \left(\left(N(v) - w - \mathcal{K}_\Psi[\rho_0^\varepsilon \mu^\varepsilon]\right)\mu^\varepsilon\right) + \partial_w \left(A\left(v,w\right)\mu^\varepsilon\right) - \partial_v^2\mu^\varepsilon = \frac{\rho_0^\varepsilon}{\varepsilon}\,\partial_v\left(\left(v - \mathcal{V}^\varepsilon\right)\mu^\varepsilon\right), \tag{5.3}$$

where the averaged voltage and adaptation variables $\mathcal{U}^\varepsilon = (\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$ at a spatial location $\boldsymbol{x}$ are defined as

$$\begin{cases} \mathcal{V}^\varepsilon(t,\boldsymbol{x}) = \displaystyle\int_{\mathbb{R}^2} v\;\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,, \\ \mathcal{W}^\varepsilon(t,\boldsymbol{x}) = \displaystyle\int_{\mathbb{R}^2} w\;\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,. \end{cases} \tag{5.4}$$

Previous works already went through the analysis of the asymptotic $\varepsilon \ll 1$ and it was proven that in this regime the voltage distribution undergoes a concentration phenomenon. We mention [79] in which is investigated this asymptotic in a deterministic setting using relative entropy methods and also [196] in which authors study this model in a spatially homogeneous framework following a Hamilton-Jacobi approach. These works conclude that as ε vanishes, μ^ε converges as follows

$$\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \underset{\varepsilon\to 0}{\longrightarrow} \delta_{\mathcal{V}(t,\boldsymbol{x})}(v)\otimes\bar{\mu}(t,\boldsymbol{x},w)\,,$$

where the couple $(\mathcal{V},\bar{\mu})$ solves

$$\begin{cases} \partial_t \mathcal{V} = N(\mathcal{V}) - \mathcal{W} - \mathcal{L}_{\rho_0}[\mathcal{V}]\,, \\ \partial_t \bar{\mu} + \partial_w\left(A(\mathcal{V}\,,\,w)\,\bar{\mu}\right) = 0\,, \end{cases} \tag{5.5}$$

with

$$\mathcal{W} = \int_{\mathbb{R}} w\,\bar{\mu}\,(t\,,\,\boldsymbol{x}\,,\,w)\,\mathrm{d}w\,,$$

and where $\mathcal{L}_{\rho_0}[\mathcal{V}]$ is a non local operator given by

$$\mathcal{L}_{\rho_0}[\mathcal{V}] = \mathcal{V}\,\Psi *_r \rho_0 - \Psi *_r (\rho_0\mathcal{V})\,,$$

where $*_r$ is a shorthand notation for the convolution on the right side of any function g with Ψ

$$\Psi *_r g(\boldsymbol{x}) = \int_K \Psi(\boldsymbol{x},\boldsymbol{x}')\,g(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\,.$$

Then, the concentration profile of μ^ε around $\delta_{\mathcal{V}}$ was investigated in [24]. The strategy consists in considering the following re-scaled version ν^ε of μ^ε

$$\mu^\varepsilon\,(t,\,\boldsymbol{x},\,\boldsymbol{u})\;=\;\frac{1}{\theta^\varepsilon}\,\nu^\varepsilon\left(t,\,\boldsymbol{x},\,\frac{v-\mathcal{V}^\varepsilon}{\theta^\varepsilon},\,w-\mathcal{W}^\varepsilon\right),\tag{5.6}$$

where θ^ε shall be interpreted as the concentration rate of μ^ε around its mean value $\mathcal{V}^\varepsilon$. Under the scaling $\theta^\varepsilon\;=\;\sqrt{\varepsilon}$, it is proven that

$$\nu^\varepsilon\;\underset{\varepsilon\to 0}{\sim}\;\nu\;:=\;\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}\,,\tag{5.7}$$

where $\bar{\nu}$ solves the following linear transport equation

$$\partial_t\,\bar{\nu}\;-\;b\,\partial_w\,(w\,\bar{\nu})\;=\;0\,,\tag{5.8}$$

and where the Maxwellian $\mathcal{M}_{\rho_0^\varepsilon}$ is defined as

$$\mathcal{M}_{\rho_0^\varepsilon(\boldsymbol{x})}(v)=\sqrt{\frac{\rho_0^\varepsilon(\boldsymbol{x})}{2\pi}}\exp\left(-\,\rho_0^\varepsilon(\boldsymbol{x})\,\frac{|v|^2}{2}\right).$$

The latter convergence translates on μ^ε as follows

$$\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\;\underset{\varepsilon\to 0}{\sim}\;\mathcal{M}_{\rho_0\,|\theta^\varepsilon|^{-2}}\,(v\;-\;\mathcal{V})\otimes\bar{\mu}(t,\boldsymbol{x},\boldsymbol{u})\,,\tag{5.9}$$

with $\theta^\varepsilon\;=\;\sqrt{\varepsilon}$. More precisely, it was proven in [24] that (5.9) occurs up to an error of order ε in the sense of weak convergence in some probability space. Our goal here is to strengthen the results obtained in [24] by providing strong convergence estimates for (5.9). The general strategy consists in deducing (5.9) from (5.7). The main difficulties to achieve this is twofold. On the one hand, since the norms associated to strong topology are usually not scaling invariant, the time homogeneous scaling $\theta^\varepsilon\;=\;\sqrt{\varepsilon}$ comes down to considering well-prepared initial conditions. Therefore we find an appropriate scaling θ^ε which enables to treat general initial condition. On the other hand, the proof is made challenging by the cross terms between v and w in (5.3). This issue is analogous to the difficulty induced by the free transport operator in the context of kinetic theory [103, 137, 138]. In our context, we propagate regularity in order to overcome this difficulty and obtain error estimates.

This article is organized as follows. We start with Section 5.2, in which we carry out an heuristic in order to derive the appropriate scaling θ^ε and then state our two main results. We first provide convergence estimates for μ^ε in a L^1 setting, which is the natural space to consider for such type of conservative problem (see Theorem 5.2). This result is a direct consequence of the convergence of the re-scaled distribution ν^ε (see Theorem 5.1) which is obtained in Section 5.3. Then we propose convergence estimates in a weighted L^2 setting (see Theorem 5.4). Once again this result is the consequence of the convergence of ν^ε (see Theorem 5.3) provided in Section 5.4. We emphasize that the latter result allows us to recover the optimal convergence rates obtained in [24] and to achieve pointwise convergence estimates with respect to time. This analysis is in line with [168], which focuses on the regime of weak interactions between neurons (this corresponds to the asymptotic $\varepsilon \to +\infty$ in equation (5.3)).

5.2 Heuristic and main results

As mentioned before the time homogeneous scaling $\theta^\varepsilon = \sqrt{\varepsilon}$ in the definition (5.6) of ν^ε comes down to considering well-prepared initial conditions. We seek for a stronger result which also applies for ill-prepared initial conditions. To overcome this difficulty, our strategy consists in adding the following constraint on the concentration rate

$$\theta^\varepsilon\,(t\,=\,0)\,=\,1\,.$$

In this setting, the equation on ν^ε is obtained performing the following change of variable

$$(t\,,\,v\,,\,w) \mapsto \left(t\,,\,\frac{v-\mathcal{V}^\varepsilon}{\theta^\epsilon}\,,\,w-\mathcal{W}^\varepsilon\right) \tag{5.10}$$

in equation (5.3). Following computations detailed in [24], it yields

$$\partial_t\,\nu^\varepsilon\,+\,\mathrm{div}_{\boldsymbol{u}}\left[\,\mathbf{b}_0^\varepsilon\,\nu^\varepsilon\right]\,=\,\frac{1}{|\theta^\varepsilon|^2}\,\partial_v\left[\left(\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\theta^\varepsilon|^2+\frac{\rho_0^\varepsilon}{\varepsilon}\,|\theta^\varepsilon|^2\right)v\,\nu^\varepsilon+\partial_v\nu^\varepsilon\right], \tag{5.11}$$

where $\mathbf{b}_0^\varepsilon$ is given by

$$\mathbf{b}_0^\varepsilon\,(t,\boldsymbol{x},\boldsymbol{u})\,=\,\begin{pmatrix}(\theta^\varepsilon)^{-1}\,B_0^\varepsilon\,(t,\boldsymbol{x},\theta^\varepsilon\,v,w)\\[2ex] A_0\,(\theta^\varepsilon v,w)\end{pmatrix} \tag{5.12}$$

and B_0^ε is defined as

$$B_0^\varepsilon(t\,,\,\boldsymbol{x}\,,\,\boldsymbol{u})\,=\,N(\mathcal{V}^\varepsilon\,+\,v)\,-\,N(\mathcal{V}^\varepsilon)\,-\,w\,-\,v\,\Psi *_r \rho_0^\varepsilon(\boldsymbol{x})\,-\,\mathcal{E}\,(\mu^\varepsilon)\,,$$

with $\mathcal{E}\,(\mu^\varepsilon)$ the following error term

$$\mathcal{E}\,(\mu^\varepsilon\,(t,\boldsymbol{x},\,\cdot\,))\,=\,\int_{\mathbb{R}^2} N(v)\,\mu^\varepsilon\,(t,\boldsymbol{x},\boldsymbol{u})\;\mathrm{d}\boldsymbol{u}\,-\,N\left(\mathcal{V}^\varepsilon\,(t,\boldsymbol{x})\right)\,, \tag{5.13}$$

and where A_0 is the linear version of A

$$A_0(\boldsymbol{u}) = A(\boldsymbol{u}) - A(\mathbf{0})\,.$$

Considering the leading order in (5.11) and since we expect concentration with Gaussian profile $\mathcal{M}_{\rho_0^\varepsilon}$, θ^ε should verify

$$\begin{cases}\dfrac{1}{2}\,\dfrac{\mathrm{d}}{\mathrm{d}t}\,|\theta^\varepsilon|^2\,+\,\dfrac{\rho_0^\varepsilon}{\varepsilon}\,|\theta^\varepsilon|^2\,=\,\rho_0^\varepsilon\,,\\[2ex] \theta^\varepsilon\,(t\,=\,0)\,=\,1\,,\end{cases}$$

whose solution is given by the following explicit formula

$$\theta^\varepsilon(t,\,\boldsymbol{x})^2 \;=\; \varepsilon\,(1\;-\;\exp\left(-(\,2\,\rho_0^\varepsilon(\boldsymbol{x})\,t\,)\,/\,\varepsilon\right))\;+\;\exp\left(-(\,2\,\rho_0^\varepsilon(\boldsymbol{x})\,t\,)\,/\,\varepsilon\right). \tag{5.14}$$

Therefore, we obtain a time dependent θ^ε, which is of order $\sqrt{\varepsilon}$, up to an exponentially decaying correction to authorize ill-prepared initial conditions. With this choice, the equation on ν^ε rewrites :

$$\partial_t\,\nu^\varepsilon\;+\;\mathrm{div}_{\boldsymbol{u}}\left[\,\mathbf{b}_0^\varepsilon\,\nu^\varepsilon\,\right]\;=\;\frac{1}{|\theta^\varepsilon|^2}\,\mathcal{F}_{\rho_0^\varepsilon}\left[\,\nu^\varepsilon\,\right]\,, \tag{5.15}$$

where $\mathbf{b}_0^\varepsilon$ is given by (5.12) and the Fokker-Planck operator is defined as

$$\mathcal{F}_{\rho_0^\varepsilon}\left[\,\nu^\varepsilon\,\right]\;=\;\partial_v\left[\,\rho_0^\varepsilon\,v\,\nu^\varepsilon+\partial_v\,\nu^\varepsilon\,\right]\,.$$

Let us now precise our assumptions on the initial data. We suppose the following uniform boundedness condition on the spatial distribution ρ_0^ε

$$\rho_0^\varepsilon\in\mathscr{C}^0\,(\,K\,)\quad\text{and}\quad m_*\le\rho_0^\varepsilon\le 1/m_*\,, \tag{5.16}$$

as well as moment assumptions on the initial data

$$\begin{cases}\displaystyle\sup_{\boldsymbol{x}\in K}\int_{\mathbb{R}^2}|\boldsymbol{u}|^{2p}\mu_0^\varepsilon\,(\boldsymbol{x},\boldsymbol{u})\;\mathrm{d}\boldsymbol{u}\;\le\;m_p\,, & (5.17\text{a})\\[2ex] \displaystyle\int_{K\times\mathbb{R}^2}|\boldsymbol{u}|^{2pr'}\,\rho_0^\varepsilon(\boldsymbol{x})\,\mu_0^\varepsilon\,(\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\;\le\;\overline{m}_p\,, & (5.17\text{b})\end{cases}$$

where p and r' are given in (5.1b) and (5.2), for constants $m_*, m_p, \overline{m}_p$ uniform with respect to ε.

Since well posedeness of the mean-field equation (5.3) and the limiting model (5.5) is well known, we do not discuss it here and refer to [24, Theorems 2.3 and 2.6] for a precise discussion on that matter. To apply these results, we suppose the following assumptions which are not uniform with respect to ε on μ_0^ε

$$\begin{cases}\displaystyle\sup_{\boldsymbol{x}\in K}\int_{\mathbb{R}^2}e^{|\boldsymbol{u}|^2/2}\,\mu_0^\varepsilon(\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u}<\;+\infty\,,\\[2ex] \displaystyle\sup_{\boldsymbol{x}\in K}\left\|\nabla_{\boldsymbol{u}}\sqrt{\mu_0^\varepsilon}(\boldsymbol{x},\cdot)\,\right\|_{L^2(\mathbb{R}^2)}\;<\;+\infty\,,\end{cases} \tag{5.18}$$

and for the limiting problem (5.5), we suppose

$$(\mathcal{V}_0\,,\,\bar{\mu}_0)\in\mathscr{C}^0\,(K)\times\mathscr{C}^0\left(K\,,\,L^1\,(\mathbb{R})\right)\,. \tag{5.19}$$

All along our analysis, we denote by τ_{w_0} the translation by w_0 with respect to the w-variable, for any given w_0 in $\mathbb{R}$

$$\tau_{w_0}\,\nu\,(t,\boldsymbol{x},v,w)\;=\;\nu\,(t,\boldsymbol{x},v,w+w_0)\,.$$

5.2.1 L^1 convergence result

In the following result, we provide explicit convergence rates for ν^ε towards the asymptotic concentration profile of the neural network's distribution μ^ε in the regime of strong interactions in a L^1 setting. We will use the notation

$$L^\infty_{\boldsymbol{x}} L^1_{\boldsymbol{u}} := L^\infty\left(K\,,\, L^1\left(\mathbb{R}^2\right)\right), \quad \text{and} \quad L^\infty_{\boldsymbol{x}} L^1_{w} := L^\infty\left(K\,,\, L^1\left(\mathbb{R}\right)\right).$$

We prove that the profile of concentration with respect to v is Gaussian and we also characterize the limiting distribution with respect to the adaptation variable w. We denote by H the Boltzmann entropy, defined for all function $\mu : \mathbb{R}^2 \to \mathbb{R}^+$ as follows

$$H\left[\,\mu\,\right] = \int_{\mathbb{R}^2} \mu \ln\left(\mu\right) \mathrm{d}\boldsymbol{u}\,.$$

Theorem 5.1. *Under assumptions* (5.1a)-(5.1b) *on the drift N, assumption* (5.2) *on the interaction kernel Ψ, consider the unique sequence of solutions $(\mu^\varepsilon)_{\varepsilon>0}$ to* (5.3) *with initial conditions satisfying assumptions* (5.16)-(5.18) *and the solution $\bar{\nu}$ to equation* (5.8) *with an initial condition $\bar{\nu}_0$ such that*

$$\bar{\nu}_0 \in L^\infty\left(K\,,\, W^{2\,,\,1}\left(\mathbb{R}\right)\right)\,, \quad \textit{and} \quad \sup_{\boldsymbol{x}\in K} \int_{\mathbb{R}} \left| w\, \partial_w\, \bar{\nu}_0(\boldsymbol{x}, w)\right| \mathrm{d}w < +\infty\,. \tag{5.20}$$

Moreover, suppose that there exists a positive constant m_1 such that for all $(\gamma, w_0) \in (\mathbb{R}^*)^2$

$$\sup_{\varepsilon>0}\left(\frac{1}{|\gamma|}\left\|\nu^\varepsilon_0 \,-\, \tau_{\gamma v}\, \nu^\varepsilon_0\right\|_{L^\infty_{\boldsymbol{x}} L^1_{\boldsymbol{u}}} + \frac{1}{|w_0|}\left\|\nu^\varepsilon_0 \,-\, \tau_{w_0}\, \nu^\varepsilon_0\right\|_{L^\infty_{\boldsymbol{x}} L^1_{\boldsymbol{u}}}\right) \leq\, m_1\,, \tag{5.21}$$

and a positive constant m_2 such that

$$\sup_{\varepsilon>0} \left\| H\left[\,\nu^\varepsilon_0\,\right]\right\|_{L^\infty(K)} \,\leq\, m_2^2\,. \tag{5.22}$$

Then, there exists a positive constant C independent of ε such that for all ε less than 1*, it holds*

$$\left\|\,\nu^\varepsilon \,-\, \mathcal{M}_{\rho^\varepsilon_0} \otimes \bar{\nu}\,\right\|_{L^\infty(K\,,\,L^1([\,0\,,\,t\,]\times\mathbb{R}^2))} \leq\, 2\sqrt{2}\,t\, \left\|\,\bar{\nu}^\varepsilon_0 \,-\, \bar{\nu}_0\right\|^{1/2}_{L^\infty_{\boldsymbol{x}} L^1_w} +\, \sqrt{\varepsilon}\,\left(4\sqrt{t}\, m_2 + C\, e^{b\,t}\right),$$

for all time $t \geq 0$. In particular, under the compatibility assumption

$$\left\|\,\bar{\nu}^\varepsilon_0 \,-\, \bar{\nu}_0\right\|^{1/2}_{L^\infty_{\boldsymbol{x}} L^1_w} \underset{\varepsilon\to 0}{=} O\left(\sqrt{\varepsilon}\,\right)\,,$$

it holds

$$\sup_{t\,\in\,\mathbb{R}^+}\left(e^{-b\,t}\left\|\,\nu^\varepsilon \,-\, \mathcal{M}_{\rho^\varepsilon_0} \otimes \bar{\nu}\,\right\|_{L^\infty(K\,,\,L^1([\,0\,,\,t\,]\times\mathbb{R}^2))}\right) \underset{\varepsilon\to 0}{=} O\left(\sqrt{\varepsilon}\,\right)\,.$$

In this result, the constant C only depends on m_1, m_*, m_p and $\overline{m}_p$ (see assumptions (5.21), (5.16)-(5.17b)*) and the data of the problem $\bar{\nu}_0$, N, Ψ and A_0.*

The proof of this result is divided into two steps. First, we prove that ν^ε converges towards the following local equilibrium of the Fokker-Planck operator

$$\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon \,,$$

where $\bar{\nu}^\varepsilon$ is the marginal of ν^ε with respect to the re-scaled adaptation variable

$$\bar{\nu}^\varepsilon(t\,,\,\boldsymbol{x}\,,\,w) \,=\, \int_{\mathbb{R}} \nu^\varepsilon(t\,,\,\boldsymbol{x}\,,\,\boldsymbol{u})\,\mathrm{d}v\,,$$

and solves the following equation, obtained after integrating equation (5.15) with respect to v

$$\partial_t\,\bar{\nu}^\varepsilon \,-\, b\,\partial_w\,(w\,\bar{\nu}^\varepsilon) \,=\, -a\,\theta^\varepsilon\,\partial_w \int_{\mathbb{R}} v\,\nu^\varepsilon(\,t\,,\,\boldsymbol{x}\,,\,\boldsymbol{u}\,)\,\mathrm{d}v\,. \tag{5.23}$$

The argument relies on a rather classical free energy estimate. However, the analysis becomes more intricate when it comes to the convergence of the marginal $\bar{\nu}^\varepsilon$. As already mentioned, the proof of convergence is made challenging by cross terms between v and w in equation (5.15) inducing in equation (5.23) the following term which involves derivatives of ν^ε

$$\partial_w \int_{\mathbb{R}} v\,\nu^\varepsilon(\,t\,,\,\boldsymbol{x}\,,\,\boldsymbol{u}\,)\,\mathrm{d}v\,.$$

To overcome this difficulty, we perform a change of variable which cancels the latter source term and then conclude by proving a uniform equicontinuity estimate.
To conclude this discussion, we point out that the following condition on the initial data is sufficient in order to meet assumption (5.21)

$$\sup_{\varepsilon\,>\,0}\,\|(1+|v|)\partial_w\nu_0^\varepsilon\,\|_{L^\infty_{\boldsymbol{x}}L^1_{\boldsymbol{u}}} \,\leq\, m_1\,.$$

Indeed, for all $(\boldsymbol{x}, v, w_1) \in K \times \mathbb{R}^2$, it holds

$$\int_{\mathbb{R}} |\nu_0^\varepsilon - \tau_{w_1}\nu_0^\varepsilon|\,(\boldsymbol{x},\boldsymbol{u})\,\,\mathrm{d}w \leq |w_1| \int_{\mathbb{R}} |\partial_w\nu_0^\varepsilon\,(\boldsymbol{x},\boldsymbol{u})|\,\mathrm{d}w\,.$$

Therefore, taking the sum between the latter estimate with $w_1 = \gamma v$, divided by $|\gamma|$ and with $w_1 = w_0$, divided by $|w_0|$, integrating with respect to $v \in \mathbb{R}$ and taking the supremum over all $\boldsymbol{x} \in K$ it yields

$$\frac{1}{|\gamma|}\,\|\nu_0^\varepsilon \,-\, \tau_{\gamma v}\,\nu_0^\varepsilon\,\|_{L^\infty_{\boldsymbol{x}}L^1_{\boldsymbol{u}}} + \frac{1}{|w_0|}\,\|\nu_0^\varepsilon \,-\, \tau_{w_0}\,\nu_0^\varepsilon\,\|_{L^\infty_{\boldsymbol{x}}L^1_{\boldsymbol{u}}} \leq \|(1+|v|)\partial_w\nu_0^\varepsilon\,\|_{L^\infty_{\boldsymbol{x}}L^1_{\boldsymbol{u}}}\,,$$

for all $(\gamma, w_0) \in (\mathbb{R}^*)^2$.

We now interpret Theorem 5.1 on the solution μ^ε to equation (5.3) in the regime of strong interactions

Theorem 5.2. *Under the assumptions of Theorem 5.1 consider the unique sequence of solutions* $(\mu^\varepsilon)_{\varepsilon>0}$ *to* (5.3) *as well as the unique solution* $(\mathcal{V}\,,\,\bar{\mu})$ *to equation* (5.5) *with an initial condition* $\bar{\mu}_0$ *which fulfills assumption* (5.19)-(5.20). *Furthermore, suppose the following compatibility assumption to be fulfilled*

$$\|\,\mathcal{U}_0\,-\,\mathcal{U}_0^\varepsilon\,\|_{L^\infty(K)}\,+\,\|\,\rho_0-\rho_0^\varepsilon\,\|_{L^\infty(K)}\,+\,\|\,\bar{\mu}_0^\varepsilon\,-\,\bar{\mu}_0\|^{1/2}_{L^\infty_{\boldsymbol{x}}L^1_w}\underset{\varepsilon\to 0}{=}O\,(\,\sqrt{\varepsilon}\,)\,. \qquad (5.24)$$

There exists $(C\,,\,\varepsilon_0)\in\left(\mathbb{R}_+^*\right)^2$ *such that for all* ε *less than* ε_0*, it holds*

$$\|\,\mu^\varepsilon\,-\,\mu\,\|_{L^\infty(K\,,\,L^1([\,0,\,t\,]\times\mathbb{R}^2))}\,\le\,C\,e^{C\,t}\,\sqrt{\varepsilon}\,,\quad \forall\,t\,\in\,\mathbb{R}^+\,,$$

where the limit μ *is given by*

$$\mu\,:=\,\mathcal{M}_{\rho_0\,|\theta^\varepsilon|^{-2}}\,(v\,-\,\mathcal{V})\otimes\bar{\mu}\,.$$

In this result, the constant C *and* ε_0 *only depend on the implicit constant in assumption* (5.24)*, on the constants* m_1*,* m_2 m_**,* m_p *and* $\overline{m}_p$ *(see assumptions* (5.21)-(5.22) *and* (5.16)-(5.17b)*) and on the data of the problem* $\bar{\mu}_0$*,* N*,* Ψ *and* A_0*.*

Proof. Since the norm $\|\,\cdot\,\|_{L^1(\mathbb{R}^2)}$ is unchanged by the re-scaling (5.10), this theorem is a straightforward consequence of Theorem 5.1 and Proposition 5.5, which ensures the convergence of the macroscopic quantities $(\mathcal{V}^\varepsilon,\mathcal{W}^\varepsilon)$. □

On the one hand we obtain L^1 in time convergence result, which is a consequence of our method, which relies on a free energy estimate for solutions to (5.15). This is somehow similar to what is obtained in various classical kinetic models. Let us mention for instance the diffusive limit for collisional Vlasov-Poisson [103, 137, 165]. On the other hand, we obtain the convergence rate $O(\sqrt{\varepsilon})$ instead of the optimal convergence rate, which should be $O(\varepsilon)$ as rigorously proven for weak convergence metrics (see [24], Theorem 2.7). This is due to the fact that we use the Csizár-Kullback inequality to close our L^1 convergence estimates. Therefore it seems quite unlikely to recover the optimal convergence rate in a L^1 setting. This motivates our next result, on the L^2 convergence (Theorems 5.3 and 5.4), in which pointwise in time convergence is achieved.

5.2.2 Weighted L^2 convergence result

In this section, we provide a pair of result analog to Theorems 5.1 and 5.2 this time in a weighted L^2 setting. Since our approach relies on propagating w-derivatives, we introduce the following functional framework

$$\mathscr{H}^k\,(m^\varepsilon)\,=\,L^\infty\left(K_{\boldsymbol{x}},H^k_w\,(m^\varepsilon_{\boldsymbol{x}})\right)\,,$$

equipped with the norm

$$\|\,\nu\,\|_{\mathscr{H}^k(m^\varepsilon)}\,=\,\sup_{\boldsymbol{x}\in K}\left\{\|\,\nu\,(\boldsymbol{x}\,,\,\cdot\,)\,\|_{H^k_w(m^\varepsilon_{\boldsymbol{x}})}\right\}\,,$$

where $H^k_w(m^\varepsilon_{\boldsymbol{x}})$ denotes the weighted Sobolev space with index $k \in \mathbb{N}$ whose norm is given by

$$\| \nu \|^2_{H^k_w(m^\varepsilon_{\boldsymbol{x}})} \,=\, \sum_{l \leq k} \int_{\mathbb{R}^2} \left| \partial^l_w \, \nu(\boldsymbol{u}) \right|^2 \, m^\varepsilon_{\boldsymbol{x}}(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} \,,$$

and where the weight $m^\varepsilon_{\boldsymbol{x}}$ is given by

$$m^\varepsilon_{\boldsymbol{x}}(\boldsymbol{u}) \,=\, \frac{2\,\pi}{\sqrt{\rho^\varepsilon_0\,\kappa}} \,\exp\left(\frac{1}{2} \left(\rho^\varepsilon_0(\boldsymbol{x})\,|v|^2 \,+\, \kappa\,|w|^2 \right) \right), \tag{5.25}$$

for some exponent $\kappa > 0$ which will be prescribed later. We also introduce the associated weight with respect to the adaptation variable

$$\overline{m}(w) \,=\, \sqrt{\frac{2\,\pi}{\kappa}} \,\exp\left(\frac{\kappa}{2}\,|w|^2 \right).$$

and denote by $\mathscr{H}^k\,(\overline{m})$ the corresponding functional space associated to the marginal $\bar{\nu}$, depending only on $(\boldsymbol{x}, w) \in K \times \mathbb{R}$.

Hence, the following result tackles the convergence of ν^ε in the L^2 weighted setting

Theorem 5.3. *Under assumptions* (5.1a)-(5.1b) *on the drift* N *and the additional assumption*

$$\sup_{|v|\,\geq\,1} \left(v^2\,\omega(v) \,-\, C_0\,N'(v) \right) \,<\, +\infty\,, \tag{5.26}$$

for all positive constant $C_0 > 0$, *supposing assumption* (5.2) *on the interaction kernel* Ψ, *consider the unique sequence of solutions* $(\mu^\varepsilon)_{\varepsilon\,>\,0}$ *to* (5.3) *with initial conditions satisfying assumptions* (5.16)-(5.18) *and the solution* $\bar{\nu}$ *to equation* (5.8) *with an initial condition* $\bar{\nu}_0$. *Furthermore, consider an exponent* κ *which verifies the condition*

$$\kappa \,\in\, \left(\frac{1}{2\,b}\,, +\infty \right)\,, \tag{5.27}$$

and consider a rate α_* *lying in* $\left(0\,, 1 - (2b\kappa)^{-1} \right)$. *There exists a positive constant* C *independent of* ε *such that for all* ε *between* 0 *and* 1 *the following results hold true*

1. *consider* k *in* $\{0\,,1\}$ *and suppose that the sequence* $(\nu^\varepsilon_0)_{\varepsilon\,>\,0}$ *verifies*

$$\sup_{\varepsilon\,>\,0} \| \,\nu^\varepsilon_0\, \|_{\mathscr{H}^{k+1}(m^\varepsilon)} \,<\, +\infty\,, \tag{5.28}$$

and that $\bar{\nu}_0$ *verifies*

$$\bar{\nu}_0 \in \mathscr{H}^k(\overline{m})\,. \tag{5.29}$$

Then for all time t *in* $\mathbb{R}^+$ *it holds*

$$\begin{aligned} &\| \,\nu^\varepsilon(t) \,-\, \nu(t)\, \|_{\mathscr{H}^k(m^\varepsilon)} \\ &\leq e^{Ct} \left(\| \,\bar{\nu}^\varepsilon_0 \,-\, \bar{\nu}_0\, \|_{\mathscr{H}^k(\overline{m})} \,+\, C\, \| \,\nu^\varepsilon_0\, \|_{\mathscr{H}^{k+1}(m^\varepsilon)} \left(\sqrt{\varepsilon} \,+\, \min\left\{ 1\,, e^{-\alpha_* \frac{t}{\varepsilon}} \varepsilon^{-\frac{\alpha_*}{2m_*}} \right\} \right) \right)\,, \end{aligned}$$

where the asymptotic profile ν *is given by* (5.7) *;*

2. *suppose assumption* (5.28) *with index* $k = 1$ *and assumption* (5.29) *with index* $k = 0$*, it holds for all time* $t \geq 0$

$$\| \bar{\nu}^\varepsilon(t) - \bar{\nu}(t) \|_{\mathscr{H}^0(\overline{m})} \leq e^{Ct} \left(\| \bar{\nu}_0^\varepsilon - \bar{\nu}_0 \|_{\mathscr{H}^0(\overline{m})} + C \, \| \nu_0^\varepsilon \|_{\mathscr{H}^2(m^\varepsilon)} \, \varepsilon \sqrt{|\ln \varepsilon| + 1} \right).$$

In this theorem, the positive constant C *only depends on* κ, α_*, m_*, m_p, $\overline{m}_p$ *(see assumptions* (5.16), (5.17a) *and* (5.17b)*) and on the data of the problem :* N, A_0 *and* Ψ*.*

The proof of this result is provided in Section 5.4 and relies on regularity estimates for the solution ν^ε to equation (5.15). These regularity estimates allow us to bound the source term which appears in the right hand side of equation (5.23).

We now interpret the latter theorem in terms μ^ε. Let us emphasize that since m^ε defined by (5.25) depends on ε through the spatial distribution ρ_0^ε, we introduce weights which do not depend on ε anymore and which are meant to upper and lower bound m^ε. We consider $(\boldsymbol{x}, \boldsymbol{u})$ lying in $K \times \mathbb{R}^2$ and define

$$\begin{cases} m_{\boldsymbol{x}}^-(\boldsymbol{u}) = (\rho_0(\boldsymbol{x})\,\kappa)^{-\frac{1}{2}} \exp\left(\dfrac{1}{8} \left(\rho_0(\boldsymbol{x})\,|v|^2 + \kappa\,|w|^2 \right) \right), \\ \overline{m}^-(w) = \kappa^{-\frac{1}{2}} \exp\left(\dfrac{\kappa}{8}\,|w|^2 \right), \\ m_{\boldsymbol{x}}^+(\boldsymbol{u}) = (\rho_0(\boldsymbol{x})\,\kappa)^{-\frac{1}{2}} \exp\left(2 \left(\rho_0(\boldsymbol{x})\,|v|^2 + \kappa\,|w|^2 \right) \right), \\ \overline{m}^+(w) = \kappa^{-\frac{1}{2}} \exp\left(2\,\kappa\,|w|^2 \right). \end{cases}$$

With these notations, our result reads as follows

Theorem 5.4. *Under the assumptions of Theorem 5.3 consider the unique sequence of solutions* $(\mu^\varepsilon)_{\varepsilon > 0}$ *to* (5.3) *and the solution* $(\mathcal{V}, \bar{\mu})$ *to* (5.5) *with initial condition* $(\mathcal{V}, \bar{\mu})$ *satisfying* (5.19)*. The following results hold true*

1. *Consider* k *in* $\{0, 1\}$ *and suppose*

$$\sup_{\varepsilon > 0} \| \mu_0^\varepsilon \|_{\mathscr{H}^{k+1}(m^+)} < +\infty, \tag{5.30}$$

as well as the following compatibility assumption

$$\| \mathcal{U}_0 - \mathcal{U}_0^\varepsilon \|_{L^\infty(K)} + \| \rho_0 - \rho_0^\varepsilon \|_{L^\infty(K)} + \| \bar{\mu}_0^\varepsilon - \bar{\mu}_0 \|_{\mathscr{H}^k(\overline{m}^+)} \underset{\varepsilon \to 0}{=} O(\varepsilon). \tag{5.31}$$

Moreover, suppose that there exists a constant C *such that*

$$\sup_{\varepsilon > 0} \| \bar{\mu}_0 - \tau_{w_0} \bar{\mu}_0 \|_{\mathcal{H}^k(\overline{m}^+)} \leq C\,|w_0|, \quad \forall\, w_0 \in \mathbb{R}. \tag{5.32}$$

Then, for all $i \in \mathbb{N}$ and under the constraint $\alpha_* < \min\{m_*/2\,,\,1-(2b\kappa)^{-1}\}$, there exists $(C_i\,,\,\varepsilon_0) \in \left(\mathbb{R}_+^*\right)^2$ such that for all ε less than ε_0, it holds for all $t \in \mathbb{R}_+$,

$$\|\,(v\,-\,\mathcal{V})^i\,(\mu^\varepsilon\,-\,\mu)\,(t)\,\|_{\mathcal{H}^k(m^-)}\,\leq\,C_i\,e^{C_i\left(t+\varepsilon\,e^{C_i\,t}\right)}\left(\varepsilon^{\frac{i}{2}+\frac{1}{4}}\,+\,e^{-\,\alpha_*\frac{t}{\varepsilon}}\,\varepsilon^{-\frac{1}{2}}\right),$$

where the limit μ is given by

$$\mu\,=\,\mathcal{M}_{\rho_0\,|\theta^\varepsilon|^{-2}}\,(v\,-\,\mathcal{V})\otimes\bar{\mu}\,.$$

2. *Suppose assumption* (5.30) *with $k=1$, assumption* (5.32) *with $k=0$ and*

$$\|\,\mathcal{U}_0-\mathcal{U}_0^\varepsilon\,\|_{L^\infty(K)}+\|\,\rho_0-\rho_0^\varepsilon\,\|_{L^\infty(K)}+\|\,\bar{\mu}_0^\varepsilon-\bar{\mu}_0\,\|_{\mathscr{H}^0(\overline{m}^+)}\underset{\varepsilon\to 0}{=}O\left(\varepsilon\sqrt{|\ln\varepsilon\,|}\right).$$

There exists $(C\,,\,\varepsilon_0) \in \left(\mathbb{R}_+^*\right)^2$ such that for all ε less than ε_0, it holds

$$\|\,\bar{\mu}^\varepsilon(t)\,-\,\bar{\mu}(t)\,\|_{\mathscr{H}^0(\overline{m}^-)}\,\leq\,C\,e^{Ct}\,\varepsilon\,\sqrt{|\ln\varepsilon\,|}\,,\quad\forall\,t\in\mathbb{R}_+\,.$$

This result is a straightforward consequence of Theorem 5.3 and the convergence estimates for the macroscopic quantities given by item (1) Proposition 5.5. We postpone the proof to Section 5.4.3 and make a few comments. On the one hand, we achieve pointwise in time convergence estimates, which is an improvement in comparison to our result in the L^1 setting. This is made possible thanks to the regularity results obtained for ν^ε, which we were not able to obtain in the L^1 setting. On the other hand, we recover the optimal convergence rate for the marginal $\bar{\mu}^\varepsilon$ of μ^ε towards the limit $\bar{\mu}$, up to a logarithmic correction. The logarithmic correction arises due to the fact that we do not consider well prepared initial data (see Proposition 5.21 for more details). In the statement (1), we prove convergence with rate $O(\,\varepsilon^i\,)$ for all i. This is specific to the structure of the weighted L^2 spaces in this result.

5.2.3 Useful estimates

Before proving our main results, we remind here uniform estimates with respect to ε, already established in [24], for the moments of μ^ε and for the relative energy given by

$$\begin{cases} M_q\,[\,\mu^\varepsilon\,]\,(t,\boldsymbol{x})\,:=\,\displaystyle\int_{\mathbb{R}^2}|\boldsymbol{u}|^q\,\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,\\[2ex] D_q\,[\,\mu^\varepsilon\,]\,(t,\boldsymbol{x})\,:=\,\displaystyle\int_{\mathbb{R}^2}|v-\mathcal{V}^\varepsilon(t,\boldsymbol{x})|^q\,\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,\end{cases}$$

where $q \geq 2$.

Proposition 5.5. *Under assumptions* (5.1a)-(5.1b) *on the drift N,* (5.2) *on Ψ,* (5.16)-(5.17b) *on the initial conditions μ_0^ε consider the unique solutions μ^ε and $(\mathcal{V},\,\bar{\mu})$ to* (5.3) *and* (5.5)*. Furthermore, define the initial macroscopic error as*

$$\mathcal{E}_{\mathrm{mac}}\,=\,\|\,\mathcal{U}_0\,-\,\mathcal{U}_0^\varepsilon\,\|_{L^\infty(K)}\,+\,\|\,\rho_0-\rho_0^\varepsilon\,\|_{L^\infty(K)}\,.$$

There exists $(C\,,\,\varepsilon_0)\,\in\,\left(\mathbb{R}_*^+\right)^2$ such that

1. *for all* $\varepsilon \leq \varepsilon_0$*, it holds*

$$\| \mathcal{U}(t) - \mathcal{U}^\varepsilon(t) \|_{L^\infty(K)} \leq C \min\left(e^{C\,t} \left(\mathcal{E}_{\text{mac}} + \varepsilon \right), 1 \right), \qquad \forall\, t \in \mathbb{R}^+,$$

where $\mathcal{U}^\varepsilon$ *and* $\mathcal{U}$ *are respectively given by* (5.4) *and* (5.5)*.*

2. *For all* $\varepsilon > 0$ *and all* q *in* $[2, 2p]$ *it holds*

$$M_q[\mu^\varepsilon](t, \boldsymbol{x}) \leq C, \qquad \forall\, (t, \boldsymbol{x}) \in \mathbb{R}^+ \times K,$$

where exponent p *is given in assumption* (5.1b)*. In particular,* $\mathcal{U}^\varepsilon$ *is uniformly bounded with respect to both* $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ *and* ε*.*

3. *For all* $\varepsilon > 0$ *and all* q *in* $[2, 2p]$ *it holds*

$$D_q[\mu^\varepsilon](t, \boldsymbol{x}) \leq C \left[\exp\left(-q\, m_* \frac{t}{\varepsilon} \right) + \varepsilon^{\frac{q}{2}} \right], \quad \forall (t, \boldsymbol{x}) \in \mathbb{R}^+ \times K.$$

4. *For all* $\varepsilon > 0$ *we have*

$$| \mathcal{E}(\mu^\varepsilon(t, \boldsymbol{x}, \cdot)) | \leq C \left[\exp\left(-2\, m_* \frac{t}{\varepsilon} \right) + \varepsilon \right], \quad \forall\, (t, \boldsymbol{x}) \in \mathbb{R}^+ \times K,$$

where $\mathcal{E}$ *is defined by* (5.13)*.*

The proof of this result can be found in [24]. More precisely, we refer to [24, Proposition 4.4] for the proof of (1), [24, Proposition 3.1] for the proof of (2) , [24, Proposition 3.3] for the proof of (3) and [24, Proposition 3.5] for the proof of (4).

5.3 Convergence analysis in L^1

In this section, we prove Theorem 5.1 which ensures the convergence of ν^ε towards the asymptotic profile $\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}$ in a L^1 setting. In order to explain our argument, we outline the main steps of our approach on a simplified example : the diffusive limit for the kinetic Fokker-Planck equation. We consider the asymptotic limit $\varepsilon \to 0$ of the following linear kinetic Fokker-Planck equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f^\varepsilon = \frac{1}{\varepsilon^2} \nabla_{\boldsymbol{v}} \cdot \left(\boldsymbol{v} f^\varepsilon + \nabla_{\boldsymbol{v}} f^\varepsilon \right),$$

where $(\boldsymbol{x}, \boldsymbol{v})$ lie in the phase space $\mathbb{R}^d \times \mathbb{R}^d$. In this context, the challenge consists in proving that as ε vanishes, it holds

$$f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{v}) \underset{\varepsilon \to 0}{\sim} \mathcal{M}(\boldsymbol{v}) \otimes \rho(t, \boldsymbol{x}),$$

where ρ is a solution to the heat equation

$$\partial_t \rho \,=\, \Delta_{\boldsymbol{x}} \rho\,,$$

and where $\mathcal{M}$ stands for the standard Maxwellian distribution over $\mathbb{R}^d$. Relying on a rather classical free energy estimate, it is possible to prove that f^ε converges to the following local equilibrium of the Fokker-Planck operator

$$\mathcal{M} \otimes \rho^\varepsilon\,,$$

where the spatial density of particles ρ^ε is defined by

$$\rho^\varepsilon \,=\, \int_{\mathbb{R}^d} f^\varepsilon \,\mathrm{d}\boldsymbol{v}\,.$$

Then, the difficulty lies in proving that the spatial density of particles ρ^ε converges to ρ. The convergence analysis is made intricate by the transport operator, which keeps us from obtaining a closed equation on ρ^ε

$$\partial_t \rho^\varepsilon \,+\, \frac{1}{\varepsilon}\nabla_{\boldsymbol{x}} \cdot \int_{\mathbb{R}^d} \boldsymbol{v}\, f^\varepsilon \,\mathrm{d}\,\boldsymbol{v} \,=\, 0\,.$$

To overcome this difficulty, our strategy consists in considering the following re-scaled quantity

$$\pi^\varepsilon(t,\boldsymbol{x}) \,=\, \int_{\mathbb{R}^d} f^\varepsilon\left(t, \boldsymbol{x} - \varepsilon\,\boldsymbol{v}, \boldsymbol{v}\right)\,\mathrm{d}\boldsymbol{v}\,.$$

On this simplified example, the advantage of considering π^ε instead of ρ^ε is straightforward as it turns out that π^ε is an exact solution of the limiting equation. Indeed, changing variables in the equation on f^ε and integrating with respect to $\boldsymbol{v}$, we obtain

$$\partial_t \pi^\varepsilon \,=\, \Delta_{\boldsymbol{x}} \pi^\varepsilon\,.$$

Therefore, the convergence analysis comes down to proving that π^ε is close to ρ^ε. It is possible to achieve this final step taking advantage of the following estimate

$$\|\,\rho^\epsilon \,-\, \pi^\varepsilon\,\|_{L^1\left(\mathbb{R}^d\right)} \,\leq\, \mathcal{A} \,+\, \mathcal{B}\,,$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined as follows

$$\left\{\begin{aligned} &\mathcal{A} \,=\, \|\,\mathcal{M} \otimes \tau_{-\varepsilon\,\boldsymbol{v}}\, \rho^\varepsilon \,-\, \tau_{-\varepsilon\,\boldsymbol{v}}\, f^\varepsilon\,\|_{L^1\left(\mathbb{R}^{2d}\right)}\,,\\ &\mathcal{B} \,=\, \int_{\mathbb{R}^d} \mathcal{M}(\tilde{\boldsymbol{v}})\,\|\,f^\varepsilon \,-\, \tau_{-\varepsilon\,\tilde{\boldsymbol{v}}}\, f^\varepsilon\,\|_{L^1\left(\mathbb{R}^{2d}\right)}\,\mathrm{d}\tilde{\boldsymbol{v}}\,, \end{aligned}\right.$$

and where $\tau_{\boldsymbol{x}_0}$ stands for the translation of vector $\boldsymbol{x}_0$ with respect to the $\boldsymbol{x}$-variable. To estimate $\mathcal{A}$, we use the first step, which ensures that f^ε is close to $\mathcal{M} \otimes \rho^\varepsilon$. Then, to estimate $\mathcal{B}$, it is sufficient to prove equicontinuity estimates for f^ε, that is

$$\|\,f^\varepsilon \,-\, \tau_{\boldsymbol{x}_0} f^\varepsilon\,\|_{L^1\left(\mathbb{R}^{2d}\right)} \,\lesssim\, |\,\boldsymbol{x}_0\,|\,.$$

In the forthcoming analysis, we adapt this argument in our context.

5.3.1 *A priori* estimates

The main object of this section consists in deriving equicontinuity estimates for the sequence of solutions $(\nu^{\varepsilon})_{\varepsilon>0}$ to equation (5.15). To obtain this result, we make use of the following key result

Lemma 5.6. *Consider δ in $\{0\,,\,1\}$ and smooth solutions f and g to the following equations*

$$\begin{cases} \partial_t f \,+\, \mathrm{div}_{\boldsymbol{y}}\left[\,\mathbf{a}\,(t\,,\,\boldsymbol{y}\,,\,\xi)\;f\,\right] \,+\, \lambda(t)\,\mathrm{div}_{\xi}\left[\,(\mathbf{b}_1\,+\,\mathbf{b}_3)\,(t\,,\,\boldsymbol{y}\,,\,\xi)\;f\,\right] \,=\, \lambda(t)^2\,\Delta_{\xi}\,f\,,\\[1ex] \partial_t\, g \,+\, \mathrm{div}_{\boldsymbol{y}}\left[\,\mathbf{a}\,(t\,,\,\boldsymbol{y}\,,\,\xi)\;g\,\right] \,+\, \lambda(t)\,\mathrm{div}_{\xi}\left[\,\mathbf{b}_2\,(t\,,\,\boldsymbol{y}\,,\,\xi)\;g\,\right] \,=\, \delta\,\lambda(t)^2\,\Delta_{\xi}\,g\,. \end{cases}$$

set on the phase space $(t,\boldsymbol{y},\xi)\in\mathbb{R}^+\times\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$, with $d_1\,\geq\,0$ and $d_2\,\geq\,1$, where

$$\left(\mathbf{a}\,:\,\mathbb{R}_+\times\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\,\to\,\mathbb{R}^{d_1}\right)\;\;\text{and}\;\;\left(\mathbf{b}_i\,:\,\mathbb{R}_+\times\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\,\to\,\mathbb{R}^{d_2}\right),\quad i\in\{1\,,\,2\,,\,3\}\,,$$

are given vector fields and where λ is a positive valued function. Suppose that f and g have positive values and are normalized as follows

$$\int_{\mathbb{R}^{d_1+d_2}} f\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\xi \,=\, \int_{\mathbb{R}^{d_1+d_2}} g\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\xi \,=\, 1\,.$$

Then it holds for all time $t\geq 0$

$$\|\,f(t)\,-\,g(t)\,\|_{L^1\left(\mathbb{R}^{d_1+d_2}\right)} \,\leq\, 2\,\sqrt{2}\,\left(\|\,f_0\,-\,g_0\,\|^{\frac{1}{2}}_{L^1\left(\mathbb{R}^{d_1+d_2}\right)} \,+\, \left(\int_0^t \mathcal{R}(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\right)\,, \tag{5.33}$$

where $\mathcal{R}$ is defined as

$$\mathcal{R}(t) \,=\, \int_{\mathbb{R}^{d_1+d_2}} \left(\frac{1}{4}\,|\,\mathbf{b}_1\,-\,\mathbf{b}_2\,|^2\,f \,+\, \lambda\,|\,\mathrm{div}_{\xi}\left[\,\mathbf{b}_3\,g \,+\, (\delta\,-\,1)\,\lambda\,\nabla_{\xi}\,g\,\right]|\right)(t\,,\,\boldsymbol{y}\,,\,\xi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\xi\,.$$

We postpone the proof of this result to Appendix C.1. Thanks to the latter lemma, we prove the following equicontinuity estimate for solutions to (5.15)

Proposition 5.7. *Consider a sequence $(\nu^{\varepsilon})_{\varepsilon>0}$ of smooth solutions to equation* (5.15) *whose initial conditions meet assumption* (5.21). *There exists a positive constant C independent of ε such that for all $\varepsilon\,>\,0$, it holds*

$$\|\,\nu^{\varepsilon}(t,\boldsymbol{x}) - \tau_{w_0}\,\nu^{\varepsilon}(t,\boldsymbol{x})\,\|_{L^1(\mathbb{R}^2)} \leq\, C\left(\,\left|\,e^{b\,t}\,w_0\,\right| \,+\, \left|\,e^{b\,t}\,w_0\,\right|^{\frac{1}{2}}\,\right),$$

for all $(t,\boldsymbol{x},w_0)\in\mathbb{R}^+\times K\times\mathbb{R}$, where C is explicitly given by

$$C \,=\, \sqrt{\max\left(8\,m_1\,,\,1\,/\,b\right)}\,,$$

with m_1 defined in assumption (5.21).

Proof. We fix some $\boldsymbol{x}$ in K, some positive ε and consider some w_0 in $\mathbb{R}$. Then we define a re-scaled version f of ν^ε

$$f\,(t\,,\,w\,,\,v)\;=\;e^{-b\,t}\,\nu^\varepsilon\left(t\,,\,\boldsymbol{x}\,,\,v\,,\,e^{-b\,t}\,w\right)\,.$$

We compute the equation solved by f performing the change of variable

$$w\;\mapsto\;e^{-b\,t}\,w \tag{5.34}$$

in equation (5.15), this yields

$$\partial_t\,f\;+\;\partial_w\left[e^{b\,t}\,A_0\,(\theta^\varepsilon\,v\,,\,0)\,f\right]\;+\;\frac{1}{\theta^\varepsilon}\,\partial_v\left[B_0^\varepsilon\left(t\,,\,\boldsymbol{x}\,,\,\theta^\varepsilon\,v\,,\,e^{-b\,t}\,w\right)f\right]\;=\;\frac{1}{|\theta^\varepsilon|^2}\,\mathcal{F}_{\rho_0^\varepsilon}\,[\,f\,]\,,$$

where A_0 and B_0^ε are given by (5.12). Then, we define $g\;:=\;\tau_{w_0}\,f$, which solves the following equation

$$\partial_t\,g+\partial_w\left[e^{b\,t}\,A_0\,(\theta^\varepsilon\,v\,,\,0)\,g\right]+\frac{1}{\theta^\varepsilon}\,\partial_v\left[B_0^\varepsilon\left(t\,,\,\boldsymbol{x}\,,\,\theta^\varepsilon\,v\,,\,e^{-b\,t}\,(w+w_0)\right)g\right]=\frac{1}{|\theta^\varepsilon|^2}\,\mathcal{F}_{\rho_0^\varepsilon}\,[\,g\,]\,.$$

Thanks to the change of variable (5.34), coefficients inside the w-derivatives in the equations on f and g are the same. Hence, we can apply Lemma 5.6 to f and g with the following parameters

$$\begin{cases}
(\,\delta\,,\,\lambda\,)\;=\;(\,1\,,\,1\,/\,\theta^\varepsilon\,)\,,\\
\mathbf{a}\,(t\,,\,w\,,\,v)\;=\;e^{b\,t}\,A_0\,(\theta^\varepsilon\,v\,,\,0)\,,\\
\mathbf{b}_1\,(t\,,\,w\,,\,v)\;=\;B_0^\varepsilon\left(t\,,\,\boldsymbol{x}\,,\,\theta^\varepsilon\,v\,,\,e^{-b\,t}\,w\right)\;-\;\dfrac{1}{\theta^\varepsilon}\,\rho_0^\varepsilon(\boldsymbol{x})\,v\,,\\
(\,\mathbf{b}_2\,,\,\mathbf{b}_3\,)\;=\;(\,\tau_{w_0}\,\mathbf{b}_1\,,\,0\,)\,.
\end{cases}$$

According to (5.33) in Lemma 5.6, it holds

$$\|\,f(t)\;-\;g(t)\,\|_{L^1(\mathbb{R}^2)}\;\leq\;2\,\sqrt{2}\;\|\,f_0\;-\;g_0\,\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}}\;+\;\left(\frac{1\;-\;e^{-2\,b\,t}}{b}\right)^{\frac{1}{2}}\,|w_0|\,,\quad\forall\,t\in\mathbb{R}^+\,.$$

Therefore, according to assumption (5.21), we obtain the result after inverting the change of variable (5.34) and taking the supremum over all $\boldsymbol{x}$ in K.

□

We conclude this section providing regularity estimates for the limiting distribution $\bar{\nu}$ with respect to the adaptation variable, which solves (5.8). The proof for this result is mainly computational since we have an explicit formula for the solutions to equation (5.8).

Proposition 5.8. *Consider some $\bar{\nu}_0$ satisfying assumption* (5.20)*. The solution $\bar{\nu}$ to equation* (5.8) *with initial condition $\bar{\nu}_0$ verifies*

$$\| \bar{\nu}(t) \|_{L^\infty(K\,,W^{2\,,1}(\mathbb{R}))} \leq \exp(2\,b\,t)\, \| \bar{\nu}_0 \|_{L^\infty(K\,,W^{2\,,1}(\mathbb{R}))}\,, \quad \forall\, t \in \mathbb{R}^+\,,$$

and

$$\| w\,\partial_w\, \bar{\nu}(t) \|_{L^\infty(K\,,L^1(\mathbb{R}))} = \| w\,\partial_w\, \bar{\nu}_0 \|_{L^\infty(K\,,L^1(\mathbb{R}))}\,, \quad \forall\, t \in \mathbb{R}^+\,.$$

Proof. Since $\bar{\nu}$ solves (5.8), it is given by the following formula

$$\bar{\nu}(t, \boldsymbol{x}\,, w) = e^{bt}\, \bar{\nu}_0 \left(\boldsymbol{x}\,, e^{bt}\, w \right)\,, \quad \forall\, (t, \boldsymbol{x}) \in \mathbb{R}^+ \times K\,.$$

Consequently, we easily obtain the expected result. □

We are now ready to prove the first convergence result on ν^ε.

5.3.2 Proof of Theorem 5.1

The proof is divided in three steps. First, we prove that the solution ν^ε to (5.15) converges to the local equilibrium

$$\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon\,,$$

thanks to a free energy estimate. Then, as in the example developed at the beginning of Section 5.3, we introduce an intermediate quantity $\bar{g}^\varepsilon$, which converges to the solution $\bar{\nu}$ to equation (5.8). At last, we prove that $\bar{g}^\varepsilon$ is close to $\bar{\nu}^\varepsilon$ thanks to the equicontinuity estimate given in Proposition 5.7 and therefore conclude that the marginal $\bar{\nu}^\varepsilon$ converges towards $\bar{\nu}$.

Convergence of ν^ε towards $\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon$: free energy estimate

In this section, we investigate the time evolution of the free energy along the trajectories of equation (5.15). It is defined for all $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ as

$$E\,[\,\nu^\varepsilon(t, \boldsymbol{x})\,] = \int_{\mathbb{R}^2} \nu^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \ln \left(\frac{\nu^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u})}{\mathcal{M}_{\rho_0^\varepsilon(\boldsymbol{x})}(v)} \right) \mathrm{d}\boldsymbol{u}\,.$$

More precisely, our interest lies in its decay rate, which is given by the Fisher information

$$I\left[\,\nu^\varepsilon(t, \boldsymbol{x})\,|\,\mathcal{M}_{\rho_0^\varepsilon(\boldsymbol{x})}\,\right] := \int_{\mathbb{R}^2} \left| \partial_v \ln \left(\frac{\nu^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u})}{\mathcal{M}_{\rho_0^\varepsilon(\boldsymbol{x})}(v)} \right) \right|^2 \nu^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u})\, \mathrm{d}\boldsymbol{u}\,.$$

The reason for our interest is that the latter quantity controls the following relative entropy

$$H\left[\,\nu^\varepsilon(t, \boldsymbol{x})\,|\,\mathcal{M}_{\rho_0^\varepsilon}(\boldsymbol{x}) \otimes \bar{\nu}^\varepsilon(t, \boldsymbol{x})\,\right] = \int_{\mathbb{R}^2} \nu^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \ln \left(\frac{\nu^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u})}{\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u})} \right) \mathrm{d}\boldsymbol{u}\,,$$

which itself controls the L^1-distance between ν^ε and $\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon$. This allows to deduce the following result

Proposition 5.9. *Under assumptions* (5.1a)-(5.1b) *on the drift* N *and* (5.2) *on the interaction kernel* Ψ*, consider a sequence of solutions* $(\mu^\varepsilon)_{\varepsilon>0}$ *to* (5.3) *with initial conditions satisfying assumptions* (5.16)-(5.17b) *and* (5.22)*. Then for all* $\varepsilon \leq 1$ *, it holds*

$$\left\| \nu^\varepsilon - \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon \right\|_{L^\infty(K\,,\,L^1([\,0,t\,]\times\mathbb{R}^2))} \leq \sqrt{\varepsilon}\left(2\,m_2\sqrt{t} + C\,(t+1)\right), \quad \forall\, t \geq 0\,,$$

where m_2 *is given in assumption* (5.22)*. In this result, the constant* C *only depends on* m_**,* m_p *and* $\overline{m}_p$ *(see assumptions* (5.16)-(5.17b)*) and the data of the problem* N*,* Ψ *and* A_0*.*

Proof. All along this proof, we choose some $\boldsymbol{x}$ lying in K and we omit the dependence with respect to $(t\,,\,\boldsymbol{x})$ when the context is clear. We compute the time derivative of $E\,[\,\nu^\varepsilon\,]$ multiplying equation (5.15) by $\ln\left(\nu^\varepsilon\,/\,\mathcal{M}_{\rho_0^\varepsilon}\right)$. After integrating by part the stiffer term, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,E\,[\,\nu^\varepsilon\,] + \frac{1}{|\theta^\varepsilon|^2}\,I\left[\,\nu^\varepsilon\,|\,\mathcal{M}_{\rho_0^\varepsilon}\,\right] = \mathcal{A}\,,$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = -\int_{\mathbb{R}^2} \mathrm{div}_{\boldsymbol{u}}\left[\,\mathbf{b}_0^\varepsilon\,\nu^\varepsilon\,\right]\ln\left(\frac{\nu^\varepsilon}{\mathcal{M}_{\rho_0^\varepsilon}}\right)\mathrm{d}\boldsymbol{u}\,.$$

After an integration by part, $\mathcal{A}$ rewrites as follows

$$\mathcal{A} = \frac{1}{\theta^\varepsilon}\int_{\mathbb{R}^2} B_0^\varepsilon\,(\,t\,,\,\boldsymbol{x}\,,\,\theta^\varepsilon\,v,\,w\,)\;\partial_v\left[\ln\left(\frac{\nu^\varepsilon}{\mathcal{M}_{\rho_0^\varepsilon}}\right)\right]\nu^\varepsilon\,\mathrm{d}\boldsymbol{u} + b\,,$$

where B_0^ε is given by (5.12). According to items (2) and (4) in Proposition 5.5 , $\mathcal{V}^\varepsilon$ and $\mathcal{E}(\mu^\varepsilon)$ are uniformly bounded with respect to both $(t,\boldsymbol{x}) \in \mathbb{R}^+\times K$ and $\varepsilon > 0$. Furthermore, according to assumptions (5.2) and (5.16) on Ψ and ρ_0^ε, $\Psi *_r \rho_0^\varepsilon$ is uniformly bounded with respect to both $\boldsymbol{x} \in K$ and $\varepsilon > 0$. Consequently, applying Young's inequality, assumption (5.1b) and since N is locally Lipschitz, we obtain

$$\mathcal{A} \leq \frac{1}{2\,|\theta^\varepsilon|^2}\,I\left[\,\nu^\varepsilon\,|\,\mathcal{M}_{\rho_0^\varepsilon}\,\right] + C\left(1 + \int_{\mathbb{R}^2}\left(\,|\,\theta^\varepsilon\,v\,|^{2p} + w^2\right)\nu^\varepsilon\,\mathrm{d}\boldsymbol{u}\right),$$

for some positive constant C only depending on m_*, m_p, $\overline{m}_p$ and the data of the problem : N, A and Ψ. Then we invert the change of variable (5.10) in the integral in the right-hand side of the latter inequality and apply item (2) in Proposition 5.5. In the end, it yields

$$\mathcal{A} \leq \frac{1}{2\,|\theta^\varepsilon|^2}\,I\left[\,\nu^\varepsilon\,|\,\mathcal{M}_{\rho_0^\varepsilon}\,\right] + C\,.$$

Consequently, we end up with the following differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,E\,[\,\nu^\varepsilon\,] + \frac{1}{2\,|\theta^\varepsilon|^2}\,I\left[\,\nu^\varepsilon\,|\,\mathcal{M}_{\rho_0^\varepsilon}\,\right] \leq C\,,$$

Then we substitute the Fisher information with the relative entropy in the latter inequality according to the Gaussian logarithmic Sobolev inequality, which reads as follows (see [105])

$$2\,H\left[\,\nu^\varepsilon(t,\boldsymbol{x})\,|\,\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon(t,\boldsymbol{x})\right]\,\leq\,I\left[\,\nu^\varepsilon(t,\boldsymbol{x})\,|\,\mathcal{M}_{\rho_0^\varepsilon(\boldsymbol{x})}\right]\,,$$

and we integrate between 0 and t to get

$$\int_0^t\frac{1}{|\theta^\varepsilon(s)|^2}\,H\left[\,\nu^\varepsilon(s,\boldsymbol{x})\,|\,\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon(s,\boldsymbol{x})\right]\,\mathrm{d}s\,\leq\,E\left[\,\nu_0^\varepsilon(\boldsymbol{x})\,\right]-E\left[\,\nu^\varepsilon(t,\boldsymbol{x})\,\right]\,+\,C\,t\,.$$

In the latter inequality, we bound $-E\left[\,\nu^\varepsilon(t,\boldsymbol{x})\,\right]$ thanks to the following estimate, obtained using Jensen's inequality

$$-E\left[\,\nu^\varepsilon(t,\boldsymbol{x})\,\right]\,\leq\,-\int_{\mathbb{R}^2}\nu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\,\ln\left(\mathcal{M}_1(\mathcal{W}^\varepsilon+w)\right)\mathrm{d}\boldsymbol{u}\,.$$

In the right hand side of the latter inequality, we replace ν^ε with μ^ε according to (5.6) and invert the change of variable (5.10), this yields

$$-E\left[\,\nu^\varepsilon(t,\boldsymbol{x})\,\right]\,\leq\,\frac{1}{2}\int_{\mathbb{R}^2}\mu^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\left(\ln(2\pi)+|w|^2\right)\mathrm{d}\boldsymbol{u}\,,$$

which, after applying item (2) in Proposition 5.5 to estimate the latter right hand side, ensures

$$\int_0^t\frac{1}{|\theta^\varepsilon(s)|^2}\,H\left[\,\nu^\varepsilon(s,\boldsymbol{x})\,|\,\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon(s,\boldsymbol{x})\right]\,\mathrm{d}s\,\leq\,E\left[\,\nu_0^\varepsilon(\boldsymbol{x})\,\right]\,+\,C\,(t+1)\,.$$

To estimate $E\left[\,\nu_0^\varepsilon(\boldsymbol{x})\,\right]$ in the latter inequality, we replace ν_0^ε with μ_0^ε according to (5.6) and invert the change of variable (5.10) at time $t=0$

$$E\left[\,\nu_0^\varepsilon(\boldsymbol{x})\,\right]=H\left[\,\nu_0^\varepsilon(\boldsymbol{x})\,\right]+\frac{1}{2}\int_{\mathbb{R}^2}\mu_0^\varepsilon(\boldsymbol{x},\boldsymbol{u})\left(\rho_0^\varepsilon(\boldsymbol{x})|\mathcal{V}_0^\varepsilon(\boldsymbol{x})-v|^2+\ln(2\pi)-\ln\left(\rho_0^\varepsilon(\boldsymbol{x})\right)\right)\mathrm{d}\boldsymbol{u}\,.$$

Then, we bound $H\left[\,\nu_0^\varepsilon(\boldsymbol{x})\,\right]$, $\rho_0^\varepsilon(\boldsymbol{x})$ and moments of μ_0^ε thanks to assumptions (5.22), (5.16) and (5.17a) respectively, which yields

$$\int_0^t\frac{1}{|\theta^\varepsilon(s)|^2}\,H\left[\,\nu^\varepsilon(s,\boldsymbol{x})\,|\,\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon(s,\boldsymbol{x})\right]\,\mathrm{d}s\,\leq\,m_2^2\,+\,C\,(t+1)\,.$$

To estimate the left hand side in the latter relation, we use the explicit formula (5.14) for θ^ε, which ensures that as long as ε is less than 1 and s is greater than T^ε, where T^ε is given by

$$T^\varepsilon\,=\,\frac{\varepsilon}{2\,m_*}\,|\ln(\varepsilon)|\,,$$

we have

$$\frac{1}{2\,\varepsilon}\,\leq\,\frac{1}{|\theta^\varepsilon(s)|^2}\,.$$

Consequently, we obtain

$$\int_{T^\varepsilon}^{t} H\left[\nu^\varepsilon(s,\boldsymbol{x})\,|\,\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon(s,\boldsymbol{x})\right]\,\mathrm{d}s \,\leq\, 2\,\varepsilon\, m_2^2 \,+\, C\,\varepsilon\,(t\,+\,1)\,.$$

Then, we substitute the relative entropy with the L^1-norm according to Csizár-Kullback inequality

$$\left\|\,\nu^\varepsilon(s,\boldsymbol{x}) \,-\, \mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon(s,\boldsymbol{x})\,\right\|^2_{L^1(\mathbb{R}^2)} \,\leq\, 2\,H\left[\nu^\varepsilon(s,\boldsymbol{x})\,|\,\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon(s,\boldsymbol{x})\right]\,,$$

and take the supremum over all $\boldsymbol{x}$ in K. After taking the square root, it yields

$$\sup_{\boldsymbol{x}\in K}\int_{T^\varepsilon}^{t}\left\|\,\nu^\varepsilon \,-\, \mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon\,\right\|_{L^1(\mathbb{R}^2)}(s,\boldsymbol{x})\,\mathrm{d}s \,\leq\, 2\,\sqrt{\varepsilon\,t}\,m_2 \,+\, C\,\sqrt{\varepsilon}\,(t+1)\,,\quad \forall\, t\,\geq\,0\,,$$

To conclude, we notice that since equation (5.15) is conservative, it holds

$$\sup_{\boldsymbol{x}\in K}\int_{0}^{T^\varepsilon}\left\|\,\nu^\varepsilon \,-\, \mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}^\varepsilon\,\right\|_{L^1(\mathbb{R}^2)}(s,\boldsymbol{x})\,\mathrm{d}s \,\leq\, 2\,T^\varepsilon \,\leq\, C\,\sqrt{\varepsilon}\,.$$

We sum the last two estimates to obtain the result. □

Convergence of $\bar{g}^\varepsilon$ towards $\bar{\nu}$

As in the example developed at the beginning of this section, we consider the following re-scaled version g^ε of ν^ε

$$\nu^\varepsilon\,(t\,,\,\boldsymbol{x}\,,\,v\,,\,w) \,=\, g^\varepsilon\,(t\,,\,\boldsymbol{x}\,,\,v\,,\,w\,+\,\gamma^\varepsilon(t,\boldsymbol{x})\,v)\,,$$

where γ^ε is given by

$$\gamma^\varepsilon\,(t\,,\,\boldsymbol{x}) \,=\, \frac{a\,\varepsilon}{\rho_0^\varepsilon(\boldsymbol{x})}\,\theta^\varepsilon\,(t\,,\,\boldsymbol{x})\,.$$

Operating the following change of variable in equation (5.15)

$$(t\,,\,v\,,\,w)\mapsto(t\,,\,v\,,\,w\,+\,\gamma^\varepsilon\,v)\,, \tag{5.35}$$

and integrating the equation with respect to v, the equation on the marginal $\bar{g}^\varepsilon$ of g^ε defined as

$$\bar{g}^\varepsilon\,(t\,,\,\boldsymbol{x}\,,\,w) \,=\, \int_{\mathbb{R}} g^\varepsilon\,(t\,,\,\boldsymbol{x}\,,\,v\,,\,w)\,\mathrm{d}v\,,$$

reads as follows

$$\partial_t\,\bar{g}^\varepsilon + \frac{a\,\varepsilon}{\rho_0^\varepsilon}\,\partial_w\left[\int_{\mathbb{R}}(B_0^\varepsilon\,(t,\,\boldsymbol{x},\,\theta^\varepsilon v,\,w-\gamma^\varepsilon\,v)+b\,\theta^\varepsilon\,v)\,g^\varepsilon\,\mathrm{d}v\right]-\left(\frac{a\,\varepsilon}{\rho_0^\varepsilon}\right)^2\partial_w^2\,\bar{g}^\varepsilon \,=\, \partial_w\left[\,b\,w\,\bar{g}^\varepsilon\,\right], \tag{5.36}$$

where B_0^ε is defined by (5.12). As in our example, the equation on $\bar{g}^\varepsilon$ is consistent with the limiting equation (5.8) as ε vanishes, this enables to prove that $\bar{g}^\varepsilon$ converges towards $\bar{\nu}$

Proposition 5.10. *Under assumptions* (5.1a)-(5.1b) *on the drift N and* (5.2) *on the interaction kernel Ψ, consider a sequence of solutions $(\mu^\varepsilon)_{\varepsilon>0}$ to* (5.3) *with initial conditions satisfying assumption* (5.16)-(5.17b) *and the solution $\bar{\nu}$ to equation* (5.8) *with initial condition $\bar{\nu}_0$ satisfying assumption* (5.20). *There exists a positive constant C independent of ε such that for all ε less than* 1, *it holds*

$$\| \bar{g}^\varepsilon(t) - \bar{\nu}(t) \|_{L^\infty_{\boldsymbol{x}} L^1_w} \leq 2\sqrt{2}\, \| \bar{g}^\varepsilon_0 - \bar{\nu}_0 \|^{\frac{1}{2}}_{L^\infty_{\boldsymbol{x}} L^1_w} + C\, e^{b\,t} \sqrt{\varepsilon}\,, \quad \forall\, t \in \mathbb{R}^+ .$$

In this result, the constant C only depends on m_*, m_p and $\overline{m}_p$ (see assumptions (5.16)-(5.17b)*) and the data of the problem $\bar{\nu}_0$, N, Ψ and A_0.*

Proof. All along this proof, we choose some $\boldsymbol{x}$ lying in K and some positive ε ; we omit the dependence with respect to $(t, \boldsymbol{x})$ when the context is clear. Since $\bar{\nu}$ and $\bar{g}^\varepsilon$ solve respectively equations (5.8) and (5.36), Lemma 5.6 applies with the following parameters

$$\begin{cases} (\, d_1\,,\, d_2\,,\, \delta\,,\, \lambda\,) \,=\, (\,0\,,\,1\,,\,0\,,\, a\,\varepsilon\,/\,\rho_0^\varepsilon\,)\,, \\ \mathbf{b}_2\,(t\,,\,w) \,=\, -\,\dfrac{\rho_0^\varepsilon}{a\,\varepsilon}\, b\,w\,, \\ \mathbf{b}_1\,(t\,,\,w) \,=\, \mathbf{b}_2\,(t\,,\,w) \,+\, \displaystyle\int_{\mathbb{R}} (B_0^\varepsilon\,(t,\, \boldsymbol{x},\, \theta^\varepsilon\, v,\, -\gamma^\varepsilon\, v) + b\,\theta^\varepsilon\, v)\, \dfrac{g^\varepsilon}{\bar{g}^\varepsilon}\, \mathrm{d}v\,, \\ \mathbf{b}_3\,(t\,,\,w) \,=\, -\,w\,, \end{cases}$$

where B_0^ε is given by (5.12). According to (5.33) in Lemma 5.6, it holds

$$\| \bar{g}^\varepsilon(t) - \bar{\nu}(t) \|_{L^1(\mathbb{R})} \leq 2\sqrt{2} \left(\| \bar{g}^\varepsilon_0 - \bar{\nu}_0 \|^{\frac{1}{2}}_{L^1(\mathbb{R})} + \left(\int_0^t \mathcal{R}_1(s) + \mathcal{R}_2(s)\, \mathrm{d}s \right)^{\frac{1}{2}} \right),$$

for all $t \in \mathbb{R}^+$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are given by

$$\begin{cases} \mathcal{R}_1(t) \,=\, \dfrac{1}{4} \displaystyle\int_{\mathbb{R}} \left| \int_{\mathbb{R}} (B_0^\varepsilon\,(t\,,\, \boldsymbol{x}\,,\, \theta^\varepsilon\, v\,,\, -\,\gamma^\varepsilon\, v) \,+\, b\,\theta^\varepsilon\, v\,)\, \dfrac{g^\varepsilon}{\bar{g}^\varepsilon}\, \mathrm{d}v \right|^2 \bar{g}^\varepsilon \mathrm{d}w\,, \\ \mathcal{R}_2(t) \,=\, \dfrac{a\,\varepsilon}{\rho_0^\varepsilon} \displaystyle\int_{\mathbb{R}} |\, \partial_w\, [\, w\,\bar{\nu}\,]| \,+\, \dfrac{a\,\varepsilon}{\rho_0^\varepsilon} \left| \partial_w^2\, \bar{\nu} \right| \mathrm{d}w\,. \end{cases}$$

We estimate $\mathcal{R}_1$ according to Jensen's inequality

$$\mathcal{R}_1(t) \leq \frac{1}{4} \int_{\mathbb{R}^2} |B_0^\varepsilon\,(t\,,\, \boldsymbol{x}\,,\, \theta^\varepsilon\, v\,,\, -\,\gamma^\varepsilon\, v) \,+\, b\,\theta^\varepsilon\, v|^2\, g^\varepsilon\, \mathrm{d}v\, \mathrm{d}w\,.$$

Then we bound B_0^ε : on the one hand $\mathcal{V}^\varepsilon$ is uniformly bounded with respect to both $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ and $\varepsilon > 0$ according (2) in Proposition 5.5, on the other hand according to assumptions (5.2) and (5.16) on Ψ and ρ_0^ε, $\Psi *_r \rho_0^\varepsilon$ is uniformly bounded with respect to both $\boldsymbol{x} \in K$ and $\varepsilon > 0$. Consequently, applying Young's inequality, assumption (5.1b) and since N is locally Lipschitz, we obtain

$$\mathcal{R}_1(t) \leq C \int_{\mathbb{R}^2} \left(|\, \theta^\varepsilon\, v\,|^{2p} + |\, \theta^\varepsilon\, v\,|^2 + |\, \mathcal{E}\,(\mu^\varepsilon)\,|^2 \right) g^\varepsilon \mathrm{d}\boldsymbol{u} \,,$$

as long as ε is less than 1 to ensure that γ^ε given by (5.35) is less than $a\ \theta^\varepsilon \,/\, m_*$. We invert the changes of variables (5.35) and (5.10) and apply items (3) and (4) in Proposition 5.5, this yields

$$\mathcal{R}_1(t) \leq C \left(e^{-2\rho_0^\varepsilon(\boldsymbol{x})t/\varepsilon} + \varepsilon \right) .$$

Then to estimate $\mathcal{R}_2$, we apply Proposition 5.8 to bound $\bar{\nu}$ and assumption (5.16) to lower bound ρ_0^ε, it yields

$$\mathcal{R}_2(t) \leq C\, \varepsilon\, e^{2\,b\,t} .$$

We gather the former computations and take the supremum over all $\boldsymbol{x}$ in K : it yields the expected result. □

Convergence of $\bar{\nu}^\varepsilon$ towards $\bar{\nu}$

In this section, we gather the result from the last steps to deduce that $\bar{\nu}^\varepsilon$ converges towards $\bar{\nu}$.

Proposition 5.11. *Under the assumptions of Theorem 5.1, there exists a positive constant C independent of ε such that for all ε less than 1, it holds*

$$\|\, \bar{\nu}^\varepsilon - \bar{\nu}\,\|_{L^\infty(K\,,\,L^1([\,0,\,t\,]\times\mathbb{R}))} \leq 2\sqrt{2}\,t\, \|\, \bar{\nu}_0^\varepsilon - \bar{\nu}_0\,\|_{L^\infty_{\boldsymbol{x}} L^1_w}^{\frac{1}{2}} + 2\sqrt{\varepsilon\, t}\, m_2 + C\, e^{b\,t} \sqrt{\varepsilon} \,,$$

for all $t \in \mathbb{R}^+$. In this result, the constant C only depends on m_1, m_*, m_p and $\overline{m}_p$ (see assumptions (5.16)-(5.17b)*) and the data of the problem $\bar{\nu}_0$, N, Ψ and A_0.*

Proof. We choose some $\boldsymbol{x} \in K$ and for all $t \geq 0$, we consider the following triangular inequality

$$\|\, \bar{\nu}^\varepsilon - \bar{\nu}\,\|_{L^\infty(K\,,\,L^1([\,0,\,t\,]\times\mathbb{R}))} \leq \sup_{\boldsymbol{x}\in K} \int_0^t \mathcal{A}(s,\boldsymbol{x}) + \mathcal{B}(s,\boldsymbol{x})\,\mathrm{d}s \,,$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{cases} \mathcal{A}(s,\boldsymbol{x}) = \|\, \bar{\nu}^\varepsilon(s,\boldsymbol{x}) - \bar{g}^\varepsilon(s,\boldsymbol{x})\,\|_{L^1(\mathbb{R})} \,, \\ \mathcal{B}(s,\boldsymbol{x}) = \|\, \bar{g}^\varepsilon(s,\boldsymbol{x}) - \bar{\nu}(s,\boldsymbol{x})\,\|_{L^1(\mathbb{R})} \,. \end{cases}$$

We estimate $\mathcal{B}$ applying Proposition 5.10, which ensures

$$\mathcal{B}(s,\boldsymbol{x}) \leq 2\sqrt{2}\, \|\, \bar{g}_0^\varepsilon - \bar{\nu}_0\|_{L^\infty_{\boldsymbol{x}} L^1_w}^{\frac{1}{2}} + C\, e^{b\,s} \sqrt{\varepsilon} \,, \quad \forall (s,\boldsymbol{x}) \in \mathbb{R}^+ \times K \,.$$

To bound $\| \bar{g}_0^\varepsilon - \bar{\nu}_0\|_{L^\infty_{\boldsymbol{x}} L^1_w}$, we first apply the following triangular inequality

$$\| \bar{g}_0^\varepsilon - \bar{\nu}_0\|_{L^\infty_{\boldsymbol{x}} L^1_w} \leq \| \bar{g}_0^\varepsilon - \bar{\nu}_0^\varepsilon\|_{L^\infty_{\boldsymbol{x}} L^1_w} + \| \bar{\nu}_0^\varepsilon - \bar{\nu}_0\|_{L^\infty_{\boldsymbol{x}} L^1_w} \,,$$

and then estimate $\| \bar{g}_0^\varepsilon - \bar{\nu}_0^\varepsilon\|_{L^\infty_{\boldsymbol{x}} L^1_w}$ replacing g_0^ε with ν_0^ε in the definition of $\bar{g}_0^\varepsilon$ according to the change of variable (5.35), that is

$$\| \bar{g}_0^\varepsilon - \bar{\nu}_0^\varepsilon\|_{L^\infty_{\boldsymbol{x}} L^1_w} = \sup_{\boldsymbol{x}\in K} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \nu_0^\varepsilon\,(\boldsymbol{x}, v, w - \gamma_0^\varepsilon\, v) - \nu_0^\varepsilon\,(\boldsymbol{x}, v, w)\,\mathrm{d}v \right| \mathrm{d}w\,.$$

Applying the integral triangle inequality in the latter relation, we deduce

$$\| \bar{g}_0^\varepsilon - \bar{\nu}_0^\varepsilon\|_{L^\infty_{\boldsymbol{x}} L^1_w} \leq \sup_{\boldsymbol{x}\in K} \left\| \nu_0^\varepsilon - \tau_{-\gamma_0^\varepsilon v}\, \nu_0^\varepsilon \right\|_{L^1(\mathbb{R}^2)}\,.$$

Since γ_0^ε is given by (5.35) and according to assumption (5.16) on ρ_0^ε, it holds $|\gamma_0^\varepsilon| \leq C\,\varepsilon$. Hence, we deduce

$$\| \bar{g}_0^\varepsilon - \bar{\nu}_0^\varepsilon\|_{L^\infty_{\boldsymbol{x}} L^1_w} \leq C\,\varepsilon \sup_{\boldsymbol{x}\in K} \frac{1}{|\gamma_0^\varepsilon|} \left\| \nu_0^\varepsilon - \tau_{-\gamma_0^\varepsilon v}\, \nu_0^\varepsilon \right\|_{L^1(\mathbb{R}^2)}\,.$$

Applying assumption (5.21) to bound the right hand side in the latter estimate, we obtain

$$\| \bar{g}_0^\varepsilon - \bar{\nu}_0^\varepsilon\|_{L^\infty_{\boldsymbol{x}} L^1_w} \leq C\,\varepsilon\,.$$

Gathering the latter computations, we deduce

$$\mathcal{B}(s, \boldsymbol{x}) \leq 2\sqrt{2}\, \| \bar{\nu}_0^\varepsilon - \bar{\nu}_0\|^{\frac{1}{2}}_{L^\infty_{\boldsymbol{x}} L^1_w} + C\, e^{b\,s} \sqrt{\varepsilon}\,.$$

We integrate the latter estimate between 0 and t and take the supremum over all $\boldsymbol{x} \in K$, it yields

$$\sup_{\boldsymbol{x}\in K} \int_0^t \mathcal{B}(s, \boldsymbol{x})\,\mathrm{d}s \leq 2\sqrt{2}\,t\, \| \bar{\nu}_0^\varepsilon - \bar{\nu}_0\|^{\frac{1}{2}}_{L^\infty_{\boldsymbol{x}} L^1_w} + C\, e^{b\,t} \sqrt{\varepsilon}\,.$$

To estimate $\mathcal{A}$, we replace g^ε with ν^ε in the definition of $\bar{g}^\varepsilon$ according to the change of variable (5.35), that is

$$\mathcal{A}(s, \boldsymbol{x}) = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \nu^\varepsilon\,(s, \boldsymbol{x}, v, w) - \nu^\varepsilon\,(s, \boldsymbol{x}, v, w - \gamma^\varepsilon\, v)\,\mathrm{d}v \right| \mathrm{d}w\,,$$

and then consider the following decomposition

$$\mathcal{A}(s, \boldsymbol{x}) \leq \mathcal{A}_1(s, \boldsymbol{x}) + \mathcal{A}_2(s, \boldsymbol{x})\,,$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined as follows

$$\begin{cases} \mathcal{A}_1(s, \boldsymbol{x}) = \displaystyle\int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} \mathcal{M}_{\rho_0^\varepsilon}(\tilde{v})\,(\nu^\varepsilon(s, \boldsymbol{x}, v, w) - \nu^\varepsilon(s, \boldsymbol{x}, v, w - \gamma^\varepsilon \tilde{v}))\,\mathrm{d}\tilde{v}\,\mathrm{d}v \right| \mathrm{d}w, \\[2ex] \mathcal{A}_2(s, \boldsymbol{x}) = \displaystyle\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathcal{M}_{\rho_0^\varepsilon}(v) \bar{\nu}^\varepsilon(s, \boldsymbol{x}, w - \gamma^\varepsilon v) - \nu^\varepsilon(s, \boldsymbol{x}, v, w - \gamma^\varepsilon v)\,\mathrm{d}v \right| \mathrm{d}w\,. \end{cases}$$

Applying the integral triangle inequality in the latter relations, we obtain

$$\begin{cases} \mathcal{A}_1(s,\boldsymbol{x}) \,\leq\, \displaystyle\int_{\mathbb{R}} \mathcal{M}_{\rho_0^\varepsilon}(\tilde{v}) \left\| \nu^\varepsilon \,-\, \tau_{-(\gamma^\varepsilon\,\tilde{v})}\, \nu^\varepsilon \right\|_{L^1(\mathbb{R}^2)} (s,\boldsymbol{x})\,\mathrm{d}\tilde{v}\,, \\[2ex] \mathcal{A}_2(s,\boldsymbol{x}) \,\leq\, \displaystyle\int_{\mathbb{R}^2} \left| \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon \,-\, \nu^\varepsilon \right| (s,\boldsymbol{x},v,w-\gamma^\varepsilon v)\,\mathrm{d}\boldsymbol{u}\,. \end{cases}$$

To estimate $\mathcal{A}_2$, we first perform the change of variable $w \leftarrow w - \gamma^\varepsilon v$ in the right hand side of the latter inequality, this yields

$$\mathcal{A}_2(s,\boldsymbol{x}) \,\leq\, \left\| \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon(s,\boldsymbol{x}) \,-\, \nu^\varepsilon(s,\boldsymbol{x}) \right\|_{L^1(\mathbb{R}^2)}\,.$$

Integrating the latter estimate from 0 to t, taking the supremum over all $\boldsymbol{x} \in K$ and applying Proposition 5.9 to the right hand side, we deduce

$$\int_0^t \mathcal{A}_2(s,\boldsymbol{x})\,\mathrm{d}s \,\leq\, 2\sqrt{\varepsilon\, t}\, m_2 \,+\, C\,\sqrt{\varepsilon}\,(t+1)\,.$$

To estimate $\mathcal{A}_1$, we apply Proposition 5.7 which yields

$$\mathcal{A}_1(s,\boldsymbol{x}) \,\leq\, C\sqrt{\varepsilon}\, e^{b\,s}\,.$$

After integrating the latter estimate and taking the supremum over all $\boldsymbol{x} \in K$, we deduce

$$\sup_{\boldsymbol{x}\in K} \int_0^t \mathcal{A}_1(s,\boldsymbol{x})\,\mathrm{d}s \,\leq\, C\sqrt{\varepsilon}\, e^{b\,t}\,.$$

We obtain the result gathering the former estimates. □

Theorem 5.1 is obtained taking the sum between the estimates in Propositions 5.9 and 5.11.

5.4 Convergence analysis in weighted L^2 spaces

In this section, we derive convergence estimates for μ^ε in a weighted L^2 functional framework. We take advantage of the variational structure of L^2 spaces in order to derive uniform regularity estimates for μ^ε. These key estimates are the object of the following section

5.4.1 *A priori* estimates

The main purpose of this section is to propagate the $\mathscr{H}^k$-norms along the trajectories of equation (5.15) uniformly with respect to ε. We outline the strategy in the case of the $\mathscr{H}^0$-norm. Its time derivative along the trajectories of equation (5.15) is obtained multiplying (5.15) by $\nu^\varepsilon\, m^\varepsilon$ and integrating with respect to $\boldsymbol{u}$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)} \,=\, \frac{1}{(\theta^\varepsilon)^2}\left\langle\, \mathcal{F}_{\rho_0^\varepsilon}\,[\,\nu^\varepsilon\,]\,,\,\nu^\varepsilon\,\right\rangle_{L^2(m^\varepsilon)} \,-\, \langle\,\mathrm{div}_{\boldsymbol{u}}\,[\,\mathbf{b}_0^\varepsilon\,\nu^\varepsilon\,]\,,\,\nu^\varepsilon\,\rangle_{L^2(m^\varepsilon)}\,.$$

We first point out that according to the following lemma, the term associated to the Fokker-Planck operator is dissipative and is consequently a helping term in the upcoming analysis

Lemma 5.12. *For all $\boldsymbol{x}$ in K, it holds*

$$\left\langle \mathcal{F}_{\rho_0^\varepsilon(\boldsymbol{x})}\left[\,\nu\,\right]\,,\,\nu \right\rangle_{L^2(m^\varepsilon_{\boldsymbol{x}})} \;=\; -\,\mathcal{D}_{\rho_0^\varepsilon(\boldsymbol{x})}\left[\,\nu\,\right]\,\leq\,0\,,$$

for all $\nu\,\in\,H^1\left(m^\varepsilon_{\boldsymbol{x}}\right)$, where the dissipation $\mathcal{D}_{\rho_0^\varepsilon}$ is given by

$$\mathcal{D}_{\rho_0^\varepsilon(\boldsymbol{x})}\left[\,\nu\,\right]\,=\,\int_{\mathbb{R}^2}\left|\partial_v\left(\,\nu\, m^\varepsilon_{\boldsymbol{x}}\,\right)\right|^2\,\left(m^\varepsilon_{\boldsymbol{x}}\right)^{-1}\,\mathrm{d}\boldsymbol{u}\,\geq\,0\,.$$

Proof. The Fokker-Planck operator rewrites as follows

$$\mathcal{F}_{\rho_0^\varepsilon(\boldsymbol{x})}\left[\,\nu\,\right]\,=\,\partial_v\left[\left(m^\varepsilon_{\boldsymbol{x}}\right)^{-1}\,\partial_v\left(\nu\, m^\varepsilon_{\boldsymbol{x}}\right)\right].$$

Consequently, the result is obtained integrating $\mathcal{F}_{\rho_0^\varepsilon(\boldsymbol{x})}\left[\,\nu\,\right]$ against $\nu\, m^\varepsilon$ with respect to $\boldsymbol{u}$ and then integrating by part with respect to v.

□

Therefore, the main challenge is to control the contribution of the transport operator $\mathrm{div}_{\boldsymbol{u}}\,\mathbf{b}_0^\varepsilon$ with the dissipation $\mathcal{D}_{\rho_0^\varepsilon}$ brought by the Fokker-Planck operator, which is done in the following lemma.

Lemma 5.13. *Under assumptions* (5.1a)-(5.1b) *and* (5.26) *on the drift N,* (5.2) *on the interaction kernel Ψ, consider a sequence of solutions $(\mu^\varepsilon)_{\varepsilon>0}$ to* (5.3) *with initial conditions satisfying assumptions* (5.16)-(5.17b)*. Then, for any α greater than $1/(2b\kappa)$, there exists a constant $C>0$ such that for all $\varepsilon>0$, we have*

$$-\,\left\langle\,\mathrm{div}_{\boldsymbol{u}}\left[\,\mathbf{b}_0^\varepsilon\,\nu\,\right]\,,\,\nu\,\right\rangle_{L^2(m^\varepsilon_{\boldsymbol{x}})}\;\leq\;\frac{\alpha}{\left(\theta^\varepsilon\right)^2}\,\mathcal{D}_{\rho_0^\varepsilon(\boldsymbol{x})}\left[\,\nu\,\right]\,+\,C\,\|\,\nu\,\|^2_{L^2(m^\varepsilon_{\boldsymbol{x}})}\,,$$

for all $(t\,,\,\boldsymbol{x})\,\in\,\mathbb{R}^+\times K$ and all $\nu\,\in\,H^1\left(m^\varepsilon_{\boldsymbol{x}}\right)$, where κ appears in the definition (5.25) *of $m^\varepsilon_{\boldsymbol{x}}$.*

Before getting into the heart of the proof, we point out that as long as the latter lemma holds with some α less than 1, the sum of the estimates in the two latter Lemmas yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)}\,\leq\,C\,\|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)}\,,$$

which ensures that the $\mathscr{H}^0$-norm is propagated along the curves of (5.15) uniformly with respect to ε. We follow the exact same strategy in order to propagate the $\mathscr{H}^k$-norms when k is not 0 : see Proposition 5.16 for more details. Moreover, we emphasize that the constraint (5.27) on κ in Theorem 5.3 arises from the lower bound on α in Lemma 5.13.

Remark 5.14. *Due to the structure of the space $L^2(m^\varepsilon)$, we added the confining assumption* (5.26) *on the drift N to Theorem 5.3. Our proof of Lemma 5.13 crucially relies on this assumption ; it is the only time that we use it as well.*

Proof. All along this proof, we consider some $\varepsilon > 0$ and some $(t, \boldsymbol{x})$ in $\mathbb{R}_+ \times K$; we omit the dependence with respect to $\boldsymbol{x}$ when the context is clear. Furthermore, we choose some ν in $H^1(m^\varepsilon_{\boldsymbol{x}})$. Since $\mathbf{b}^\varepsilon_{\mathbf{0}}$ is given by (5.12), we have

$$- \langle \operatorname{div}_{\boldsymbol{u}} [\, \mathbf{b}^\varepsilon_0 \, \nu \,] \, , \, \nu \, \rangle_{L^2(m^\varepsilon)} \, = \, \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 \, ,$$

where

$$\begin{cases} \mathcal{A}_1 \, = \, \dfrac{1}{\theta^\varepsilon} \displaystyle\int_{\mathbb{R}^2} \partial_v \, [\, w \, \nu \,] \, \nu \, m^\varepsilon \, \mathrm{d}\boldsymbol{u} \, - \, \int_{\mathbb{R}^2} \partial_w \, [\, A_0 \, (\theta^\varepsilon v, \, w) \, \nu \,] \, \nu \, m^\varepsilon \, \mathrm{d}\boldsymbol{u} \, , \\ \mathcal{A}_2 \, = \, - \dfrac{1}{\theta^\varepsilon} \displaystyle\int_{\mathbb{R}^2} \partial_v \, [\, (N(\mathcal{V}^\varepsilon + \theta^\varepsilon v) \, - \, N(\mathcal{V}^\varepsilon)) \, \nu \,] \, \nu \, m^\varepsilon \, \mathrm{d}\boldsymbol{u} \, , \\ \mathcal{A}_3 \, = \, \displaystyle\int_{\mathbb{R}^2} \partial_v \left[\left(v \, \Psi *_r \rho^\varepsilon_0 \, + \, \mathcal{E}(\mu^\varepsilon) \, (\theta^\varepsilon)^{-1} \right) \nu \right] \, \nu \, m^\varepsilon \, \mathrm{d}\boldsymbol{u} \, . \end{cases}$$

To estimate $\mathcal{A}_1$, we take advantage of the confining properties of A_0. When it comes to $\mathcal{A}_2$, the estimate relies on the confining properties of the non-linear drift N. The last term $\mathcal{A}_3$ gathers the lower order terms and adds no difficulty.

We start with $\mathcal{A}_1$, which rewrites as follows after exact computations and an integration by part,

$$\mathcal{A}_1 \, = \, - \int_{\mathbb{R}^2} w \, \nu \, (m^\varepsilon)^{\frac{1}{2}} \, \frac{1}{\theta^\varepsilon} \, \partial_v \, [\nu \, m^\varepsilon] \, (m^\varepsilon)^{-\frac{1}{2}} \, \mathrm{d}\boldsymbol{u} + \frac{1}{2} \int_{\mathbb{R}^2} \left(\kappa w A_0 \, (\theta^\varepsilon v, w) - \partial_w \, A_0 \right) |\nu|^2 \, m^\varepsilon \mathrm{d}\boldsymbol{u} \, .$$

According to the definition of A_0 and applying Young's inequality, we obtain

$$\mathcal{A}_1 \leq \frac{1}{2\eta_2 \, (\theta^\varepsilon)^2} \, \mathcal{D}_{\rho^\varepsilon_0} \, [\, \nu \,] + \int_{\mathbb{R}^2} \left(\frac{C\kappa^2}{\eta_1} \, |\theta^\varepsilon v|^2 + \left(C\eta_1 + \frac{\eta_2 - b\kappa}{2} \right) w^2 \right) |\nu|^2 \, m^\varepsilon \mathrm{d}\boldsymbol{u} + C \|\nu\|^2_{L^2(m^\varepsilon)},$$

for all positive η_1 and η_2 and for some positive constant C. We set $\alpha_- \, = \, (\alpha + 1/(2b\kappa))/2$, $\eta_2 = 1/(2\alpha_-)$ and $\eta_1 \, = \, (b\kappa \, - \, \eta_2)\,/(2C)$ which is positive according to the condition on α in Lemma 5.13. With this choice, we have $C\eta_1 + (\eta_2 \, - \, b\kappa)/2 \, = \, 0$ and consequently, we obtain

$$\mathcal{A}_1 \, \leq \, \frac{\alpha_-}{(\theta^\varepsilon)^2} \, \mathcal{D}_{\rho^\varepsilon_0} \, [\, \nu \,] \, + \, C \int_{\mathbb{R}^2} |\theta^\varepsilon v|^2 \, |\nu|^2 \, m^\varepsilon \, \mathrm{d}\boldsymbol{u} \, + \, C \|\nu\|^2_{L^2(m^\varepsilon)} \, , \tag{5.37}$$

for some positive constant C only depending on A_0, κ and α.

To estimate $\mathcal{A}_2$, we take advantage of the super-linear decay of N (see assumption (5.1a)) in order to control the terms growing at most linearly. We emphasize that the decaying property of N is prescribed at infinity. Consequently, it may not have confining property on bounded sets. Hence, the main point here consists in isolating the domain

where N decays super linearly.
After some exact computations and an integration by part, $\mathcal{A}_2$ rewrites

$$\mathcal{A}_2 \;=\; \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\rho_0^\varepsilon}{\theta^\varepsilon}\,v\,\left(N(\mathcal{V}^\varepsilon+\theta^\varepsilon v)\;-\;N(\mathcal{V}^\varepsilon)\right)\;-\;N'\left(\mathcal{V}^\varepsilon+\theta^\varepsilon v\right)\right]\,|\nu|^2\,m^\varepsilon\,\mathrm{d}\boldsymbol{u}\,.$$

We consider some $R > 0$ and split the former expression in three different parts

$$\mathcal{A}_2 \;=\; \mathcal{A}_{21} \;+\; \mathcal{A}_{22} \;+\; \mathcal{A}_{23}\,,$$

where

$$\begin{cases}
\mathcal{A}_{21} \;=\; \dfrac{\rho_0^\varepsilon}{2\theta^\varepsilon}\displaystyle\int_{\mathbb{R}^2}\mathbb{1}_{|\theta^\varepsilon v|\,>\,R}\,v\,\left(N(\mathcal{V}^\varepsilon+\theta^\varepsilon v)\;-\;N(\mathcal{V}^\varepsilon)\right)\,|\nu|^2\,m^\varepsilon\,\mathrm{d}\boldsymbol{u}\,,\\[2ex]
\mathcal{A}_{22} \;=\; \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2}\mathbb{1}_{|\theta^\varepsilon v|\,\leq\,R}\left[\dfrac{\rho_0^\varepsilon}{\theta^\varepsilon}\,v\left(N(\mathcal{V}^\varepsilon+\theta^\varepsilon v)\;-\;N(\mathcal{V}^\varepsilon)\right)\;-\;N'\left(\mathcal{V}^\varepsilon+\theta^\varepsilon v\right)\right]\,|\nu|^2\,m^\varepsilon\,\mathrm{d}\boldsymbol{u}\,,\\[2ex]
\mathcal{A}_{23} \;=\; -\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2}\mathbb{1}_{|\theta^\varepsilon v|>R}\,N'\left(\mathcal{V}^\varepsilon+\theta^\varepsilon v\right)\,|\nu|^2\,m^\varepsilon\,\mathrm{d}\boldsymbol{u}\,.
\end{cases}$$

The first term $\mathcal{A}_{21}$ corresponds to the the contribution of N on the domain where it decays super-linearly. Consequently, $\mathcal{A}_{21}$ is non positive for R great enough. We take advantage of the helping term $\mathcal{A}_{21}$ to control $\mathcal{A}_{22}$, which corresponds to the contribution of N on bounded sets. We estimate $\mathcal{A}_3$ taking advantage of the confining term $\mathcal{A}_{21}$ coupled with the confining assumption (5.26) on N.

Let us estimate $\mathcal{A}_{21}$. According to item (2) in Proposition 5.5, $\mathcal{V}^\varepsilon$ is uniformly bounded with respect to both $(t,\,\boldsymbol{x}) \in \mathbb{R}^+ \times K$ and ε. Therefore, since N is continuous, we bound $N(\mathcal{V}^\varepsilon)$ by a constant in the following expression

$$\mathbb{1}_{|\theta^\varepsilon v|>R}\,\theta^\varepsilon v\left(N(\mathcal{V}^\varepsilon+\theta^\varepsilon v)-N(\mathcal{V}^\varepsilon)\right)\;\leq$$
$$\mathbb{1}_{|\theta^\varepsilon v|>R}\left(|\mathcal{V}^\varepsilon+\theta^\varepsilon v|^2\,\omega(\mathcal{V}^\varepsilon+\theta^\varepsilon v)\,\frac{\theta^\varepsilon v}{\mathcal{V}^\varepsilon+\theta^\varepsilon v}+C\,|\theta^\varepsilon v|\right),$$

where ω is given in assumption (5.1a) and where the constant C depends on both N and the uniform upper bound on $\mathcal{V}^\varepsilon$. Since $\mathcal{V}^\varepsilon$ is uniformly bounded, there exists R large enough such that

$$\frac{1}{2}\;\leq\;\frac{\theta^\varepsilon v}{\mathcal{V}^\varepsilon+\theta^\varepsilon v}\,,$$

for all $|\theta^\varepsilon v| > R$. Furthermore, since $\mathcal{V}^\varepsilon$ is uniformly bounded and according to assumption (5.1a), there exists R large enough such that

$$\mathbb{1}_{|\theta^\varepsilon v|\,>\,R}\left(\frac{1}{2}\,|\mathcal{V}^\varepsilon+\theta^\varepsilon v|^2\,\omega(\mathcal{V}^\varepsilon+\theta^\varepsilon v)\;\;+\;C\,|\theta^\varepsilon v|\right)\;\leq\;0\,.$$

From now on, **we fix** R such that the latter two relations hold true. For such R, it holds

$$\mathcal{A}_{21} \,\leq\, \int_{\mathbb{R}^2} \left(\frac{m_*}{4}\, \mathbb{1}_{|\theta^\varepsilon v| \,>\, R}\ |\mathcal{V}^\varepsilon + \theta^\varepsilon v|^2\, \omega(\mathcal{V}^\varepsilon + \theta^\varepsilon v) \,+\, C\, |\theta^\varepsilon v| \right) |\nu|^2\, m^\varepsilon\, \mathrm{d}\boldsymbol{u}\,,$$

where we used that $\theta^\varepsilon \leq 1$ and where m_* is the lower bound of ρ_0^ε given by assumption (5.16). We note that the radius R depends only on N and the uniform bound on $|\mathcal{V}^\varepsilon|$. Furthermore, we introduce the following notation

$$\mathcal{N} \,=\, \int_{\mathbb{R}^2} \mathbb{1}_{|\theta^\varepsilon v| \,>\, R}\ |\mathcal{V}^\varepsilon + \theta^\varepsilon v|^2\, \omega(\mathcal{V}^\varepsilon + \theta^\varepsilon v)\ |\nu|^2\, m^\varepsilon\, \mathrm{d}\boldsymbol{u} \,\leq\, 0 \quad \text{when} \quad R \gg 1\,.$$

The term $\mathcal{N}$ corresponds to the contribution of N on the domain where it has super-linear decaying properties and according to assumption (5.1a), it is non positive for R sufficiently large. Therefore, we use $\mathcal{N}$ to control the contribution of the other terms. With this notation, the estimate on $\mathcal{A}_{21}$ rewrites

$$\mathcal{A}_{21} \,\leq\, C \int_{\mathbb{R}^2} |\theta^\varepsilon v|\ |\nu|^2\, m^\varepsilon\, \mathrm{d}\boldsymbol{u} \,+\, \frac{m_*}{4}\, \mathcal{N}\,,$$

where C and R only depend on N and the uniform bound on $|\mathcal{V}^\varepsilon|$.

We turn to $\mathcal{A}_{22}$. Since N has $\mathscr{C}^1$ regularity and relying item (2) in Proposition 5.5 , which ensures that $\mathcal{V}^\varepsilon$ stays uniformly bounded, we obtain

$$\mathcal{A}_{22} \,\leq\, C\, \frac{\rho_0^\varepsilon}{2} \int_{\mathbb{R}^2} |v|^2\ |\nu|^2\, m^\varepsilon\, \mathrm{d}\boldsymbol{u} \,+\, C \|\nu\|^2_{L^2(m^\varepsilon)}\,,$$

where C is a positive constant which may depend on m_*, N and the uniform bound on $|\mathcal{V}^\varepsilon|$. We estimate the quadratic term in v in the latter inequality thanks to the following relation

$$\frac{1}{2} \int_{\mathbb{R}^2} \left(\rho_0^\varepsilon\, |v|^2 \,-\, 1 \right) |\nu|^2\, m^\varepsilon\, \mathrm{d}\boldsymbol{u} \,=\, \int_{\mathbb{R}^2} v\, \nu\, \partial_v \left(\nu\, m^\varepsilon \right) \mathrm{d}\boldsymbol{u}\,,$$

which is obtained after exact computations and an integration by part in the right hand side of the latter equality. We apply Young's inequality to the former relation and in the end it yields

$$\mathcal{A}_{22} \,\leq\, \frac{\eta}{(\theta^\varepsilon)^2}\, \mathcal{D}_{\rho_0^\varepsilon}\left[\, \nu \,\right] \,+\, C \left(\frac{1}{\eta} \int_{\mathbb{R}^2} |\theta^\varepsilon v|^2\, |\nu|^2\, m^\varepsilon\, \mathrm{d}\boldsymbol{u} \,+\, \|\nu\|^2_{L^2(m^\varepsilon)} \right)\,,$$

for all positive η and for some positive constant C depending on m_*, N and the uniform bound on $|\mathcal{V}^\varepsilon|$.

We estimate the last term $\mathcal{A}_{23}$ taking advantage of the confining properties corresponding to $\mathcal{N}$. Indeed we have

$$\mathcal{A}_{23} \,+\, \frac{m_*}{8}\, \mathcal{N} \,=\, \int_{\mathbb{R}^2} \mathbb{1}_{|\theta^\varepsilon v| > R} \left(\frac{m_*}{8}\ |v'|^2\, \omega(v') - \frac{1}{2} N'\left(v'\right) \right) |\nu|^2\, m^\varepsilon\, \mathrm{d}\boldsymbol{u}\,,$$

where we used the shorthand notation $v' = \mathcal{V}^\varepsilon + \theta^\varepsilon v$. Hence, according to assumption (5.26), we deduce

$$\mathcal{A}_{23} + \frac{m_*}{8}\mathcal{N} \leq C\|\nu\|^2_{L^2(m^\varepsilon)},$$

for some positive constant C depending on m_* and N. Gathering these computations, we obtain

$$\mathcal{A}_2 \leq \frac{\eta}{(\theta^\varepsilon)^2}\mathcal{D}_{\rho_0^\varepsilon}[\,\nu\,] + C\left(\frac{1}{\eta}+1\right)\int_{\mathbb{R}^2}|\theta^\varepsilon v|^2\,|\nu|^2\,m^\varepsilon\,\mathrm{d}\boldsymbol{u} + C\|\nu\|^2_{L^2(m^\varepsilon)} + \frac{m_*}{8}\mathcal{N}, \quad (5.38)$$

for all $\eta > 0$ and where C is a positive constant which may depend on m_*, R, N and the uniform bound on $|\mathcal{V}^\varepsilon|$.

We turn to $\mathcal{A}_3$, which gathers terms of lower-order. We integrate by part and apply Young's inequality. It yields

$$\mathcal{A}_3 \leq \frac{\eta}{(\theta^\varepsilon)^2}\mathcal{D}_{\rho_0^\varepsilon}[\,\nu\,] + \frac{1}{2\eta}\left(|\Psi *_r \rho_0^\varepsilon|^2\int_{\mathbb{R}^2}|\theta^\varepsilon v|^2\,|\nu|^2\,m^\varepsilon\,\mathrm{d}\boldsymbol{u} + |\mathcal{E}(\mu^\varepsilon)|^2\,\|\nu\|^2_{L^2(m^\varepsilon)}\right),$$

for all positive η. According to item (4) in Proposition 5.5 , $\mathcal{E}(\mu^\varepsilon)$ is uniformly bounded with respect to both $(t,\boldsymbol{x}) \in \mathbb{R}^+ \times K$ and $\varepsilon > 0$. Furthermore, according to assumptions (5.2) and (5.16) on Ψ and ρ_0^ε, $\Psi *_r \rho_0^\varepsilon$ is uniformly bounded with respect to both $\boldsymbol{x} \in K$ and $\varepsilon > 0$. Consequently, we obtain

$$\mathcal{A}_3 \leq \frac{\eta}{(\theta^\varepsilon)^2}\mathcal{D}_{\rho_0^\varepsilon}[\,\nu\,] + \frac{C}{\eta}\left(\int_{\mathbb{R}^2}|\theta^\varepsilon v|^2\,|\nu|^2\,m^\varepsilon\,\mathrm{d}\boldsymbol{u} + \|\nu\|^2_{L^2(m^\varepsilon)}\right), \quad (5.39)$$

for some positive constant C which may depend on m_* (see assumption (5.16)), m_p and $\overline{m}_p$ (see assumptions (5.17a) and (5.17b)) and the data of our problem N, Ψ and A_0.

Gathering estimates (5.37), (5.38) and (5.39), it yields

$$-\langle\,\mathrm{div}_{\boldsymbol{u}}[\,\mathbf{b}_0^\varepsilon\,\nu\,]\,,\,\nu\,\rangle \leq \frac{\alpha_- + 2\eta}{(\theta^\varepsilon)^2}\mathcal{D}_{\rho_0^\varepsilon}[\,\nu\,] + C\left(1+\frac{1}{\eta}\right)\int_{\mathbb{R}^2}\left(|\theta^\varepsilon v|^2+1\right)|\nu|^2\,m^\varepsilon\mathrm{d}\boldsymbol{u} + \frac{m_*}{8}\mathcal{N},$$

for all positive η. Hence, we choose $2\eta = \alpha - \alpha_-$. Therefore, replacing $\mathcal{N}$ by its definition, the former estimate rewrites

$$\begin{aligned}-\langle\mathrm{div}_{\boldsymbol{u}}[\,\mathbf{b}_0^\varepsilon\,\nu\,]\,,\,\nu\rangle \leq\;& \frac{\alpha}{(\theta^\varepsilon)^2}\mathcal{D}_{\rho_0^\varepsilon}[\,\nu\,]\\ &+ \int_{\mathbb{R}^2}\left(C\left(|\theta^\varepsilon v|^2+1\right) + \mathbb{1}_{|\theta^\varepsilon v|>R}\,\frac{m_*}{8}\,|v'|^2\,\omega(v')\right)|\nu|^2\,m^\varepsilon\mathrm{d}\boldsymbol{u},\end{aligned}$$

where we used the shorthand notation $v' = \mathcal{V}^\varepsilon + \theta^\varepsilon v$. To conclude, we estimate the right-hand side in the latter inequality applying assumption (5.1a) on N. Since $\mathcal{V}^\varepsilon$ is uniformly bounded, we obtain

$$-\langle\,\mathrm{div}_{\boldsymbol{u}}[\,\mathbf{b}_0^\varepsilon\,\nu\,]\,,\,\nu\,\rangle_{L^2(m^\varepsilon)} \leq \frac{\alpha}{(\theta^\varepsilon)^2}\mathcal{D}_{\rho_0^\varepsilon}[\,\nu\,] + C\|\nu\|^2_{L^2(m^\varepsilon)},$$

for some constant C only depending on α, κ, m_*, m_p, $\overline{m}_p$ and the data of the problem : N, A_0 and Ψ. □

We also mention the following general result, which may be interpreted as a Poincaré inequality in the functional space $L^2\,(m^\varepsilon)$

Lemma 5.15. *For all $\boldsymbol{x} \in K$ and all function ν in $H^1_w\,(m^\varepsilon_{\boldsymbol{x}})$, hold the following estimates*

$$\|\,\nu\,\|_{L^2(m^\varepsilon_{\boldsymbol{x}})} \,\leq\, \frac{1}{\sqrt{\kappa}}\,\|\,\partial_w\,\nu\,\|_{L^2(m^\varepsilon_{\boldsymbol{x}})} \quad \text{and} \quad \|\,w\,\nu\,\|_{L^2(m^\varepsilon_{\boldsymbol{x}})} \,\leq\, \frac{2}{\kappa}\,\|\,\partial_w\,\nu\,\|_{L^2(m^\varepsilon_{\boldsymbol{x}})}\,.$$

Proof. The proof relies on the following relation

$$\frac{1}{2}\int_{\mathbb{R}^2}\left(1\,+\,\kappa\,w^2\right)|\,\nu\,|^2\;m^\varepsilon_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,=\,-\int_{\mathbb{R}^2} w\,\nu\,\partial_w\,\nu\,m^\varepsilon_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,$$

which is obtained after an integration by part in the right-hand side of the equality. From the latter relation we obtain the result applying Young's inequality in the right-hand side for the first estimate and Cauchy-Schwarz inequality for the second one.

□

From Lemma 5.13, we deduce regularity estimates for the solution ν^ε to equation (5.15). The main challenge consists in propagating the $\mathscr{H}^0$-norm. Then we easily adapt our analysis to the case of the $\mathscr{H}^k$-norms, when k is greater than 0. Indeed, the w-derivatives of ν^ε solve equation (5.15) with additional source terms which we are able to control with the dissipation brought by the Fokker-Planck operator. More precisely, equation (5.15) on ν^ε reads as follows

$$\partial_t\,\nu^\varepsilon\,+\,\mathscr{A}^\varepsilon\,[\,\nu^\varepsilon\,]\,=\,0\,,$$

where the operator $\mathscr{A}^\varepsilon$ is given by

$$\mathscr{A}^\varepsilon\,[\,\nu^\varepsilon\,]\,=\,\mathrm{div}_{\boldsymbol{u}}\,[\,\mathbf{b}^\varepsilon_0\,\nu^\varepsilon\,]\,-\,\frac{1}{(\theta^\varepsilon)^2}\,\mathcal{F}_{\rho^\varepsilon_0}\,[\,\nu^\varepsilon\,]\,. \tag{5.40}$$

With this notation, the equations on the w-derivatives read as follows

$$\partial_t\,h^\varepsilon\,+\,\mathscr{A}^\varepsilon\,[\,h^\varepsilon\,]\,=\,\frac{1}{\theta^\varepsilon}\,\partial_v\,\nu^\varepsilon\,+\,b\,h^\varepsilon\,, \tag{5.41}$$

where $h^\varepsilon\,=\,\partial_w\,\nu^\varepsilon$, and

$$\partial_t\,g^\varepsilon\,+\,\mathscr{A}^\varepsilon\,[\,g^\varepsilon\,]\,=\,\frac{2}{\theta^\varepsilon}\,\partial_v\,h^\varepsilon\,+\,2\,b\,g^\varepsilon\,, \tag{5.42}$$

where g^ε is given by $\partial^2_w\,\nu^\varepsilon$.

Proposition 5.16. *Under assumptions* (5.1a)-(5.1b) *and* (5.26) *on the drift* N*,* (5.2) *on the interaction kernel* Ψ*, consider a sequence of smooth solutions* $(\mu^\varepsilon)_{\varepsilon>0}$ *to* (5.3) *with initial conditions satisfying assumptions* (5.16)-(5.17b) *and* (5.28) *with an exponent* κ *greater than* $1/(2b)$*. Then, there exists a constant* $C > 0$*, such that, for all* $\varepsilon > 0$*, we have*

$$\left\| \partial_w^k \nu^\varepsilon(t\,,\,\boldsymbol{x}) \right\|_{L^2(m^\varepsilon_{\boldsymbol{x}})} \,\leq\, e^{Ct} \left\| \partial_w^k \nu_0^\varepsilon(\boldsymbol{x}) \right\|_{L^2(m^\varepsilon_{\boldsymbol{x}})}\,, \quad \forall\,(t,\boldsymbol{x}) \,\in\, \mathbb{R}^+ \times K\,,$$

for all k *in* $\{0,1,2\}$*.*

Proof. We start with $k = 0$. We compute the time derivative of $\|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)}$ multiplying equation (5.15) by $\nu^\varepsilon\, m$ and integrating with respect to $\boldsymbol{u}$. After integrating by part the stiffer term, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)} \,+\, \frac{1}{(\theta^\varepsilon)^2}\,\mathcal{D}_{\rho_0^\varepsilon}\,[\,\nu^\varepsilon\,] \,=\, -\,\langle\,\mathrm{div}_{\boldsymbol{u}}\,[\,\mathbf{b}_0^\varepsilon\,\nu^\varepsilon\,]\,,\,\nu^\varepsilon\,\rangle_{L^2(m^\varepsilon)}\,,$$

for all $\varepsilon > 0$ and all $(t,\,\boldsymbol{x}) \in \mathbb{R}_+ \times K$. Since κ is greater than $1/(2b)$, we apply Lemma 5.13 with $\alpha = 1$. This leads to the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)} \,\leq\, C\,\|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)}\,,$$

for some constant C only depending on κ, m_*, m_p, $\overline{m}_p$ and on the data of the problem : N, A_0 and Ψ. According to Gronwall's lemma, it yields

$$\|\,\nu^\varepsilon(t\,,\,\boldsymbol{x})\,\|_{L^2(m^\varepsilon_{\boldsymbol{x}})} \,\leq\, e^{Ct}\,\|\,\nu^\varepsilon(0\,,\,\boldsymbol{x})\,\|_{L^2(m^\varepsilon_{\boldsymbol{x}})}\,, \quad \forall\,(t,\boldsymbol{x}) \,\in\, \mathbb{R}^+ \times K\,.$$

Let us now treat the case $k = 1$. We write $h^\varepsilon = \partial_w\,\nu^\varepsilon$. We compute the derivative of $\|\,h^\varepsilon\,\|^2_{L^2(m^\varepsilon)}$ multiplying equation (5.41) by $h^\varepsilon\, m^\varepsilon$ and integrating with respect to $\boldsymbol{u}$. After integrating by part the stiffer term, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,h^\varepsilon\,\|^2_{L^2(m^\varepsilon)} \,+\, \frac{1}{(\theta^\varepsilon)^2}\,\mathcal{D}_{\rho_0^\varepsilon}\,[\,h^\varepsilon\,] \,=\, -\,\langle\,\mathrm{div}_{\boldsymbol{u}}\,[\,\mathbf{b}_0^\varepsilon\,h^\varepsilon\,]\,,\,h^\varepsilon\,\rangle_{L^2(m^\varepsilon)} \,+\, b\,\|\,h^\varepsilon\,\|^2_{L^2(m^\varepsilon)} \,+\, \mathcal{B}\,,$$

for all $\varepsilon > 0$ and all $(t,\,\boldsymbol{x}) \in \mathbb{R}_+ \times K$, where $\mathcal{B}$ is given by

$$\mathcal{B} \,=\, \frac{1}{\theta^\varepsilon}\int_{\mathbb{R}^2} \partial_v\,\nu^\varepsilon\,h^\varepsilon\,m^\varepsilon\,\mathrm{d}\boldsymbol{u}\,.$$

We estimate $\mathcal{B}$ integrating by part and applying Young's inequality. It yields

$$\mathcal{B} \,\leq\, \frac{C}{\eta}\,\|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)} + \frac{\eta}{(\theta^\varepsilon)^2}\,\mathcal{D}_{\rho_0^\varepsilon}\,[\,h^\varepsilon\,]\,.$$

for some positive constant C and for all positive η. Then we apply Lemma 5.15, which yields

$$\mathcal{B} \,\leq\, \frac{C}{\eta}\,\|\,h^\varepsilon\,\|^2_{L^2(m^\varepsilon)} + \frac{\eta}{(\theta^\varepsilon)^2}\,\mathcal{D}_{\rho_0^\varepsilon}\,[\,h^\varepsilon\,]\,,$$

and conclude this step following the same method as in the former step of the proof.

The last case $k = 2$ relies on the same arguments as the former step. Indeed, equation (5.42) on $\partial_w^2 \nu^\varepsilon$ is the same as equation (5.41) on $\partial_w \nu^\varepsilon$ up to a constant. Consequently, we skip the details and conclude this proof.

□

Due to the cross terms between the v and w variables in equation (5.3), we are led to estimate mixed quantities of the form $w^{k_1} \partial_w^{k_2} \nu^\varepsilon$. These estimates are easily obtained from Proposition 5.16 and Lemma 5.15

Corollary 5.17. *Under the assumptions of Proposition 5.16, we consider* $(k\,,\,k_1\,,\,k_2)$ *in* $\mathbb{N}^3$ *such that* $k_1 + k_2 = k$ *and* $k \leq 2$. *there exists a constant* $C > 0$, *such that, for all* $\varepsilon > 0$, *we have*

$$\left\| \left(w^{k_1} \partial_w^{k_2} \right) \nu^\varepsilon(t\,,\,\boldsymbol{x}) \right\|_{L^2(m^\varepsilon_{\boldsymbol{x}})} \leq \left(\frac{2}{\kappa} \right)^{k_1} e^{Ct} \left\| \partial_w^k \nu_0^\varepsilon(\boldsymbol{x}) \right\|_{L^2(m^\varepsilon_{\boldsymbol{x}})}\,, \quad \forall\,(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K\,,$$

Proof. We consider $(k\,,\,k_1\,,\,k_2)$ in $\mathbb{N}^3$ such that $k_1 + k_2 = k$ and $k \leq 2$ and point out that according to Lemma 5.15, we have

$$\left\| \left(w^{k_1} \partial_w^{k_2} \right) \nu^\varepsilon(t\,,\,\boldsymbol{x}) \right\|_{L^2(m^\varepsilon_{\boldsymbol{x}})} \leq \left(\frac{2}{\kappa} \right)^{k_1} \left\| \partial_w^k \nu^\varepsilon(t\,,\,\boldsymbol{x}) \right\|_{L^2(m^\varepsilon_{\boldsymbol{x}})}\,.$$

Consequently, we obtain the result applying Proposition 5.16.

□

We conclude this section with providing regularity estimates for the limiting distribution $\bar{\nu}$ with respect to the adaptation variable, which solves (5.8). The proof for this result is mainly computational since we have an explicit formula for the solutions to equation (5.8).

Lemma 5.18. *Consider some index* k *lying in* $\{0\,,\,1\}$ *and some* $\bar{\nu}_0$ *lying in* $\mathscr{H}^k\,(\overline{m})$. *The solution* $\bar{\nu}$ *to equation* (5.8) *with initial condition* $\bar{\nu}_0$ *verifies*

$$\|\,\bar{\nu}(t)\,\|_{\mathscr{H}^k(\overline{m})} \leq \exp\left(\left(k + \frac{1}{2} \right) b\,t \right) \|\,\bar{\nu}_0\,\|_{\mathscr{H}^k(\overline{m})}\,, \quad \forall\, t \in \mathbb{R}^+\,.$$

Proof. Since $\bar{\nu}$ solves (5.8), it is given by the following formula

$$\bar{\nu}_{t,\,\boldsymbol{x}}(w) = e^{bt}\,\bar{\nu}_{0,\,\boldsymbol{x}}\left(e^{bt}\,w \right)\,, \quad \forall\,(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K\,.$$

Consequently, we easily obtain the expected result. □

5.4.2 Proof of Theorem 5.3

In the forthcoming analysis we quantify the convergence of ν^ε towards the asymptotic profile ν given by

$$\nu \,=\, \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}\,,$$

in the functional spaces $\mathscr{H}^k\left(m^\varepsilon\right)$. We introduce the orthogonal projection of ν^ε onto the space of function with marginal $\mathcal{M}_{\rho_0^\varepsilon}$ with respect to the voltage variable

$$\Pi\,\nu^\varepsilon \,=\, \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon\,.$$

Furthermore, we consider the orthogonal component $\nu_\perp^\varepsilon$ of ν^ε with respect to the latter projection

$$\nu_\perp^\varepsilon \,=\, \nu^\varepsilon \,-\, \Pi\,\nu^\varepsilon\,.$$

With these notations we have

$$\|\,\nu^\varepsilon \,-\, \nu\,\|^2_{\mathscr{H}^k(m^\varepsilon)} \,=\, \|\,\nu_\perp^\varepsilon\,\|^2_{\mathscr{H}^k(m^\varepsilon)} \,+\, \|\,\bar{\nu}^\varepsilon \,-\, \bar{\nu}\,\|^2_{\mathscr{H}^k(\overline{m})}\,,$$

for k in $\{0\,,\,1\}$. Therefore, we prove that $\nu_\perp^\varepsilon$ and $\bar{\nu}^\varepsilon \,-\, \bar{\nu}$ vanish as ε goes to zero in both $\mathscr{H}^0$ and $\mathscr{H}^1$.

Estimates for $\nu_\perp^\varepsilon$

Our strategy relies on the same arguments as the ones we developed in the former section to prove Proposition 5.16. Indeed, the equation satisfied by $\nu_\perp^\varepsilon$ is the same equation as equation (5.15) solved by ν^ε with additional source terms. It reads as follows

$$\partial_t\,\nu_\perp^\varepsilon \,+\, \mathscr{A}^\varepsilon\,[\,\nu_\perp^\varepsilon\,] \,=\, \mathscr{S}\,[\,\nu^\varepsilon\,,\,\Pi\,\nu^\varepsilon\,]\,, \tag{5.43}$$

where the operator $\mathscr{A}^\varepsilon$ is given by (5.40) and the source terms are given by

$$\mathscr{S}^\varepsilon\,[\,\nu^\varepsilon\,,\,\Pi\,\nu^\varepsilon\,] \,=\, \partial_w\left[a\,\theta^\varepsilon\int v\,\nu^\varepsilon\,\mathrm{d}v\,\mathcal{M}_{\rho_0^\varepsilon} \,-\, b\,w\,\Pi\,\nu^\varepsilon\right] \,-\, \mathrm{div}_{\boldsymbol{u}}\,[\,\mathbf{b}_0^\varepsilon\,\Pi\,\nu^\varepsilon\,]\,.$$

Consequently, our strategy consists in estimating the source terms using the regularity estimates provided by Proposition 5.16. Then, we adapt our analysis to the case of $\partial_w\,\nu_\perp^\varepsilon$, which we write $h_\perp^\varepsilon \,=\, \partial_w\,\nu_\perp^\varepsilon$ and which solves again the same equation up to extra terms that add no difficulty

$$\partial_t\,h_\perp^\varepsilon \,+\, \mathscr{A}^\varepsilon\,[\,h_\perp^\varepsilon\,] \,=\, \mathscr{S}\,[\,h^\varepsilon\,,\,\Pi\,h^\varepsilon\,] \,+\, b\,h_\perp^\varepsilon \,+\, \frac{1}{\theta^\varepsilon}\,\partial_v\,\nu^\varepsilon\,, \tag{5.44}$$

where we used the notation $h^\varepsilon \,=\, \partial_w\,\nu^\varepsilon$.

Proposition 5.19. *Under assumptions* (5.1a)-(5.1b) *and* (5.26) *on the drift* N *and* (5.2) *on the interaction kernel* Ψ*, consider a sequence of solutions* $(\mu^\varepsilon)_{\varepsilon>0}$ *to* (5.3) *with initial conditions satisfying assumptions* (5.16)-(5.17b) *and* (5.28) *with an index* k *in* $\{0\,,\,1\}$ *and an exponent* κ *greater than* $1/(2b)$*. Then, for all* ε *between* 0 *and* 1 *and any* α_* *lying in* $\left(0\,,\,1-(2b\kappa)^{-1}\right)$*, there exists a constant* $C>0$*, independent of* ε*, such that*

$$\|\nu^\varepsilon_\perp(t)\|_{\mathscr{H}^k(m^\varepsilon)} \,\leq\, e^{Ct}\,\|\,\nu^\varepsilon_0\,\|_{\mathscr{H}^{k+1}(m^\varepsilon)}\,\left(C\,\sqrt{\varepsilon}\,+\,\min\left\{1\,,\,e^{-\alpha_* t/\varepsilon}\,\varepsilon^{-\alpha_*/(2\,m_*)}\right\}\right)\,,$$

for all $t\geq 0$.

Proof. We first treat the case $k=0$. All along this step of the proof, we consider some $\varepsilon>0$ and some $(t,\,\boldsymbol{x})$ in $\mathbb{R}_+\times K$; we omit the dependence with respect to $(t,\boldsymbol{x})$ when the context is clear. We compute the time derivative of $\|\,\nu^\varepsilon_\perp\,\|^2_{L^2(m^\varepsilon)}$ multiplying equation (5.43) by $\nu^\varepsilon_\perp\,m^\varepsilon$ and integrating with respect to $\boldsymbol{u}$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nu^\varepsilon_\perp\|^2_{L^2(m^\varepsilon)} \,=\, \langle\,\mathscr{S}^\varepsilon\,[\,\nu^\varepsilon\,,\,\Pi\,\nu^\varepsilon\,]\,,\,\nu^\varepsilon_\perp\,\rangle_{L^2(m^\varepsilon)} \,-\, \langle\,\mathscr{A}^\varepsilon\,[\,\nu^\varepsilon_\perp\,]\,,\,\nu^\varepsilon_\perp\,\rangle_{L^2(m^\varepsilon)}\,,$$

and we split the contribution of the source terms as follows

$$\langle\,\mathscr{S}^\varepsilon\,[\,\nu^\varepsilon\,,\,\Pi\,\nu^\varepsilon\,]\,,\,\nu^\varepsilon_\perp\,\rangle_{L^2(m^\varepsilon)} \,=\, \mathcal{A}_1\,+\,\mathcal{A}_2\,,$$

where the terms $\mathcal{A}_1$ and $\mathcal{A}_2$ are given by

$$\begin{cases} \mathcal{A}_1 \,=\, -\dfrac{1}{\theta^\varepsilon}\displaystyle\int_{\mathbb{R}^2}\partial_v\,[\,B^\varepsilon_0\,(t\,,\,\boldsymbol{x}\,,\,\theta^\varepsilon\,v\,,\,w)\,\Pi\,\nu^\varepsilon\,]\;\nu^\varepsilon_\perp\,m^\varepsilon\,\mathrm{d}\boldsymbol{u}\,,\\[2ex] \mathcal{A}_2 \,=\, -a\,\theta^\varepsilon\displaystyle\int_{\mathbb{R}^2} v\,\partial_w\,[\,\Pi\,\nu^\varepsilon\,]\;\nu^\varepsilon_\perp\,m^\varepsilon\,\mathrm{d}\boldsymbol{u}\,, \end{cases}$$

where B^ε_0 is given by (5.12).

Let us estimate $\mathcal{A}_1$. After an integration by part, this term rewrites as follows

$$\mathcal{A}_1 \,=\, \int_{\mathbb{R}^2} B^\varepsilon_0\,(t\,,\,\boldsymbol{x}\,,\,\theta^\varepsilon\,v\,,\,w)\;\Pi\,\nu^\varepsilon\;(m^\varepsilon)^{\frac{1}{2}}\;\frac{1}{\theta^\varepsilon}\,\partial_v\,[\,\nu^\varepsilon_\perp\,m^\varepsilon\,]\;(m^\varepsilon)^{-\frac{1}{2}}\;\mathrm{d}\boldsymbol{u}\,.$$

According to items (2) and (4) in Proposition 5.5 , $\mathcal{V}^\varepsilon$ and $\mathcal{E}\,(\mu^\varepsilon)$ are uniformly bounded with respect to both $(t,\,\boldsymbol{x})\in\mathbb{R}^+\times K$ and ε. Moreover, according to assumptions (5.2) and (5.16), $\Psi *_r \rho^\varepsilon_0$ stays uniformly bounded with respect to both $\boldsymbol{x}\in K$ and ε as well. Consequently, applying Young's inequality to the former relation and using assumption (5.1b), which ensures that N has polynomial growth, we obtain

$$\mathcal{A}_1 \,\leq\, \frac{\eta}{(\theta^\varepsilon)^2}\,\mathcal{D}_{\rho^\varepsilon_0}\,[\,\nu^\varepsilon_\perp\,]\,+\,\frac{C}{\eta}\int_{\mathbb{R}^2}\left(|\theta^\varepsilon v|^{2p}\,+\,|w|^2\,+\,1\right)\,|\overline{\nu}^\varepsilon|^2\;\mathcal{M}_{\rho^\varepsilon_0}\,\overline{m}\,\mathrm{d}\boldsymbol{u}\,,$$

for all positive η and for some positive constant C which only depends on m_*, m_p and $\overline{m}_p$ and the data of the problem N, Ψ and A_0. Taking advantage of the properties of the Maxwellian $\mathcal{M}_{\rho_0^\varepsilon}$ and since ρ_0^ε meets assumption (5.16), the latter estimate simplifies into

$$\mathcal{A}_1 \,\leq\, \frac{\eta}{(\theta^\varepsilon)^2}\,\mathcal{D}_{\rho_0^\varepsilon}\left[\,\nu_\perp^\varepsilon\,\right] + \frac{C}{\eta}\left(\|\,\bar{\nu}^\varepsilon\,\|^2_{L^2(\overline{m})} + \|\,w\,\bar{\nu}^\varepsilon\,\|^2_{L^2(\overline{m})}\right).$$

Furthermore, according to Jensen's inequality, it holds

$$\|\,\bar{\nu}^\varepsilon\,\|^2_{L^2(\overline{m})} + \|\,w\,\bar{\nu}^\varepsilon\,\|^2_{L^2(\overline{m})} \,\leq\, \|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)} + \|\,w\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)}\,.$$

Therefore, we apply Corollary 5.17 and obtain the following estimate for $\mathcal{A}_1$

$$\mathcal{A}_1 \,\leq\, \frac{\eta}{(\theta^\varepsilon)^2}\,\mathcal{D}_{\rho_0^\varepsilon}\left[\,\nu_\perp^\varepsilon\,\right] + \frac{C}{\eta}\,e^{Ct}\,\|\,\nu_0^\varepsilon\,\|^2_{H^1_w(m^\varepsilon)}\,.$$

To estimate $\mathcal{A}_2$, we apply the Cauchy-Schwarz inequality, use the properties of the Maxwellian $\mathcal{M}_{\rho_0^\varepsilon}$ and assumption (5.16). It yields

$$\mathcal{A}_2 \,\leq\, C\,\left(\|\,\partial_w\,\bar{\nu}^\varepsilon\,\|^2_{L^2(m^\varepsilon)} + \|\,\nu_\perp^\varepsilon\,\|^2_{L^2(m^\varepsilon)}\right).$$

According to the same remark as in the former step, it holds

$$\|\,\partial_w\,\bar{\nu}^\varepsilon\,\|^2_{L^2(\overline{m})} + \|\,\nu_\perp^\varepsilon\,\|^2_{L^2(m^\varepsilon)} \,\leq\, \|\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)} + \|\,\partial_w\,\nu^\varepsilon\,\|^2_{L^2(m^\varepsilon)}\,,$$

hence, applying Proposition 5.16, we obtain

$$\mathcal{A}_2 \,\leq\, C\,e^{Ct}\,\|\,\nu_0^\varepsilon\,\|^2_{H^1_w(m^\varepsilon)}\,.$$

To evaluate the contribution of $\mathscr{A}^\varepsilon$ we replace it by its definition (5.40) and integrate by part the stiffer term. It yields

$$-\,\langle\,\mathscr{A}^\varepsilon\left[\,\nu_\perp^\varepsilon\,\right],\,\nu_\perp^\varepsilon\,\rangle_{L^2(m^\varepsilon)} \,=\, -\,\frac{1}{(\theta^\varepsilon)^2}\mathcal{D}_{\rho_0^\varepsilon}\left[\,\nu_\perp^\varepsilon\,\right] - \langle\,\mathrm{div}_{\boldsymbol{u}}\left[\,\mathbf{b}_0^\varepsilon\,\nu_\perp^\varepsilon\,\right],\,\nu_\perp^\varepsilon\,\rangle_{L^2(m^\varepsilon)}\,.$$

In order to close the estimate, we apply Lemma 5.13 and Proposition 5.16 to control the term associated to linear transport. It yields

$$-\,\langle\,\mathrm{div}_{\boldsymbol{u}}\left[\,\mathbf{b}_0^\varepsilon\,\nu_\perp^\varepsilon\,\right],\,\nu_\perp^\varepsilon\,\rangle_{L^2(m^\varepsilon)} \,\leq\, \frac{\alpha}{(\theta^\varepsilon)^2}\,\mathcal{D}_{\rho_0^\varepsilon}\left[\,\nu_\perp^\varepsilon\,\right] + C\,e^{Ct}\,\|\,\nu_0^\varepsilon\,\|^2_{L^2(m^\varepsilon)}\,,$$

for all positive constant α greater than $1/(2b\kappa)$. Consequently, the former computations lead to the following differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nu_\perp^\varepsilon\|^2_{L^2(m^\varepsilon)} + \frac{1-\alpha-\eta}{(\theta^\varepsilon)^2}\,\mathcal{D}_{\rho_0^\varepsilon}\left[\,\nu_\perp^\varepsilon\,\right] \,\leq\, \frac{C}{\eta}\,e^{Ct}\,\|\,\nu_0^\varepsilon\,\|^2_{H^1_w(m^\varepsilon)}\,.$$

Based on our assumptions, it holds $1/(2b\kappa) < 1$ therefore we may choose α and η such that $\alpha_* = 1-\alpha-\eta$ lies in $\left]\, 0\,,\, 1-(2b\kappa)^{-1}\,\right[$. Furthermore, in our context, the Gaussian-Poincaré inequality [98] rewrites

$$\|\nu_\perp^\varepsilon\|^2_{L^2(m^\varepsilon)} \,\leq\, \mathcal{D}_{\rho_0^\varepsilon}\,[\,\nu_\perp^\varepsilon\,]\,.$$

According to the latter remarks, the former inequality rewrites

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nu_\perp^\varepsilon\|^2_{L^2(m^\varepsilon)} \,+\, \frac{2\,\alpha_*}{(\theta^\varepsilon)^2}\,\|\nu_\perp^\varepsilon\|^2_{L^2(m^\varepsilon)} \,\leq\, C\,e^{Ct}\,\|\,\nu_0^\varepsilon\,\|^2_{H^1_w(m^\varepsilon)}\,.$$

We multiply this estimate by

$$\exp\left(2\,\alpha_*\,\int_0^t \frac{1}{(\theta^\varepsilon)^2}\,\mathrm{d}s\right),$$

and integrate between 0 and t. In the end, we deduce the following inequality

$$\|\,\nu_\perp^\varepsilon\,\|^2_{L^2(m^\varepsilon)} \,\leq\, \|\,\nu_0^\varepsilon\,\|^2_{H^1_w(m^\varepsilon)}\,e^{-2\,\alpha_*\,I(t)}\left(1\,+\,C\int_0^t e^{Cs}\,e^{2\,\alpha_*\,I(s)}\,\mathrm{d}s\right),$$

where I is given by

$$I(t) \,=\, \int_0^t \frac{1}{(\theta^\varepsilon)^2}\,\mathrm{d}s\,.$$

Taking advantage of the ODE solved by θ^ε (see (5.14)), we compute explicitly I

$$I(t) \,=\, \frac{t}{\varepsilon} \,+\, \frac{1}{2\,\rho_0^\varepsilon}\ln\left(\theta^\varepsilon(t)^2\right).$$

Consequently, the latter estimate rewrites

$$\|\,\nu_\perp^\varepsilon\,\|^2_{L^2(m^\varepsilon)} \,\leq\, \|\,\nu_0^\varepsilon\,\|^2_{H^1_w(m^\varepsilon)}\,e^{-2\,\alpha_*\,\frac{t}{\varepsilon}}\left(\left(\theta^\varepsilon(t)\right)^{-2\,\frac{\alpha_*}{\rho_0^\varepsilon}} + C\int_0^t e^{Cs}\,e^{2\,\alpha_*\,\frac{s}{\varepsilon}}\,\left(\frac{\theta^\varepsilon(s)}{\theta^\varepsilon(t)}\right)^{2\,\frac{\alpha_*}{\rho_0^\varepsilon}}\mathrm{d}s\right).$$

Then we notice that according to the explicit formula (5.14) for θ^ε, given that ε lies in $(0\,,\,1)$, we have on the one hand

$$e^{-2\,\alpha_*\,\frac{t}{\varepsilon}}\,\left(\theta^\varepsilon(t)\right)^{-2\,\frac{\alpha_*}{\rho_0^\varepsilon}} \,\leq\, \min\left(1\,,\,e^{-2\,\alpha_*\,\frac{t}{\varepsilon}}\,\varepsilon^{-\,\frac{\alpha_*}{\rho_0^\varepsilon}}\right),$$

and on the other hand

$$\left(\frac{\theta^\varepsilon(s)}{\theta^\varepsilon(t)}\right)^{2\,\frac{\alpha_*}{\rho_0^\varepsilon}} \,\leq\, \min\left(2\,e^{2\,\alpha_*\,\frac{t-s}{\varepsilon}}\,,\,C\left(1\,+\,e^{-2\,\alpha_*\,\frac{s}{\varepsilon}}\,\varepsilon^{-\,\frac{\alpha_*}{\rho_0^\varepsilon}}\right)\right).$$

We inject these bounds and take the supremum over all $\boldsymbol{x}$ in K in the latter estimate. In the end, we obtain the estimate for the case where $k=0$ in Proposition 5.19.

We turn to the case $k = 1$ in Proposition 5.19. We make use of the shorthand notation $h^\varepsilon_\perp = \partial_w \nu^\varepsilon_\perp$. We compute the time derivative of $\| h^\varepsilon_\perp \|^2_{L^2(m^\varepsilon)}$ multiplying equation (5.44) by $h^\varepsilon_\perp m^\varepsilon$ and integrating with respect to $\boldsymbol{u}$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \| h^\varepsilon_\perp \|^2_{L^2(m^\varepsilon)} = \langle \mathscr{S}^\varepsilon [h^\varepsilon , \Pi h^\varepsilon] , h^\varepsilon_\perp \rangle_{L^2(m^\varepsilon)} - \langle \mathscr{A}^\varepsilon [h^\varepsilon_\perp] , h^\varepsilon_\perp \rangle_{L^2(m^\varepsilon)}$$

$$+ b \| h^\varepsilon_\perp \|^2_{L^2(m^\varepsilon)} + \mathcal{A} ,$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{1}{\theta^\varepsilon} \int_{\mathbb{R}^2} \partial_v \nu^\varepsilon h^\varepsilon_\perp m^\varepsilon \, \mathrm{d}\boldsymbol{u} .$$

We estimate $\mathcal{A}$ integrating by part with respect to v and applying Young's inequality. After applying Proposition 5.16, it yields

$$\mathcal{A} \leq \frac{1}{4\eta} e^{Ct} \| \nu^\varepsilon_0 \|^2_{L^2(m^\varepsilon)} + \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho^\varepsilon_0} [h^\varepsilon_\perp] ,$$

for some positive constant C and all positive η. Then we follow the same argument as in the last step and obtain the expected result.

□

Estimate for $(\bar{\nu}^\varepsilon - \bar{\nu})$

It solves the following equation

$$\partial_t (\bar{\nu}^\varepsilon - \bar{\nu}) - b \, \partial_w [w \, (\bar{\nu}^\varepsilon - \bar{\nu})] = -a \, \theta^\varepsilon \, \partial_w \left[\int_{\mathbb{R}} v \, \nu^\varepsilon \, \mathrm{d}v \right] , \tag{5.45}$$

obtained taking the difference between equations (5.8) and (5.23). It is the same equation as (5.8) solved by $\bar{\nu}$ with the additional source term on the right-hand side of the latter equation. Consequently, our strategy consists in estimating the source term. We point out that since the source term is weighted by θ^ε, it sufficient to prove that it is bounded in order to obtain convergence. However, it is not hard to check that the source term cancels if we replace ν^ε by its projection $\Pi \nu^\varepsilon$

$$\partial_w \left[\int_{\mathbb{R}} v \, \Pi \, \nu^\varepsilon \, \mathrm{d}v \right] = 0 .$$

Consequently, based on the estimates obtained in the first step on $\nu^\varepsilon_\perp$ (see Proposition 5.19), we expect

$$\theta^\varepsilon \, \partial_w \left[\int_{\mathbb{R}} v \, \nu^\varepsilon \, \mathrm{d}v \right] \underset{\varepsilon \to 0}{=} O(\varepsilon) .$$

This formal approach was already rigorously justified in a weak convergence setting in [24]. In our setting and due to the structure of the source term, we use regularity estimates to achieve the latter convergence rate.

Lemma 5.20. *Consider a sequence of solutions* $(\mu^\varepsilon)_{\varepsilon>0}$ *to* (5.3) *with initial conditions satisfying assumption* (5.28) *with an index* k *in* $\{0\,,\,1\}$ *as well as the solution* $\bar{\nu}$ *to equation* (5.8) *with some initial condition* $\bar{\nu}_0$ *lying in* $\mathscr{H}^k(\overline{m})$*. The following estimate holds for all positive* ε

$$\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|_{H^k(\overline{m})}\,\leq\,e^{C\,t}\left(\|\,\bar{\nu}^\varepsilon_0\,-\,\bar{\nu}_0\,\|_{H^k(\overline{m})}\,+\,C\int_0^t e^{-C\,s}\,\theta^\varepsilon\,\|\,\nu^\varepsilon_\perp\|_{H^{k+1}_w(m^\varepsilon)}\,\mathrm{d}s\right),$$

for all $(t,\boldsymbol{x})\in\mathbb{R}^+\times K$*, where* k *lies in* $\{0\,,\,1\}$ *and for some positive constant* C *only depending on* κ*,* m_* *and* A*.*

Proof. We start with the case $k=0$. We consider some $\varepsilon>0$ and some $(t,\,\boldsymbol{x})$ in $\mathbb{R}_+\times K$; we omit the dependence with respect to $(t,\boldsymbol{x})$ when the context is clear. We compute the time derivative of $\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|^2_{L^2(\overline{m})}$ multiplying equation (5.45) by $(\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,)\,\overline{m}$ and integrating with respect to w. We integrate by part the term associated to linear transport and end up with the following relation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|^2_{L^2(\overline{m})}=\frac{b}{2}\int_{\mathbb{R}}\left(1-\kappa\,w^2\right)|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,|^2\,\overline{m}\,\mathrm{d}w-a\,\theta^\varepsilon\int_{\mathbb{R}^2}v\,\partial_w\,\nu^\varepsilon_\perp\,(\bar{\nu}^\varepsilon\,-\,\bar{\nu})\,\overline{m}\,\mathrm{d}\boldsymbol{u}\,.$$

According to Cauchy-Schwarz inequality and applying assumption (5.16), the source term admits the bound

$$-\,a\,\theta^\varepsilon\int_{\mathbb{R}^2}v\,\partial_w\,\nu^\varepsilon_\perp\,(\bar{\nu}^\varepsilon\,-\,\bar{\nu})\,\overline{m}\,\mathrm{d}\boldsymbol{u}\,\leq\,C\,\theta^\varepsilon\,\|\,h^\varepsilon_\perp\,\|_{L^2(m^\varepsilon)}\,\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|_{L^2(\overline{m})}\,,$$

for some positive constant C only depending on A and m_*. Furthermore, we bound the term associated to linear transport using that the polynomial $1\,-\,\kappa\,w^2$ is upper-bounded over $\mathbb{R}$. Gathering the former considerations we end up with the following differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|^2_{L^2(\overline{m})}\,\leq\,C\,\left(\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|^2_{L^2(\overline{m})}\,+\,\theta^\varepsilon\,\|\,\partial_w\,\nu^\varepsilon_\perp\,\|_{L^2(m^\varepsilon)}\,\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|_{L^2(\overline{m})}\right),$$

for some positive constant C only depending on κ, m_* and A. we divide the latter inequality by $\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|_{L^2(\overline{m})}$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|_{L^2(\overline{m})}\,\leq\,C\,\left(\|\,\bar{\nu}^\varepsilon\,-\,\bar{\nu}\,\|_{L^2(\overline{m})}\,+\,\theta^\varepsilon\,\|\,\partial_w\,\nu^\varepsilon_\perp\,\|_{L^2(m^\varepsilon)}\right),$$

We conclude this step applying Gronwall's Lemma.

We treat the case $k\,=\,1$ applying the same method. Indeed, $\bar{\nu}^\varepsilon\,-\,\bar{\nu}$ and $\partial_w\,(\bar{\nu}^\varepsilon\,-\,\bar{\nu})$ solve the same equation up to an additional source term which adds no difficulty.

□

Proposition 5.21. *Under assumptions* (5.1a)*-*(5.1b) *and* (5.26) *on the drift* N *and* (5.2) *on the interaction kernel* Ψ*, consider a sequence of solutions* $(\mu^\varepsilon)_{\varepsilon>0}$ *to* (5.3) *with initial conditions satisfying assumptions* (5.16)*-*(5.17b) *as well as the solution* $\bar{\nu}$ *to equation* (5.8) *with some initial condition* $\bar{\nu}_0$ *and an exponent* κ *greater than* $1/(2b)$*. Then there exists a constant* $C>0$*, such that for all* $\varepsilon\in(0,1)$*, the following statements hold*

1. *suppose that assumptions* (5.28)-(5.29) *are fulfilled with an index k in $\{0\,,\,1\}$, then for all $t \geq 0$,*

$$\|\,\bar{\nu}^{\varepsilon}(t)\,-\,\bar{\nu}(t)\,\|_{\mathscr{H}^{k}(\overline{m})}\,\leq\,e^{Ct}\left(\|\bar{\nu}^{\varepsilon}_0\,-\,\bar{\nu}_0\|_{\mathscr{H}^{k}(\overline{m})}\,+\,C\,\|\,\nu^{\varepsilon}_0\,\|_{\mathscr{H}^{k+1}(m^{\varepsilon})}\,\sqrt{\varepsilon}\right)\,;$$

2. *supposing assumption* (5.28) *with index $k\,=\,1$ and assumption* (5.29) *with index $k\,=\,0$, it holds for all $t \geq 0$,*

$$\|\,\bar{\nu}^{\varepsilon}(t)\,-\,\bar{\nu}(t)\,\|_{\mathscr{H}^{0}(\overline{m})}\,\leq\,e^{Ct}\left(\|\bar{\nu}^{\varepsilon}_0\,-\,\bar{\nu}_0\|_{\mathscr{H}^{0}(\overline{m})}\,+\,C\,\|\,\nu^{\varepsilon}_0\,\|_{\mathscr{H}^{2}(m^{\varepsilon})}\,\varepsilon\,\sqrt{|\ln\varepsilon\,|\,+\,1}\right)\,.$$

Proof. We prove item (2) in the latter proposition. According to Lemma 5.13, we have

$$\|\,\bar{\nu}^{\varepsilon}\,-\,\bar{\nu}\,\|_{L^2(\overline{m})}\,\leq\,e^{C\,t}\left(\|\,\bar{\nu}^{\varepsilon}_0\,-\,\bar{\nu}_0\,\|_{L^2(\overline{m})}\,+\,C\int_0^t e^{-C\,s}\,\theta^{\varepsilon}\,\|\,\nu^{\varepsilon}_{\perp}\|_{H^1_w(m^{\varepsilon})}\,\mathrm{d}s\right)\,.$$

Therefore, the proof comes down to estimating the integral in the right-hand side of the latter inequality

$$\mathcal{A}\,:=\,\int_0^t e^{-C\,s}\,\theta^{\varepsilon}\,\|\,\nu^{\varepsilon}_{\perp}\|_{H^1_w(m^{\varepsilon})}\,\mathrm{d}s\,.$$

We apply the second estimate in Proposition 5.19 and Cauchy-Schwarz inequality. This yields

$$\mathcal{A}\,\leq\,C\,\|\,\nu^{\varepsilon}_0\,\|_{H^2_w(m^{\varepsilon})}\left(\int_0^t|\theta^{\varepsilon}|^2\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\varepsilon\,+\,\min\left\{1\,,\,e^{-2\,\alpha_*\frac{s}{\varepsilon}}\,\varepsilon^{-\frac{\alpha_*}{m_*}}\right\}\,\mathrm{d}s\right)^{1/2}\,.$$

Then we inject the following estimate in the latter inequality

$$\min\left\{1\,,\,e^{-2\,\alpha_* s/\varepsilon}\,\varepsilon^{-\alpha_*/m_*}\right\}\,\leq\,\mathbb{1}_{\left\{s\,\leq\,-\frac{1}{2m_*}\varepsilon\ln\varepsilon\right\}}\,+\,\mathbb{1}_{\left\{s\,>\,-\frac{1}{2m_*}\varepsilon\ln\varepsilon\right\}}\,e^{-2\,\alpha_*\,\frac{s}{\varepsilon}}\,\varepsilon^{-\frac{\alpha_*}{m_*}}\,.$$

Moreover, we use (5.14) to compute the time integral of $|\theta^{\varepsilon}|^2$. In the end, we obtain

$$\mathcal{A}\,\leq\,C\,\|\,\nu^{\varepsilon}_0\,\|_{H^2_w(m^{\varepsilon})}\,\varepsilon\,\sqrt{t\,+\,1}\,\sqrt{|\ln\varepsilon\,|\,+\,1}\,.$$

Hence, we obtain the expected result taking the supremum over all $\boldsymbol{x}$ in K in the latter estimate.

Item (1) in Proposition 5.21 is obtained following the same method excepted that we estimate $\mathcal{A}$ using Proposition 5.16 with index k instead of Proposition 5.19.

□

Let us now conclude the proof of Theorem 5.3. On the one hand, we observe that item (2) corresponds to item (2) of Proposition 5.21. On the other hand, item (1) is obtained by gathering the estimate in Proposition 5.19 and item (1) in Proposition 5.21.

5.4.3 Proof of Theorem 5.4

All along this proof, we consider some ε_0 small enough so that the following condition is fulfilled

$$\| \rho_0^\varepsilon - \rho_0 \|_{L^\infty(K)} < m_* / 2 ,$$

for all ε less than ε_0. We omit the dependence with respect to $(t, \boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$ when the context is clear. We start by proving item (1) in Theorem 5.4. Since the cases $k = 0$ and $k = 1$ are treated the same way, we only detail the case $k = 0$. We consider some integer i and take some ε less than ε_0. Then we decompose the error as follows

$$\| (v - \mathcal{V})^i \, (\mu^\varepsilon - \mu) \, (t) \|_{\mathcal{H}^0(m^-)} \leq \mathcal{A} + \mathcal{B} ,$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{cases} \mathcal{A} = \|(v - \mathcal{V})^i \, (\mu^\varepsilon - \tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon} \, (\nu)) \, \|_{\mathcal{H}^0(m^-)} , \\ \mathcal{B} = \|(v - \mathcal{V})^i \, (\tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon} \, (\nu) - \mu) \, \|_{\mathcal{H}^0(m^-)} , \end{cases}$$

where ν is the limit of ν^ε in Theorem 5.3 and is defined by (5.7) and where the operators $\tau_{-\mathcal{U}^\varepsilon}$ and D_{θ^ε} respectively stand for the translation of vector $-\mathcal{U}^\varepsilon$ with respect to the $\boldsymbol{u}$-variable and the dilatation with parameter $(\theta^\varepsilon)^{-1}$ with respect to the v-variable, that is

$$\tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon} \, (\nu) \, (t \, , \, \boldsymbol{x} \, , \, \boldsymbol{u}) \, = \, \frac{1}{\theta^\varepsilon} \, \nu \left(t \, , \, \boldsymbol{x} \, , \, \frac{v - \mathcal{V}^\varepsilon}{\theta^\varepsilon} \, , \, w - \mathcal{W}^\varepsilon \right) .$$

The first term $\mathcal{A}$ corresponds to the convergence of the re-scaled version ν^ε of μ^ε towards ν whereas $\mathcal{B}$ corresponds to the convergence of the macroscopic quantities.

We estimate $\mathcal{A}$ as follows

$$\mathcal{A} \leq C \, (\mathcal{A}_1 + \mathcal{A}_2) \, ,$$

where C is a positive constant which only depends on i and where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given by

$$\begin{cases} \mathcal{A}_1 = \left\| (v - \mathcal{V}^\varepsilon)^i \, (\mu^\varepsilon - \tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon} \, (\nu) \,) \right\|_{\mathcal{H}^0(m^-)} , \\ \mathcal{A}_2 = \| \mathcal{V} - \mathcal{V}^\varepsilon \|^i_{L^\infty(K)} \, \| \mu^\varepsilon - \tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon} \, (\nu) \, \|_{\mathcal{H}^0(m^-)} . \end{cases}$$

According to item (2) in Proposition 5.5 , $\mathcal{V}^\varepsilon$ and $\mathcal{W}^\varepsilon$ are uniformly bounded with respect to both $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ and $\varepsilon > 0$, and $0 \leq \theta^\varepsilon \leq 1$, hence

$$m^-(\boldsymbol{x}, v, w) \leq C \, m^\varepsilon \left(\boldsymbol{x} \, , \, \frac{v - \mathcal{V}^\varepsilon}{\sqrt{2} \, \theta^\varepsilon} \, , \, w - \mathcal{W}^\varepsilon \right) ,$$

which yields

$$\mathcal{A}_1 \leq C \sup_{\boldsymbol{x} \in K} \left(\int_{\mathbb{R}^2} (v - \mathcal{V}^\varepsilon)^{2\,i} \, | \mu^\varepsilon - \tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon} \, (\nu) \, |^2 \, m^\varepsilon \left(\boldsymbol{x} \, , \, \frac{v - \mathcal{V}^\varepsilon}{\sqrt{2} \, \theta^\varepsilon} \, , \, w - \mathcal{W}^\varepsilon \right) \mathrm{d}\boldsymbol{u} \right)^{\frac{1}{2}} ,$$

for another constant $C > 0$ depending only on κ, m_*, m_p and $\overline{m}_p$ (see assumptions (5.16)-(5.17b)) and on the data of the problem N, Ψ and A_0. Then we invert the change of variable (5.10) and notice that

$$v^{2\,i}\, m^\varepsilon \left(\boldsymbol{x}\,,\, \frac{v}{\sqrt{2}}\,,\, w \right) \,\leq\, C\, m^\varepsilon\, (\boldsymbol{x}\,,\, \boldsymbol{u})\;,$$

for some constant $C > 0$ only depending on i and m_*. Consequently, we deduce

$$\mathcal{A}_1 \,\leq\, C \left\| (\theta^\varepsilon)^{i\,-\,\frac{1}{2}} \right\|_{L^\infty(K)} \|\, \nu^\varepsilon \,-\, \nu \,\|_{\mathcal{H}^0(m^\varepsilon)}\;.$$

Therefore, applying Theorem 5.3, using the compatibility assumption (5.31), and thanks to the constraint $\theta^\varepsilon\,(t=0) \,=\, 1$, which ensures

$$\|\, \nu_0^\varepsilon \,\|_{\mathscr{H}^1(m^\varepsilon)} \,\leq\, C\, \|\, \mu_0^\varepsilon \,\|_{\mathscr{H}^1(m^+)}\;,$$

for some constant C depending only on m_p, κ and m_*, we finally get

$$\mathcal{A}_1 \,\leq\, C\, e^{Ct}\, \varepsilon^{-\frac{1}{4}} \left(\varepsilon^{\frac{i}{2}} \,+\, e^{-\,i\,m_*\,\frac{t}{\varepsilon}} \right) \left(\varepsilon^{\frac{1}{2}} \,+\, e^{-\,\alpha_*\,\frac{t}{\varepsilon}}\, \varepsilon^{-\,\frac{\alpha_*}{2\,m_*}} \right)\,.$$

Moreover, since $\alpha_* \,<\, m_*\,/\,2$, we deduce

$$\mathcal{A}_1 \,\leq\, C\, e^{Ct} \left(\varepsilon^{\frac{i}{2}\,+\,\frac{1}{4}} \,+\, e^{-\,\alpha_*\,\frac{t}{\varepsilon}}\, \varepsilon^{-\,\frac{1}{2}} \right)\,.$$

To estimate $\mathcal{A}_2$, we apply item (1) in Proposition 5.5 and the compatibility assumption (5.31), which ensure

$$\|\, \mathcal{V} \,-\, \mathcal{V}^\varepsilon \,\|_{L^\infty(K)}^i \,\leq\, C\, e^{C\,t}\, \varepsilon^i\,.$$

Then we follow the same method as before. In the end, we end up with the following bound for $\mathcal{A}_2$

$$\mathcal{A}_2 \,\leq\, C\, e^{Ct} \left(\varepsilon^{i\,+\,\frac{1}{4}} \,+\, e^{-\,\alpha_*\,\frac{t}{\varepsilon}}\, \varepsilon^{i\,-\,\frac{1}{2}} \right)\,.$$

Gathering these results, we obtain the following estimate for $\mathcal{A}$

$$\mathcal{A} \,\leq\, C\, e^{Ct} \left(\varepsilon^{\frac{i}{2}\,+\,\frac{1}{4}} \,+\, e^{-\,\alpha_*\,\frac{t}{\varepsilon}}\, \varepsilon^{-\,\frac{1}{2}} \right)\,.$$

We turn to $\mathcal{B}$. Similarly as before, we apply the triangular inequality and invert the change of variable (5.10). Then we apply Proposition 5.5, which yields

$$\mathcal{B} \,\leq\, C\, e^{C\,t} \varepsilon^{-\frac{1}{4}} \left(\varepsilon^{\frac{i}{2}} \,+\, e^{-\,i\,m_*\,\frac{t}{\varepsilon}} \right) \left\| \nu \,-\, \tau_{\left(\frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon}\,,\, \mathcal{W}^\varepsilon - \mathcal{W} \right)}\, (\mathcal{M}_{\rho_0} \otimes \bar{\nu}\,) \right\|_{\mathcal{H}^0\left(D_{\sqrt{2}}\,(m^\varepsilon)\right)}\,,$$

where $D_{\sqrt{2}}\ (m^\varepsilon)$ is a short-hand notation for

$$D_{\sqrt{2}}\,(m^\varepsilon)\,(\boldsymbol{x}\,,\, \boldsymbol{u}) \,=\, m^\varepsilon \left(\boldsymbol{x}\,,\, \frac{v}{\sqrt{2}}\,,\, w \right)\,.$$

Then we decompose the right-hand side of the latter inequality as follows

$$\left\| \nu \,-\, \tau_{\left(\frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon}\,,\,\mathcal{W}^\varepsilon - \mathcal{W}\right)} \left(\mathcal{M}_{\rho_0} \otimes \bar{\nu}\right) \right\|_{\mathcal{H}^0\left(D_{\sqrt{2}}\,(m^\varepsilon)\right)} \leq \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3\,,$$

where $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are given by

$$\begin{cases} \mathcal{B}_1 \,=\, \left\| \bar{\nu} \,-\, \tau_{(\mathcal{W}^\varepsilon - \mathcal{W})}\, \bar{\nu} \right\|_{\mathcal{H}^0(\overline{m})}\,, \\ \mathcal{B}_2 \,=\, \left\| \tau_{(\mathcal{W}^\varepsilon - \mathcal{W})}\, \bar{\nu} \right\|_{\mathcal{H}^0(\overline{m})} \left\| \mathcal{M}_{\rho_0^\varepsilon} \,-\, \tau_{\left(\frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon}\right)} \mathcal{M}_{\rho_0^\varepsilon} \right\|_{\mathcal{H}^0\left(\mathcal{M}^{-1}_{\rho_0^\varepsilon}\right)}\,, \\ \mathcal{B}_3 \,=\, \left\| \tau_{(\mathcal{W}^\varepsilon - \mathcal{W})}\, \bar{\nu} \right\|_{\mathcal{H}^0(\overline{m})} \left\| \mathcal{M}_{\rho_0^\varepsilon} \,-\, \mathcal{M}_{\rho_0} \right\|_{\mathcal{H}^0\left(\mathcal{M}^{-1}_{\rho_0^\varepsilon}\right)} e^{\|\mathcal{V}^\varepsilon - \mathcal{V}\|^2_{L^\infty(K)}\,/\,(2\,m_*\,\varepsilon)}\,, \end{cases}$$

where we used that

$$D_{\sqrt{2}}\,(m^\varepsilon) \,\leq\, m^\varepsilon\,, \quad \text{and} \quad m^\varepsilon \,=\, \mathcal{M}^{-1}_{\rho_0^\varepsilon}\, \overline{m}\,.$$

Since equation (5.8) is linear, $\bar{\nu} - \tau_{w_0}\,\bar{\nu}$ also solves the equation, therefore, applying Lemma 5.18 with $w_0 \,=\, \mathcal{W}^\varepsilon - \mathcal{W}$, it yields

$$\mathcal{B}_1 \,\leq\, C\, e^{\frac{b}{2}\,t} \left\| \bar{\nu}_0 \,-\, \tau_{(\mathcal{W}^\varepsilon - \mathcal{W})}\, \bar{\nu}_0 \right\|_{\mathcal{H}^0(\overline{m})}\,.$$

Furthermore, since $\overline{m} \,\leq\, \overline{m}^+$ and relying on assumption (5.32), we deduce

$$\mathcal{B}_1 \,\leq\, C\, e^{\frac{b}{2}\,t}\, \|\, \mathcal{W}^\varepsilon \,-\, \mathcal{W}\,\|_{L^\infty(K)}\,.$$

Therefore, according to item (1) in Proposition 5.5 we conclude

$$\mathcal{B}_1 \,\leq\, C\, e^{C\,t}\, \varepsilon\,.$$

To estimate $\mathcal{B}_2$ and $\mathcal{B}_3$, we follow the same method as for $\mathcal{B}_1$: we first apply the following relation

$$\left\| \mathcal{M}_{\rho_0^\varepsilon} \,-\, \tau_{\left(\frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon}\right)} \mathcal{M}_{\rho_0^\varepsilon} \right\|_{\mathcal{H}^0\left(\mathcal{M}^{-1}_{\rho_0^\varepsilon}\right)} \,=\, \left\| e^{\rho_0^\varepsilon \left| \frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon} \right|^2} \,-\, 1 \right\|^{\frac{1}{2}}_{L^\infty(K)}\,,$$

which ensures

$$\left\| \mathcal{M}_{\rho_0^\varepsilon} \,-\, \tau_{\left(\frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon}\right)} \mathcal{M}_{\rho_0^\varepsilon} \right\|_{\mathcal{H}^0\left(\mathcal{M}^{-1}_{\rho_0^\varepsilon}\right)} \,\leq\, \left\| e^{\rho_0^\varepsilon \left| \frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon} \right|^2}\, \rho_0^\varepsilon \left| \frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon} \right|^2 \right\|^{\frac{1}{2}}_{L^\infty(K)}\,.$$

Furthermore, we apply Lemma 5.18, which ensures that

$$\left\| \tau_{(\mathcal{W}^\varepsilon - \mathcal{W})}\, \bar{\nu} \right\|_{\mathcal{H}^0(\overline{m})} \,\leq\, C\, e^{C\,t}\,.$$

Therefore, applying item (1) in Proposition 5.5, we obtain

$$\mathcal{B}_2 \,\leq\, C\, e^{C\left(t + \varepsilon\, e^{Ct}\right)}\, \varepsilon^{\frac{1}{2}}\,.$$

Then to estimate $\mathcal{B}_3$, an exact computation yields

$$\left\| \mathcal{M}_{\rho_0^\varepsilon} - \mathcal{M}_{\rho_0} \right\|_{\mathcal{H}^0\left(\mathcal{M}_{\rho_0^\varepsilon}^{-1}\right)} = \sup_{\boldsymbol{x}\in K} \left(\frac{|\,\rho_0 - \rho_0^\varepsilon\,|^2}{\sqrt{\rho_0^2 - (\rho_0 - \rho_0^\varepsilon)^2}\left(\rho_0 + \sqrt{\rho_0^2 - (\rho_0 - \rho_0^\varepsilon)^2}\right)} \right)^{\frac{1}{2}}.$$

Therefore, according to assumption (5.31) and item (1) in Proposition 5.5, which ensures that

$$e^{\|\mathcal{V}^\varepsilon - \mathcal{V}\|^2_{L^\infty(K)} / (2\, m_*\, \varepsilon)} \leq e^{C\, e^{Ct}\, \varepsilon},$$

this yields

$$\mathcal{B}_3 \leq C\, e^{C\left(t + e^{Ct}\,\varepsilon\right)}\, \varepsilon\,.$$

In the end, we deduce the following estimate for $\mathcal{B}$

$$\mathcal{B} \leq C\, e^{C\left(t + \varepsilon\, e^{Ct}\right)} \left(\varepsilon^{\frac{i}{2}+\frac{1}{4}} + e^{-i\, m_*\, \frac{t}{\varepsilon}}\, \varepsilon^{\frac{1}{4}}\right).$$

The proof for the statement (2) in Theorem 5.4 follows the same lines as the former one excepted that we apply item (2) in Theorem 5.3 instead of item (1) to quantify the convergence of $\bar{\nu}^\varepsilon$ towards $\bar{\nu}$. Therefore, we do not detail the proof.

5.5 Conclusion

This article highlights how macroscopic behavior arise in spatially extended FitzHugh-Nagumo neural networks. In the regime where strong short-range interactions dominate, we proved that the voltage distribution concentrates with Gaussian profile around an averaged value $\mathcal{V}$ whereas the adaptation variable converges towards a distribution $\bar{\mu}$. The limiting quantities $(\mathcal{V}, \bar{\mu})$ solve the coupled reaction diffusion-transport system (5.5). The novelty of this work is that we derive quantitative estimates ensuring strong convergence towards the Gaussian profile. More precisely, we provide two results. On the one hand, we prove convergence in a L^1 functional framework. Our analysis relies on a modified Boltzmann entropy (see Appendix C.1) which is original to our knowledge. On the other hand, we prove convergence in a weighted L^2 setting, in which we take advantage of the variational structure to obtain regularity estimates and recover optimal convergence rates. These results complement collaborations of the author with E. Bouin and F. Filbet dedicated to the quantitative analysis of the strong short-range interaction regime (see [24] for weak convergence estimates and [23] for uniform convergence estimates).

A natural continuation of this work concerns the link between equation (5.3) and other popular models in neuroscience. More precisely, it is of primary interest to understand how the FitzHugh-Nagumo model relates to models based on "forced action potential" such as integrate and fire [44, 55, 56, 86, 46, 201], voltage-conductance [184, 99] and time elapsed

[181, 71, 70, 170] neural models. Indeed, Hodgkin-Huxley and FitzHugh-Nagumo neural models reproduce the spiking behavior of neurons thanks to an autonomous system of ordinary differential equations whereas their "forced action potential" counterparts artificially enforce the spiking behavior. Therefore, the next step of our investigation is to derive "forced action potential" models as (biologically relevant) asymptotic limits of FitzHugh-Nagumo or Hodgkin-Huxley networks.

To conclude, our model displays structural similarities with others coming from kinetic theory such as flocking models [106, 152, 150, 193], Vlasov-Navier-Stokes models with Brinkman force [104] and Vlasov-Poisson-Fokker-Planck models [192, 103, 137]. Hence, a natural question concerns the applicability of our methods in this wider context. We partially answered this question in [22] by applying the strategy explained at the beginning of Section 5.3 to treat the diffusive limit of the Vlasov-Poisson-Fokker-Planck model. Closer to applications in Biology, we also address the applicability of our approach to the fast adaptation regime in the run and tumble model for bacterial motion, analyzed by B. Perthame *et al.* in [186].

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Chapitre 6

Concentration profiles in FitzHugh-Nagumo neural networks : A Hopf-Cole approach

In this paper we focus on a mean-field model for a spatially extended FitzHugh-Nagumo neural network. In the regime where strong and local interactions dominate, we quantify how the probability density of voltage concentrates into a Dirac distribution. Previous work investigating this question have provided relative bounds in integrability spaces. Using a Hopf-Cole framework, we derive precise L^∞ estimates using subtle explicit sub- and super- solutions which prove, with rates of convergence, that the blow up profile is Gaussian.

This work has been submitted and is available on *arXiv :2207.11010*, Concentration profiles in FitzHugh-Nagumo neural networks : A Hopf-Cole approach, with Emeric Bouin.

Contents

6.1 Introduction

The model.

Understanding the complex dynamics induced by interactions in large assemblies of neurons constitute one of the great challenge in neuroscience. As described in neuroscience textbooks [149], neurons behave and interact according to intricate chemical and electrical mechanisms. Due to the complexity of these mechanisms, it is mandatory to consider simplified models. A key step in this direction is the pioneering work of A. Hodgkin and A. Huxley [139, 140], who built an accurate model describing the membrane potential dynamics of a single nerve cell submitted to an external current. This model captures the main features of a neuron's membrane potential behaviour such as periodic patterns, relaxation toward equilibrium state as well as spiking behaviour also known as action potential ; it falls into the category of so called "voltage-conductance based models" which describe the dynamics of the membrane potential through auxiliary variables taking into account ionic exchanges between a neuron and its extra-cellular environment (see [38, 84, 149] for precise introductions to such models). Due to the complexity of this model, we focus on a simplified version introduced by R. FitzHugh and J. Nagumo [114, 174], which conserves the main features of the Hodgkin-Huxley model while remaining more tractable from a mathematical point of view. This version describes the dynamics of the membrane potential $v_t \in \mathbb{R}$ coupled with an adaptation variable $w_t \in \mathbb{R}$.

$$\begin{cases} \mathrm{d}v_t = \left(N(v_t) - w_t + I_{ext}\right)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t, \\ \\ \mathrm{d}w_t = A\left(v_t, w_t\right)\mathrm{d}t, \end{cases}$$

where N is a non-linear drift that, in the original articles of R. FitzHugh and J. Nagumo, takes the typical form $N(v) = v - v^3$ even though we shall here consider a broader class of drifts. Coefficient A is given by

$$A(v, w) = a\,v - b\,w + c\,,$$

where $a, c \in \mathbb{R}$ and $b > 0$. The Brownian motion B_t describes non deterministic fluctuations of the potential which are not taken into account by the model whereas I_{ext} stands for the current received by the neuron from its environment. Since our purpose is to describe interactions between neurons, we introduce coupling through the term I_{ext} : we consider

Ohmic interactions between neurons with spatially dependent conductance given by a connectivity kernel $\Phi : K \times K \to \mathbb{R}$, where K is a compact set of $\mathbb{R}^d$. Hence, in the case of m interacting neurons described by the triplet voltage-adaptation-position $(v_i, w_i, \boldsymbol{x}_i)_{1\leq i\leq m}$, the current received by neuron i from its neighbors is given by

$$I_{ext} \,=\, -\frac{1}{m}\sum_{j=1}^{m} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)\,(v_t^i - v_t^j)\,. \tag{6.1}$$

Therefore, we obtain the following microscopic description of a FitzHugh-Nagumo neural network

$$\begin{cases} \mathrm{d}v_t^i \,=\, \left(N(v_t^i) \,-\, w_t^i \,-\, \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)\,(v_t^i - v_t^j)\right)\,\mathrm{d}t \,+\, \sqrt{2}\,\mathrm{d}B_t^i\,, \\ \\ \mathrm{d}w_t^i \,=\, A\left(v_t^i, w_t^i\right)\,\mathrm{d}t\,, \end{cases}$$

where $i \in \{1, ..., m\}$. The mean-field limit, corresponding to $m \to +\infty$ in the latter system, was rigorously analysed in the case of FitzHugh-Nagumo neurons [77, 168] as well as in more general cases, including the Hodkin-Huxley model [3, 30, 161] ; we also mention similar works for mean-field limits with non-exchangeable systems [144, 195] and [26] for a related model in collective dynamics. It was proved that the empirical measure associated to the latter system converges towards a distribution function $f := f(t,\, \boldsymbol{x}, \boldsymbol{u})$, with $\boldsymbol{u} = (v, w) \in \mathbb{R}^2$, representing the density of neurons at time t, position $\boldsymbol{x} \in K$, with membrane potential v and adaptation variable $w \in \mathbb{R}$. The dynamics of the distribution function f are prescribed by the following mean-field equation

$$\partial_t f \,+\, \partial_v\left((N(v) - w - \mathcal{K}_\Phi[f])\; f\right) \,+\, \partial_w\left(A(v, w) f\right) \,-\, \partial_v^2 f \,=\, 0\,,$$

where the operator $\mathcal{K}_\Phi[f]$ takes into account spatial interactions and is given by

$$\mathcal{K}_\Phi[f](t,\, \boldsymbol{x}, v) \,=\, \int_{K\times\mathbb{R}^2} \Phi(\boldsymbol{x}, \boldsymbol{x}')\,(v - v')\, f(t,\, \boldsymbol{x}', \boldsymbol{u}')\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{u}'\,.$$

This model is a typical example of McKean-Vlasov equation including voltage and conductance variables ; other models of this type are available in the literature [54, 184] as well as other popular family of models including integrate-and-fire neural networks [44, 46, 56] and time-elapsed neuronal models [70, 71, 72, 170, 181]. We mention that the discrete analog of $\mathcal{K}_\Phi[f]$ given by (6.1) may be recovered replacing f in the definition of $\mathcal{K}_\Phi[f]$ by the empirical distribution of the microscopic model.

The question at hand.

In the present article, we are interested in the dynamics of the network when short-range interactions dominate : we consider a situation where the connectivity kernel Φ decomposes as follows

$$\Phi(\boldsymbol{x}, \boldsymbol{x}') \,=\, \Psi(\boldsymbol{x}, \boldsymbol{x}') \,+\, \frac{1}{\varepsilon}\,\delta_0(\boldsymbol{x} - \boldsymbol{x}')\,, \tag{6.2}$$

where the Dirac mass δ_0 accounts for short-range interactions whereas the interaction kernel Ψ models long-range interactions. The scaling parameter $\varepsilon > 0$ represents the magnitude of short-range interactions : we now write f^ε instead of f and focus on the regime $\varepsilon \ll 1$. Such decomposition was motivated from the biological point of view [162] as well as mathematically and numerically analyzed [21, 24, 39, 79, 80, 196]. Let us present the conclusions of these works. To do so, we introduce the macroscopic quantities associated to the network : the spatial distribution of neurons (which is time-homogeneous, according to an integration of the mean-field equation with respect to $\boldsymbol{u} \in \mathbb{R}^2$)

$$\rho_0^\varepsilon(\boldsymbol{x}) = \int_{\mathbb{R}^2} f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \mathrm{d}\boldsymbol{u},$$

as well as the averaged voltage and adaptation variable at spatial location $\boldsymbol{x} \in K$

$$\mathcal{U}^\varepsilon := (\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon) \ , \quad \text{with} \quad \begin{cases} \rho_0^\varepsilon\,(\boldsymbol{x})\,\mathcal{V}^\varepsilon(t,\boldsymbol{x}) & = \displaystyle\int_{\mathbb{R}^2} v\ f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \mathrm{d}\boldsymbol{u} \\ \rho_0^\varepsilon\,(\boldsymbol{x})\,\mathcal{W}^\varepsilon(t,\boldsymbol{x}) & = \displaystyle\int_{\mathbb{R}^2} w\ f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \mathrm{d}\boldsymbol{u} \end{cases} . \tag{6.3}$$

We outline that the interaction term $\mathcal{K}_\Phi[f^\varepsilon]$ admits the following simple expression in terms of the macroscopic quantities

$$\mathcal{K}_\Phi[f^\varepsilon](t, \boldsymbol{x}, v) = \Phi * \rho_0^\varepsilon(\boldsymbol{x})\, v - \Phi * (\rho_0^\varepsilon\ \mathcal{V}^\varepsilon)(t, \boldsymbol{x}),$$

where $*$ denotes the convolution on the right side of any function g with Φ

$$\Phi * g(\boldsymbol{x}) \ = \int_K \Phi(\boldsymbol{x}, \boldsymbol{x}') g(\boldsymbol{x}') \mathrm{d}\boldsymbol{x}'.$$

According to *ansatz* (6.2), we substitute Φ with $\Psi + \delta_0/\varepsilon$ in the latter expression of $\mathcal{K}_\Phi[f^\varepsilon]$; it yields

$$\mathcal{K}_\Phi[f^\varepsilon](t,\ \boldsymbol{x}, v) \ = \ \Psi * \rho_0^\varepsilon(\boldsymbol{x})\, v - \Psi * (\rho_0^\varepsilon\ \mathcal{V}^\varepsilon)(t, \boldsymbol{x}) \ + \ \frac{1}{\varepsilon} \rho_0^\varepsilon(\boldsymbol{x})\, (v - \mathcal{V}^\varepsilon(t, \boldsymbol{x}))\, .$$

Therefore, the mean-field equation may be rewritten

$$\partial_t f^\varepsilon + \mathrm{div}_{\boldsymbol{u}} \left[\mathbf{b}^\varepsilon f^\varepsilon \right] - \partial_v^2 f^\varepsilon = \frac{1}{\varepsilon} \rho_0^\varepsilon \partial_v \left[(v - \mathcal{V}^\varepsilon) f^\varepsilon \right], \tag{6.4}$$

where coefficient $\mathbf{b}^\varepsilon$ is defined for all $(t, \boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$ as

$$\mathbf{b}^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) := \begin{pmatrix} B^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \\ \\ A(\boldsymbol{u}) \end{pmatrix} = \begin{pmatrix} N(v) - w - v\, \Psi * \rho_0^\varepsilon + \Psi * (\rho_0^\varepsilon\ \mathcal{V}^\varepsilon) \\ \\ av - bw + c \end{pmatrix} . \tag{6.5}$$

To infer the asymptotic behaviour of the network in the regime of strong interactions, we look for the leading order in (6.4) : in our case, it is induced by short-range interactions between neurons, and as $\varepsilon \to 0$, we expect

$$(v - \ \mathcal{V}^\varepsilon) f^\varepsilon \underset{\varepsilon \to 0}{\to} 0,$$

to make sure that no term is singular in (6.4). Since (6.4) conserves total mass, this means that f^ε concentrates into a Dirac mass centred in $\mathcal{V}^\varepsilon$ with respect to the v-variable, that is

$$f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \underset{\varepsilon\to 0}{\approx} \delta_0\left(v-\mathcal{V}^\varepsilon(t,\boldsymbol{x})\right)\otimes F^\varepsilon(t,\boldsymbol{x},w),$$

where $\mathcal{V}^\varepsilon$ is given by (6.3) and F^ε is defined as the marginal of f^ε with respect to the voltage variable

$$F^\varepsilon\left(t,\boldsymbol{x},w\right)=\int_{\mathbb{R}} f^\varepsilon\left(t,\boldsymbol{x},\boldsymbol{u}\right)\mathrm{d}v.$$

Multiplying equation (6.4) by $\boldsymbol{u}/\rho_0^\varepsilon$ and integrating with respect to $\boldsymbol{u}$, one finds that the couple $(\mathcal{V}^\varepsilon,\mathcal{W}^\varepsilon)$ solves the following system

$$\begin{cases} \partial_t\mathcal{V}^\varepsilon = N(\mathcal{V}^\varepsilon)-\mathcal{W}^\varepsilon-\mathcal{L}_{\rho_0^\varepsilon}\left[\mathcal{V}^\varepsilon\right]+\mathcal{E}(f^\varepsilon), \\ \partial_t\mathcal{W}^\varepsilon = A(\mathcal{V}^\varepsilon,\mathcal{W}^\varepsilon), \end{cases} \tag{6.6}$$

where the error term $\mathcal{E}\left(f^\varepsilon\right)$ is given by

$$\mathcal{E}\left(f^\varepsilon\left(t,\boldsymbol{x},\cdot\right)\right) \;=\; \frac{1}{\rho_0^\varepsilon(\boldsymbol{x})}\int_{\mathbb{R}^2} N(v)f^\varepsilon\left(t,\boldsymbol{x},\boldsymbol{u}\right)\mathrm{d}\boldsymbol{u}-N(\mathcal{V}^\varepsilon), \tag{6.7}$$

and $\mathcal{L}_{\rho_0^\varepsilon}$ is a non local operator given by

$$\mathcal{L}_{\rho_0^\varepsilon}\left[\mathcal{V}^\varepsilon\right]=\mathcal{V}^\varepsilon\Psi*\rho_0^\varepsilon-\Psi*\left(\rho_0^\varepsilon\mathcal{V}^\varepsilon\right).$$

All this, in turn, implies that as ε vanishes, $(\mathcal{V}^\varepsilon,\mathcal{W}^\varepsilon)$ converges to the couple $(\mathcal{V},\mathcal{W})$, which solves

$$\begin{cases} \partial_t\mathcal{V} = N(\mathcal{V})-\mathcal{W}-\mathcal{L}_{\rho_0}[\mathcal{V}], \\ \partial_t\mathcal{W} = A(\mathcal{V},\mathcal{W}), \\ \left(\mathcal{V}\left(0,\cdot\right),\mathcal{W}\left(0,\cdot\right)\right)=\left(\mathcal{V}_0,\mathcal{W}_0\right), \end{cases} \tag{6.8}$$

where $\rho_0 = \lim_{\varepsilon\to 0}\rho_0^\varepsilon$. As of the marginal F^ε, it has been shown in [21, 24, 79] that it also converges.

In this article, we refine the latter result by investigating the concentration profile of the solution f^ε when ε goes to 0. To this end, we perform the so-called Hopf-Cole transform of f^ε

$$\phi^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) := \varepsilon\ln\left(\sqrt{\frac{2\pi\varepsilon}{\rho_0}}f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\right), \qquad \forall\,(t,\boldsymbol{x},\boldsymbol{u})\in\mathbb{R}^+\times K\times\mathbb{R}^2, \tag{6.9}$$

and study the convergence of ϕ^ε as ε goes to zero. This approach has been widely followed to study concentration phenomena occurring in selection-mutation models in population dynamics [6, 8, 37, 51, 64, 93, 156, 166, 167]. Quininao and Touboul have shown in [196] that it can lead to fruitful results in the present context. It is indeed a powerful tool to study concentration phenomena : inverting (6.9) we obtain

$$f^\varepsilon=\sqrt{\frac{\rho_0}{2\pi\varepsilon}}\exp\left(\frac{\phi^\varepsilon}{\varepsilon}\right).$$

According to the latter reformulation, we expect $\phi^\varepsilon \leq 0$ as $\varepsilon \to 0$. Furthermore, we see that the concentration points of f^ε are characterized by the level sets $\{\phi^\varepsilon = 0\}$. Here, we are specifically interested in the behaviour of ϕ^ε at $v \to +\infty$, which describes precisely the asymptotic tail of f^ε with respect to v.

Heuristics.

Let us now formally justify the convergence of ϕ^ε : injecting *ansatz* (6.9) in equation (6.4), we find that ϕ^ε solves the following Hamilton-Jacobi equation

$$\partial_t \phi^\varepsilon + \nabla_{\boldsymbol{u}} \phi^\varepsilon \cdot \mathbf{b}^\varepsilon + \varepsilon \operatorname{div}_{\boldsymbol{u}} [\mathbf{b}^\varepsilon] - \partial_v^2 \phi^\varepsilon - \rho_0^\varepsilon = \frac{1}{\varepsilon} \partial_v \left(\frac{1}{2} \rho_0^\varepsilon \, |v - \mathcal{V}^\varepsilon|^2 + \phi^\varepsilon \right) \partial_v \phi^\varepsilon . \tag{6.10}$$

Keeping only the leading order in equation (6.10), we expect

$$\phi^\varepsilon \underset{\varepsilon \to 0}{\approx} \phi \,,$$

where ϕ satisfies

$$\partial_v \left(\frac{1}{2} \, \rho_0^\varepsilon \, |v - \mathcal{V}^\varepsilon|^2 + \phi \right) \partial_v \phi \; = \; 0 \, .$$

To determine ϕ, we reformulate the latter equation in the following equivalent form

$$\partial_v \, \phi(t, \boldsymbol{x}, v) \; = \; - \, \delta(t, \boldsymbol{x}, v) \, \rho_0^\varepsilon(\boldsymbol{x}) \, (v - \mathcal{V}^\varepsilon(t, \boldsymbol{x})) \ , \quad \forall \, (t, \boldsymbol{x}, v) \in \mathbb{R}^+ \times K \times \mathbb{R} \, , \tag{6.11}$$

where $\delta(t, \boldsymbol{x}, v)$ takes values in $\{0, 1\}$. We fix some $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ and suppose on the one hand $\partial_v \phi(t, \boldsymbol{x}, \cdot)$ to be smooth with respect to v and on the other hand $\rho_0^\varepsilon(\boldsymbol{x}) > 0$. Since $\partial_v \phi(t, \boldsymbol{x}, \mathcal{V}^\varepsilon) = 0$, we may divide (6.11) by $\rho_0^\varepsilon(\boldsymbol{x}) \, (v - \mathcal{V}^\varepsilon(t, \boldsymbol{x}))$ and deduce that $\delta(t, \boldsymbol{x}, \cdot)$ is a smooth function of v. Together with the fact that it takes discrete values, this implies that it does not depend on v. After integrating (6.11) with respect to v and passing to the limit $\varepsilon \to 0$, this yields

$$\phi^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \underset{\varepsilon \to 0}{\longrightarrow} \; - \frac{\delta}{2}(t, \boldsymbol{x}) \, \rho_0(\boldsymbol{x}) \, |v - \mathcal{V}(t, \boldsymbol{x})|^2 + c(t, \boldsymbol{x}).$$

Furthermore, since our problem conserves mass, we expect for each $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$

$$\int_{v \in \mathbb{R}} \exp \left(- \frac{\delta}{2 \, \varepsilon} \, \rho_0 \, |v - \mathcal{V}|^2 + \frac{c}{\varepsilon} \right) (t, \boldsymbol{x}) \, \mathrm{d}v \; = \; \sqrt{\frac{2 \pi \varepsilon}{\rho_0(\boldsymbol{x})}} \, ,$$

for all ε. This forces $\delta(t, \boldsymbol{x}) = 1$ and $c(t, \boldsymbol{x}) = 0$; thus we obtain

$$\phi^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \underset{\varepsilon \to 0}{\longrightarrow} \; - \frac{1}{2} \, \rho_0(\boldsymbol{x}) \, |v - \mathcal{V}(t, \boldsymbol{x})|^2 \; .$$

This convergence is the object of our main result, Theorem 6.6 below, in which we provide explicit convergence rates with respect ε. Before going further, we shall be precise about the mathematical framework of this article.

Mathematical framework.

In this paragraph, we first state and motivate our assumptions on the data of the problem : N, Ψ and f_0^ε. Then we precise the notion of solution we consider for equation (6.4). We suppose that the drift $N \in \mathscr{C}^2(\mathbb{R})$ satisfies

$$\limsup_{|v|\to+\infty} \frac{N(v)}{\mathrm{sgn}(v)|v|^p} < 0, \qquad \sup_{|v|\geq 1} \left| \frac{N(v)}{|v|^p} \right| < +\infty, \tag{6.12}$$

for some $p \geq 2$, and

$$\sup_{|v|\geq 1} \left(|N''(v)| + |N'(v)| \right) |v|^{-p'} < +\infty, \tag{6.13}$$

for some $p' \geq 0$.

Remark 6.1. *On the one hand, assumption* (6.12) *is a key feature in the model proposed by R. FitzHugh and J. Nagumo : it says that N has super-linear confining properties in the sense that it decays super-linearly at infinity. On the other hand, assumption* (6.13) *is technical yet not restrictive in our case since it is satisfied when N is given by*

$$N(v) = v - v^3,$$

which is the original choice in the FitzHugh-Nagumo model. More generally, (6.12)-(6.13) *are satisfied by all drifts $P(v)$ given by*

$$P(v) \;=\; Q(v) \;-\; Cv|v|^{p-1} \,,$$

for some positive constant $C > 0$ and where Q is a polynomial function with degree less then p.

The connectivity kernel satisfies

$$\left(\Psi : \boldsymbol{x} \longmapsto \Psi\left(\boldsymbol{x}\,,\,\cdot\right)\right) \in \mathscr{C}^0\left(K, L^1\left(K\right)\right)\,, \tag{6.14}$$

and

$$\sup_{\boldsymbol{x}'\in K} \int_K \left|\Psi(\boldsymbol{x},\boldsymbol{x}')\right| \mathrm{d}\boldsymbol{x} < \; +\infty, \qquad \sup_{\boldsymbol{x}\in K} \int_K \left|\Psi(\boldsymbol{x},\boldsymbol{x}')\right|^r \mathrm{d}\boldsymbol{x}' < +\infty, \tag{6.15}$$

for some $r > 1$; in the sequel we define r' by $\frac{1}{r} + \frac{1}{r'} = 1$. Thanks to this set of assumptions on Ψ our model takes into account non-symmetric interactions between neurons [39], interactions following negative power law [73, 131, 161, 176] as well as "nearest-neighbor" type interactions [161, 177, 178].

We now specify the notion of solution we consider for equation (6.4). To this end, we state our assumptions on f_0^ε. We suppose, for each $\varepsilon > 0$

$$f_0^\varepsilon \in \mathscr{C}^0\left(K, L^1\left(\mathbb{R}^2\right)\right), \quad f^\varepsilon \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^2} f_0^\varepsilon(\boldsymbol{x},\boldsymbol{u}) \mathrm{d}\boldsymbol{u}\mathrm{d}\boldsymbol{x} = 1, \tag{6.16}$$

which ensures $\rho_0^\varepsilon \in \mathscr{C}^0\left(K\right)$. We also suppose

$$m_* \leq \rho_0^\varepsilon \leq 1/m_*, \tag{6.17}$$

for some positive constant m_* independent of ε.

Remark 6.2. *The uniform lower bound condition on ρ_0^ε seems necessary in our analysis because the problem degenerates when ρ_0^ε vanishes as it may be seen on equation* (6.4)*. However the upper bound condition on ρ_0^ε may be relaxed at the cost of loosing uniform convergence with respect to $\boldsymbol{x}$ in our main result Theorem 6.6.*

On top of that, we assume the following condition : there exist two positive constants m_p and $\overline{m}_p$, independent of ε, such that

$$\sup_{\boldsymbol{x}\in K}\int_{\mathbb{R}^2}|\boldsymbol{u}|^{2(p+p')}f_0^\varepsilon\left(\boldsymbol{x},\boldsymbol{u}\right)\mathrm{d}\boldsymbol{u}\leq m_p, \tag{6.18}$$

and such that

$$\int_{K\times\mathbb{R}^2}|\boldsymbol{u}|^{2(p+p')r'}f_0^\varepsilon\left(\boldsymbol{x},\boldsymbol{u}\right)\mathrm{d}\boldsymbol{u}\mathrm{d}\boldsymbol{x}\leq \overline{m}_p, \tag{6.19}$$

where p, p' and r' are given in (6.12), (6.13) and (6.15).

Remark 6.3. *Moment assumptions such as* (6.18)-(6.19) *are common in the literature of mean-field description for neural networks (see [86, Assumption* 2*], [44, Theorem* 2.2 *and Corollary* 2.4*], [184, Section* 5.1*]). In our context, it allows to propagate moments of the solution f^ε to* (6.4)*, which in turn provides control over macroscopic quantities* $(\mathcal{V}^\varepsilon,\mathcal{W}^\varepsilon)$ *and over the error term* $\mathcal{E}\left(f^\varepsilon\right)$ *given by* (6.7) *(see the second item of Theorem 6.10 for more details). The order of moment required is related to the drift N through exponents p and p' defined in* (6.12)-(6.13) *: this is natural in order to control* $\mathcal{E}\left(f^\varepsilon\right)$ *which displays the drift N. These assumptions might not be optimal in the sense that the order of moment may be lowered ; however we do not investigate any further this technical aspect since it is not our main interest here.*

We consider the following solutions to (6.4)

Definition 6.4. *For all $\varepsilon>0$, we say that f^ε solves* (6.4) *with initial condition f_0^ε if $f^\varepsilon\in\mathscr{C}^0\left(\mathbb{R}^+\times K,L^1\left(\mathbb{R}^2\right)\right)$ and for all $\boldsymbol{x}\in K$, $t\geq 0$, and $\varphi\in\mathscr{C}_c^\infty\left(\mathbb{R}^2\right)$, it holds*

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi(\boldsymbol{u})\left(f^\varepsilon\left(t,\boldsymbol{x},\boldsymbol{u}\right)-f_0^\varepsilon\left(\boldsymbol{x},\boldsymbol{u}\right)\right)\mathrm{d}\boldsymbol{u} &= \int_0^t\int_{\mathbb{R}^2}\left[\left(\nabla_{\boldsymbol{u}}\varphi\cdot\mathbf{b}^\varepsilon+\partial_v^2\varphi\right)f^\varepsilon\right](s,\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u}\mathrm{d}s\\ &\quad -\frac{\rho_0^\varepsilon(\boldsymbol{x})}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}\left[\partial_v\varphi\left(v-\mathcal{V}^\varepsilon\right)f^\varepsilon\right](s,\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u}\mathrm{d}s,\end{aligned}$$

where $\mathcal{V}^\varepsilon$ and $\mathbf{b}^\varepsilon$ are given by (6.3) *and* (6.4) *respectively.*

With this notion of solution, equation (6.4) is well-posed, the following result being proved in [24].

Theorem 6.5 ([24])**.** *For any* $\varepsilon > 0$*, suppose that assumptions* (6.12) *on* N*,* (6.15) *on* Ψ *and* (6.16)-(6.17) *on the initial condition are fulfilled and that* f_0^ε *also verifies*

$$\begin{cases} \sup\limits_{\boldsymbol{x}\in K} \displaystyle\int_{\mathbb{R}^2} e^{|\boldsymbol{u}|^2/2} f_0^\varepsilon(\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u} < +\infty, \\ \sup\limits_{\boldsymbol{x}\in K} \displaystyle\int_{\mathbb{R}^2} \ln\left[f_0^\varepsilon(\boldsymbol{x},\boldsymbol{u})\right] f_0^\varepsilon(\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u} < +\infty, \end{cases}$$

and

$$\sup_{\boldsymbol{x}\in K} \left\| \nabla_{\boldsymbol{u}} \sqrt{f_0^\varepsilon} \right\|^2_{L^2(\mathbb{R}^2)} < +\infty.$$

Then there exists a unique solution f^ε *to equation* (6.4) *with initial condition* f_0^ε*, in the sense of Definition 6.4 which verifies*

$$\sup_{(t,\boldsymbol{x})\in[0,T]\times K} \int_{\mathbb{R}^2} e^{|\boldsymbol{u}|^2/2} f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u} < +\infty,$$

for all $T \geq 0$*.*

Let us now state our main result.

Main result.

The following theorem states that in the regime of strong local interactions, the voltage distribution of the neural network described by (6.4) blows up into a Dirac distribution and that concentration occurs with Gaussian profile. More specifically, we prove that the Hopf-Cole transform ϕ^ε of f^ε defined by (6.9) converges to $-\rho_0\,|v\,-\mathcal{V}|^2/2$ uniformly with respect to all variables $(t,\boldsymbol{x},\boldsymbol{u})$ as ε vanishes. Furthermore, we prove that convergence occurs at rate $O(\varepsilon)$, which is (at least formally) optimal.

Theorem 6.6. *Assume* (6.12)-(6.19) *and the additional assumptions of Theorem 6.5 and Proposition 6.8. Suppose that there exists a positive constant* C *independent of* ε *such that the following compatibility assumption holds*

$$\|\mathcal{U}_0 - \mathcal{U}_0^\varepsilon\|_{L^\infty(K)} + \|\rho_0 - \rho_0^\varepsilon\|_{L^\infty(K)} \leq C\varepsilon, \tag{6.20}$$

as well as the following set of "smallness assumptions"

$$\begin{cases} \left|\phi_0^\varepsilon + \dfrac{1}{2}\rho_0\,|v-\mathcal{V}|^2 - \varepsilon n\right| \leq \varepsilon C\left(1+|\boldsymbol{u}|^2\right), \quad \forall(\boldsymbol{x},\boldsymbol{u})\in K\times\mathbb{R}^2, & \text{(6.21a)} \\ \displaystyle\int_{\mathbb{R}^2}\left(|v-\mathcal{V}_0^\varepsilon|^2 + |v-\mathcal{V}_0^\varepsilon|^{p'+1}\right) f_0^\varepsilon(\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u} \leq C\varepsilon, \quad \forall\boldsymbol{x}\in K & \text{(6.21b)} \end{cases}$$

for all $\varepsilon > 0$*, where* n *is a primitive of* $N : n'(v) = N(v)$*. Then the sequence* $(\phi^\varepsilon)_{\varepsilon>0}$ *of Hopf-Cole transforms of the solutions* $(f^\varepsilon)_{\varepsilon>0}$ *to* (6.4) *is well defined and it converges* *locally uniformly on* $\mathbb{R}^+ \times K \times \mathbb{R}^2$ *to* $-\rho_0 \left|v - \mathcal{V}\right|^2 / 2$ *with rate* ε*, where* $\mathcal{V}$ *is given by* (6.8)*. More precisely, there exist two positive constants* C *and* ε_0 *such that for all* $\varepsilon \leq \varepsilon_0$*,*

$$\left|\phi^\varepsilon + \frac{1}{2}\rho_0 \left|v - \mathcal{V}\right|^2 - \varepsilon n\right|(t, \boldsymbol{x}, \boldsymbol{u}) \leq \varepsilon C e^{Ct}\left(1 + |\boldsymbol{u}|^2\right), \quad \forall (t, \boldsymbol{x}) \in \mathbb{R}^+ \times K, \text{a.e. in } \boldsymbol{u} \in \mathbb{R}^2.$$

As a consequence, f^ε *converges uniformly to* 0 *on the compact subsets of* $\mathbb{R}^+ \times K \times \mathbb{R}^2 \setminus \{v \neq \mathcal{V}(t, \boldsymbol{x})\}$*. In the latter results, constants* C *and* ε_0 *only depend on the data of our problem :* f_0^ε *(only through the constants appearing in assumptions* (6.17)-(6.19) *and* (6.20)-(6.21b)*),* N*,* A *and* Ψ*.*

Before going further into our analysis, let us comment on our result. We first emphasize that it deals with uniform convergence with respect to all variables, which is a great improvement in comparison to former results obtained in [21, 24], where L^1, L^2 and weak convergence estimates were obtained.
We also point out that the present article is a continuation of [196], which initiated the Hopf-Cole approach in our context. In the present article, we take advantage of the particular structure of the problem, specifically of the confining properties of N provided by assumption (6.12) to obtain a rate of convergence with respect to ε.

To conclude, we emphasize that our result holds in a perturbative setting. To illustrate this remark, we consider the following initial condition, which satisfies the assumptions of Theorem 6.6

$$f_0^\varepsilon(\boldsymbol{x}, \boldsymbol{u}) = c_\varepsilon(\boldsymbol{x}) \frac{\rho_0(\boldsymbol{x})^{\frac{3}{2}}}{\pi\sqrt{2\varepsilon}} \exp\left(-\frac{\rho_0(\boldsymbol{x})}{2\,\varepsilon} \left|v - \mathcal{V}_0(\boldsymbol{x})\right|^2 - \left|w - \mathcal{W}_0(\boldsymbol{x})\right|^2 + n(v)\right),$$

where c_ε is a normalizing constant such that for all $\boldsymbol{x} \in K$, it holds

$$\int_{\mathbb{R}^2 \times K} f_0^\varepsilon(\boldsymbol{u}, \boldsymbol{x}) \mathrm{d}\boldsymbol{u} = \rho_0(\boldsymbol{x}),$$

In the latter example, we see that f^ε is already concentrated with respect to v at initial time. This restriction is quite common in articles which follow a Hopf-Cole transform approach : it is the case in most references cited above [6, 37, 93, 166, 167, 196]. We also outline that the additional term $n(v)$ in the latter example induces fast decay towards 0 of f_0^ε in the limit $v \to +\infty$. Indeed, since N has super-linear confining properties (see the first equation in assumption (6.12)), its primitive n decays like $-|v|^{p+2}$ as $v \to +\infty$. This condition arises naturally, as formally explained in the next Section **Comments on the stategy** and then rigorously justified at the beginning of Section 6.3.

Comments on the stategy.

Let us outline our strategy and the challenges in order to prove Theorem 6.6. As identified in [196], the main difficulty is induced by the drift N, which is not globally Lipschitz according to assumption (6.12). To bypass this difficulty, authors in [196] rely on a regularization argument : they consider truncated versions N^R, for $R > 0$, of N, in order to recover global Lipschitz properties, allowing to prove uniform estimates in ε on the derivatives $\nabla_{\boldsymbol{u}}\phi^{\varepsilon,R}$ of the truncated problem thanks to the Bernstein method (see [5] for a general description of the method and [6] for another application). Then, they conclude that $\phi^{\varepsilon,R}$ converges as ε goes to zero relying a compactness argument. The argument is robust and may apply to a wider range of problems. An alternate mean to carry out the proof would be to use the method of half-relaxed limits (see [7] for a general introduction of the method and [166] for an application) which applies without requiring any regularity estimates, at the cost of loosing continuity and therefore uniqueness in the limit $\varepsilon \to 0$. To recover uniqueness, we may add the additional constraint

$$\phi(t, \boldsymbol{x}, \mathcal{V}, w) \;=\; 0$$

on the limiting problem. However, proving that the limit provided by this method satisfies this constraint also requires uniform regularity estimates on the derivatives of ϕ^ε and we are back to our initial problem, since N is not globally Lipschitz.
In this article, we take advantage of the particular structure of the problem, specifically of the confining properties of N, to build a method which does not require regularity estimates and which has the advantage of providing explicit convergence rates : instead of proving uniform estimates on the derivatives of ϕ^ε, we prove uniform estimates on the first term in the expansion of ϕ^ε with respect to ε. This is made possible since this first term takes into account the non-linear fluctuations induced by N. Indeed, these non-linear variations induced by N are expected to be perturbations of order ε, as it may be seen rewriting equation (6.10) on ϕ^ε as follows

$$\frac{1}{\varepsilon}\,\partial_v\left(-\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2+\varepsilon n(v)-\phi^\varepsilon\right)\partial_v\phi^\varepsilon+\ldots=0,$$

where the correction $n(v)$ is such that $n'(v) = N(v)$ and where "..." gathers the lower order terms with respect to v and w. Hence, at least formally as ε goes to zero, we expect

$$\phi^\varepsilon \underset{\varepsilon\to 0}{\approx} -\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2+\varepsilon n(v).$$

Therefore, we consider the first term ϕ_1^ε in the expansion of ϕ^ε with respect to ε, that is

$$\phi^\varepsilon = -\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2+\varepsilon\phi_1^\varepsilon,$$

and identify its formal equivalent $\overline{\phi_1^\varepsilon}$ as ε goes to zero, which displays $n(v)$ and which depends on ε only through the macroscopic quantities $(\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$ (see Section 6.3). In Lemma 6.12, we look for super and sub-solutions to the equation solved by ϕ_1^ε with the form $\chi_\pm = \overline{\phi_1^\varepsilon} \pm \psi$. Once this is done, we apply a comparison principle in order to obtain $\chi_- \leq \phi_1^\varepsilon \leq \chi_+$, which in turns ensures

$$-\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2+\varepsilon\,\chi_- \;\leq\; \phi^\varepsilon \;\leq\; -\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2+\varepsilon\,\chi_+.$$

The last step consists in proving that $-\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2$ and $\overline{\phi_1^\varepsilon}$ converge as $\varepsilon\to 0$. This is done relying on previous results which ensure that the macroscopic quantities converge (see Theorem 6.10).

Comments and perspectives.

Two major perspectives arise from our work. First, our result holds in a perturbative setting in the sense that we need the initial data to be concentrated in order for our result to hold true. It would be interesting to cancel this constraint and to treat a general set of initial data without requiring any well-preparedness condition. To achieve this, one possibility would be to adapt the strategy adopted in [21], where was introduced a time dependent scaling in order to take into account the initial layer induced by the ill-preparedness of the initial data.

Second, an fundamental continuation consists in building on our approach in order to describe the limiting dynamics of the Hopf-Cole exponent ϕ^ε with respect to the adaptation variable w. This is very challenging since the scaling of interest for equation (6.4) is not singular in ε with respect to w, which means that w is a "slow variable" in our problem. Therefore, the limiting dynamics with respect to the adaptation variable correspond to fluctuations of ϕ^ε of order $O(\varepsilon)$. Let us provide some insights on the difficulties attached to this problem and the strategy that we have in mind to get over them. We consider ϕ_1^ε defined by (6.26), the first correction in the expansion of ϕ^ε as ε vanishes. Referring to the beginning of Section 6.3, ϕ_1^ε solves equation (6.27). Therefore, considering the leading order in equation (6.27), we expect

$$\phi_1^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \underset{\varepsilon\to 0}{\approx} \overline{\phi_1^\varepsilon}(t,\boldsymbol{x},\boldsymbol{u}) \,+\, \overline{\psi_1^\varepsilon}(t,\boldsymbol{x},w),$$

where $\overline{\phi_1^\varepsilon}$ is defined by (6.28) and where $\overline{\psi_1^\varepsilon}$ takes into account the limiting dynamics with respect to w and needs to be determined. To prove the latter convergence, we follow the same strategy as the one developed in the present article : we define ϕ_2^ε, the correction of order ε^2 in the expansion of ϕ^ε

$$\phi^\varepsilon = -\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2 + \varepsilon\left(\overline{\phi_1^\varepsilon}(t,\boldsymbol{x},\boldsymbol{u}) \,+\, \overline{\psi_1^\varepsilon}(t,\boldsymbol{x},w)\right) \,+\, \varepsilon^2\,\phi_2^\varepsilon,$$

or equivalently

$$\phi_1^\varepsilon = \overline{\phi_1^\varepsilon}(t,\boldsymbol{x},\boldsymbol{u}) \,+\, \overline{\psi_1^\varepsilon}(t,\boldsymbol{x},w) \,+\, \varepsilon\,\phi_2^\varepsilon, \tag{6.22}$$

and we derive uniform bounds with respect to ε for ϕ_2^ε. To do so, the challenge consists in designing super- and sub-solutions for the equation solved by ϕ_2^ε, which is obtained after plugging the *ansatz* (6.22) in equation (6.27)

$$\partial_t\phi_2^\varepsilon+\nabla_{\boldsymbol{u}}\phi_2^\varepsilon\cdot\widetilde{\mathbf{b}}^\varepsilon+2\,\partial_t\mathcal{V}^\varepsilon\partial_v\phi_2^\varepsilon-\partial_v^2\phi_2^\varepsilon-\varepsilon\,|\partial_v\phi_2^\varepsilon|^2+\frac{1}{\varepsilon}\left(\rho_0^\varepsilon\,(v-\mathcal{V}^\varepsilon)\,\partial_v\left(\phi_2^\varepsilon-\overline{\phi_2^\varepsilon}\right)+\mathcal{T}\left[\overline{\psi_1^\varepsilon}\right]\right)=0, \tag{6.23}$$

where $\widetilde{\mathbf{b}}^\varepsilon$ is defined as

$$\widetilde{\mathbf{b}}^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) := \begin{pmatrix} -B^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \\ \\ A(\boldsymbol{u}) \end{pmatrix},$$

where $\overline{\phi_2^\varepsilon}$ verifies

$$\partial_v \overline{\phi_2^\varepsilon}(t,\boldsymbol{x},\boldsymbol{u}) = \\ -\frac{1}{\rho_0^\varepsilon}\left(\partial_t \mathcal{V}^\varepsilon \left(\frac{N(v)-N(\mathcal{V}^\varepsilon)}{v-\mathcal{V}^\varepsilon} - N'(\mathcal{V}^\varepsilon)\right) - A_0(\boldsymbol{u}-\mathcal{U}^\varepsilon) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\varepsilon(f^\varepsilon) + a\,\partial_w \overline{\psi_1^\varepsilon}\right),$$

and where operator $\mathcal{T}$ reads

$$\mathcal{T}\left[\,\overline{\psi_1^\varepsilon}\,\right] = \partial_t \overline{\psi_1^\varepsilon} + A(\mathcal{V}^\varepsilon, w)\partial_w \overline{\psi_1^\varepsilon} + \partial_w A.$$

In order to infer the limit of $\overline{\psi_1^\varepsilon}$, we set $v = \mathcal{V}^\varepsilon$ in the leading order *w.r.t.* ε of equation (6.23). We deduce that $\mathcal{T}\left[\,\overline{\psi_1^\varepsilon}\,\right] \underset{\varepsilon\to 0}{\sim} 0$, which means that $\overline{\psi_1^\varepsilon}$ is determined by the following equation

$$\partial_t \overline{\psi_1^\varepsilon} + A(\mathcal{V}^\varepsilon, w)\partial_w \overline{\psi_1^\varepsilon} + \partial_w A = 0.$$

The latter considerations provide the corrective term of order ε in the expansion of ϕ^ε. This result is formal yet interesting. First, we recover the limiting dynamics with respect to w obtained in [21, 24]. Indeed, we notice that the function $G^\varepsilon = \exp\left(\overline{\psi_1^\varepsilon}\right)$ solves the following equation

$$\partial_t G^\varepsilon + \partial_w \left[A(\mathcal{V}^\varepsilon, w) G^\varepsilon\right] = 0,$$

which is exactly the one obtained in Section 1.2 of [21, 24]. Second, we surprisingly obtain additional corrective terms, gathered in $\overline{\phi_1^\varepsilon}$ defined by (6.28), which include cross terms between v and w as well as higher order terms with respect to v and which, at least to our knowledge, were not known in the literature. This assesses the potential of the Hopf-Cole approach initiated in [196] and continued in the present article to provide new insights for the model at hand. However, it also highlights the difficulties attached to this approach : its precision and rigidity lead to intricate and technical analysis whereas other methods focusing rather on the convergence at the level of densities yield coarser convergence result but are easier to implement.

Structure of the paper.

The remaining part of this article is organized as follows : in Section 6.2, we prove some regularity estimates for equation (6.4) in order to make our further computations rigorous : this is the object of Lemma 6.7 and Proposition 6.8. In Theorem 6.10 and Corollary 6.11, we also we recall and prove some convergence results on the macroscopic quantities $\mathcal{U}^\varepsilon$ and $\mathcal{E}(f^\varepsilon)$. Then we pass to Section 6.3, which is dedicated to the proof of Theorem 6.6. The proof relies on the key Lemma 6.12, in which we construct sub- and super-solution for equation (6.10) on ϕ^ε.

6.2 Preliminary estimates

First, we prove non-uniform in ε regularity estimates for the solutions to (6.4) in order to make later computations rigorous. Second, we recall uniform in ε moment estimates for the solutions to (6.4). We start with the following lemma, in which we prove regularity results for the macroscopic quantities $\mathcal{U}^\varepsilon$ and $\mathcal{E}(f^\varepsilon)$

Lemma 6.7. *Consider the solution f^ε to* (6.4) *provided by Theorem 6.5. For all function $\varphi \in \mathscr{C}^2\left(\mathbb{R}^2\right)$ with polynomial growth of order $l \geq 0$, that is*

$$|\varphi(\boldsymbol{u})| + |\nabla_{\boldsymbol{u}}\varphi(\boldsymbol{u})| + \left|\nabla^2_{\boldsymbol{u}}\varphi(\boldsymbol{u})\right| \underset{|\boldsymbol{u}|\to+\infty}{=} O\left(|\boldsymbol{u}|^l\right),$$

the function $\left((t,\boldsymbol{x}) \mapsto \int_{\mathbb{R}^2} \varphi(\boldsymbol{u}) f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\right)$ is continuous and has continuous time derivative over $\mathbb{R}^+ \times K$. In particular, the macroscopic quantities $\mathcal{V}^\varepsilon$ and $\mathcal{W}^\varepsilon$ given by (6.3) *and the error $\mathcal{E}\left(f^\varepsilon\right)$ given by* (6.7) *are continuous and have continuous time derivatives.*

Proof. Consider such function φ. To simplify notations we write

$$\varphi\left(f^\varepsilon\right) : (t,\boldsymbol{x}) \mapsto \int_{\mathbb{R}^2} \varphi(\boldsymbol{u}) f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}.$$

We start by proving that $\varphi\left(f^\varepsilon\right)$ is continuous. To do so, we fix some $(t,\boldsymbol{x}) \in \mathbb{R}^+ \times K$ and prove

$$\lim_{(s,\boldsymbol{y})\to(t,\boldsymbol{x})} \varphi\left(f^\varepsilon\right)(s,\boldsymbol{y}) \;=\; \varphi\left(f^\varepsilon\right)(t,\boldsymbol{x}).$$

For all $(s,\boldsymbol{y}) \in \mathbb{R}^+ \times K$, it holds

$$|\varphi\left(f^\varepsilon\right)(t,\boldsymbol{x}) - \varphi\left(f^\varepsilon\right)(s,\boldsymbol{y})| \leq \int_{\mathbb{R}^2} |\varphi(\boldsymbol{u})| \left|f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) - f^\varepsilon(s,\boldsymbol{y},\boldsymbol{u})\right|^{1/2} \left|f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) - f^\varepsilon(s,\boldsymbol{y},\boldsymbol{u})\right|^{1/2} \mathrm{d}\boldsymbol{u}.$$

Applying Cauchy-Schwarz inequality to the latter estimate, we deduce

$$|\varphi\left(f^\varepsilon\right)(t,\boldsymbol{x}) - \varphi\left(f^\varepsilon\right)(s,\boldsymbol{y})| \leq \left(\varphi^2\left(f^\varepsilon\right)(t,\boldsymbol{x}) + \varphi^2\left(f^\varepsilon\right)(s,\boldsymbol{y})\right)^{\frac{1}{2}} \left\|f^\varepsilon(t,\boldsymbol{x}) - f^\varepsilon(s,\boldsymbol{y})\right\|^{\frac{1}{2}}_{L^1(\mathbb{R}^2)}.$$

According to Theorem 6.5, $\varphi^2\left(f^\varepsilon\right)$ is locally bounded over $\mathbb{R}^+ \times K$ since f^ε has exponential moments and φ has polynomial growth. Therefore, we obtain the result since f^ε lies in $\mathscr{C}^0\left(\mathbb{R}^+ \times K, L^1\left(\mathbb{R}^2\right)\right)$ according to Definition 6.4.

We now prove that $\varphi\left(f^\varepsilon\right)$ has continuous time derivative. We multiply equation (6.4) by φ and integrate with respect to $\boldsymbol{u}$. After an integration by part, this yields

$$\partial_t \varphi\left(f^\varepsilon\right) = \xi_1\left(f^\varepsilon\right) + \left(\frac{1}{\varepsilon}\rho_0^\varepsilon \mathcal{V}^\varepsilon + \Psi * \left(\rho_0^\varepsilon\, \mathcal{V}^\varepsilon\right)\right) \partial_v \varphi\left(f^\varepsilon\right) - \left(\frac{1}{\varepsilon}\rho_0^\varepsilon + \Psi * \rho_0^\varepsilon\right) \xi_2\left(f^\varepsilon\right),$$

with

$$\begin{cases} \xi_1(\boldsymbol{u}) = \partial_v\varphi(\boldsymbol{u})\left(N(v) - w\right) + \partial_w\varphi(\boldsymbol{u})A(\boldsymbol{u}) + \partial_v^2\varphi(\boldsymbol{u}), \\ \\ \xi_2(\boldsymbol{u}) = \partial_v\varphi(\boldsymbol{u})v. \end{cases}$$

Functions $\xi_1\left(f^\varepsilon\right), \xi_2\left(f^\varepsilon\right), \partial_v\varphi\left(f^\varepsilon\right), \mathcal{V}^\varepsilon$ are continuous according to the previous step. Furthermore, we obtain that function $\Psi * \left(\rho_0^\varepsilon\, \mathcal{V}^\varepsilon\right)$ and $\Psi * \rho_0^\varepsilon$ are continuous using continuity of $\mathcal{V}^\varepsilon$ and ρ_0^ε and Assumption (6.14) on Ψ. This yields the result. □

We prove that when the initial data f_0^ε is smooth, the associated solution f^ε to (6.4) is regular.

Proposition 6.8. *Under the assumptions of Theorem 6.5, suppose in addition that f_0^ε lies in $\mathscr{C}^0\left(K, \mathscr{C}_c^\infty\left(\mathbb{R}^2\right)\right)$ and that N meets the following assumptions*

$$\sup_{|v|\geq 1}\left|N'(v)\right||v|^{1-p} < +\infty, \qquad \sup_{|v|\geq 1}\left|N'''(v)\right||v|^{-p'} < +\infty, \tag{6.24}$$

where p is given in assumption (6.12) *and p' in assumption* (6.13)*. Then the solution f^ε to equation* (6.4) *provided by Theorem 6.5 verifies*

$$f^\varepsilon \in L^\infty_{loc}\left(\mathbb{R}^+ \times K, W^{2,1}\left(\mathbb{R}^2\right)\right), \quad \partial_t f^\varepsilon \in L^\infty_{loc}\left(\mathbb{R}^+ \times K, L^1\left(\mathbb{R}^2\right)\right).$$

We postpone the proof to Appendix D.2 : it is mainly technical and relies on moment estimates on the derivatives of f^ε.

Remark 6.9. *Assumption* (6.24) *on N is purely technical but it does not constitute a limitation in our context since it is satisfied by the general class of drifts described below assumptions* (6.12)-(6.13)*, which includes the original FitzHugh-Nagumo model.*

For self-consistency, we recall a result from [24] about the control of the macroscopic quantities $(\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$ defined by (6.3) and the error term $\mathcal{E}\left(f^\varepsilon\right)$ defined by (6.7). We also provide uniform estimates with respect to ε for the moments of f^ε and for the relative energy given by

$$\begin{cases} M_q\left[f^\varepsilon\right](t,\boldsymbol{x}) := \dfrac{1}{\rho_0^\varepsilon(\boldsymbol{x})}\displaystyle\int_{\mathbb{R}^2}|\boldsymbol{u}|^q f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u}, \\ D_q\left[f^\varepsilon\right](t,\boldsymbol{x}) := \dfrac{1}{\rho_0^\varepsilon(\boldsymbol{x})}\displaystyle\int_{\mathbb{R}^2}|v - \mathcal{V}^\varepsilon(t,\boldsymbol{x})|^q f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{u}, \end{cases}$$

where $q \geq 2$.

Theorem 6.10 ([24])**.** *Under assumptions* (6.12)-(6.19) *and under the additional assumptions of Theorem 6.5, consider the solutions f^ε to* (6.4) *provided by Theorem 6.5 and the solution $\mathcal{U}$ to* (6.8)*. Furthermore, define the initial macroscopic error as*

$$\mathcal{E}_{\mathrm{mac}} = \|\mathcal{U}_0 - \mathcal{U}_0^\varepsilon\|_{L^\infty(K)} + \|\rho_0 - \rho_0^\varepsilon\|_{L^\infty(K)}.$$

There exists $(C, \varepsilon_0) \in \left(\mathbb{R}_*^+\right)^2$ such that

1. *for all* $\varepsilon \leq \varepsilon_0$*, it holds*

$$\|\mathcal{U}(t) - \mathcal{U}^\varepsilon(t)\|_{L^\infty(K)} \leq C \min\left(e^{Ct}\left(\mathcal{E}_{\text{mac}} + \varepsilon\right), 1\right), \qquad \forall t \in \mathbb{R}^+,$$

   where $\mathcal{U}^\varepsilon$ *and* $\mathcal{U}$ *are respectively given by* (6.3) *and* (6.8)*.*

2. *For all* $\varepsilon > 0$ *and all* q *in* $[2, 2(p+p')]$ *it holds*

$$M_q[f^\varepsilon](t, \boldsymbol{x}) \leq C, \qquad \forall (t, \boldsymbol{x}) \in \mathbb{R}^+ \times K,$$

   where exponent p *is given in assumption* (6.12) *and* p' *in assumption* (6.13)*. In particular,* $\mathcal{U}^\varepsilon$*,* $\partial_t \mathcal{U}^\varepsilon$ *and* $\mathcal{E}(f^\varepsilon)$ *are uniformly bounded with respect to both* $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ *and* ε*, where* $\mathcal{E}$ *is defined by* (6.7)*.*

3. *For all* $\varepsilon > 0$ *and all* q *in* $[2, 2(p+p')]$ *it holds*

$$D_q[f^\varepsilon](t, \boldsymbol{x}) \leq C\left[D_q[f^\varepsilon](0, \boldsymbol{x}) \exp\left(-q m_* \frac{t}{\varepsilon}\right) + \varepsilon^{\frac{q}{2}}\right], \quad \forall (t, \boldsymbol{x}) \in \mathbb{R}^+ \times K.$$

In this theorem, constants C *and* ε_0 *only depend on* m_p*,* $\overline{m}_p$*,* m_* *(see* (6.17)-(6.19) *) and on the data of the problem* N*,* A *and* Ψ*.*

We deduce from this result an estimate on the derivative of the error term, that will be used later in our proof.

Corollary 6.11. *Under the assumptions of Theorem 6.10 and the additional assumption* (6.21b) *on the sequence* $(f_0^\varepsilon)_{\varepsilon>0}$*, there exists a constant* $C > 0$ *such that*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f^\varepsilon(t, \boldsymbol{x}, \cdot))\right| \leq C,$$

for all $\varepsilon > 0$ *and all* $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$*, where* $\mathcal{E}(f^\varepsilon)$ *is the macroscopic error given by* (6.7)*. In this result, constant* C *only depends on* m_p*,* $\overline{m}_p$*,* m_* *(see* (6.17)-(6.19) *), on the data of the problem* N*,* A *and* Ψ *and on the constant in assumption* (6.21b)*.*

Proof. We compute the derivative of $\mathcal{E}(f^\varepsilon)$ taking the difference between equation (6.4) multiplied by N/ρ_0^ε and integrated with respect to $\boldsymbol{u}$, and the first line of (6.6) multiplied by $N'(\mathcal{V}^\varepsilon)$. After integrating by part with respect to v, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f^\varepsilon(t, \boldsymbol{x}, \cdot)) = \mathcal{A} + \mathcal{B}, \tag{6.25}$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{cases} \mathcal{A} = -\dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}^2} N'(v)\left(v - \mathcal{V}^\varepsilon(t, \boldsymbol{x})\right) f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) \mathrm{d}\boldsymbol{u}, \\ \mathcal{B} = \dfrac{1}{\rho_0^\varepsilon}\displaystyle\int_{\mathbb{R}^2} \left[\left(N' B^\varepsilon + N''\right) f^\varepsilon\right](t, \boldsymbol{x}, \boldsymbol{u}) \mathrm{d}\boldsymbol{u} - N'(\mathcal{V}^\varepsilon)\left(B^\varepsilon\left(t, \boldsymbol{x}, \mathcal{U}^\varepsilon\right) + \mathcal{E}(f^\varepsilon)\right). \end{cases}$$

The main difficulty here consists in estimating the stiffer term $\mathcal{A}$: this is what we start with. According to the definition of $\mathcal{V}^\varepsilon$, we have

$$\mathcal{A} = -\frac{1}{\varepsilon}\int_{\mathbb{R}^2} \left(N'(v) - N'\left(\mathcal{V}^\varepsilon\right)\right)\left(v - \mathcal{V}^\varepsilon\right) f^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u})\mathrm{d}\boldsymbol{u}.$$

Therefore, relying on our regularity assumptions on N, assumption (6.13) and item (2) of Theorem 6.10, which ensures that $\mathcal{V}^\varepsilon$ is uniformly bounded with respect to both $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ and $\varepsilon > 0$, we obtain some constant C such that

$$\mathcal{A} \leq \frac{C}{\varepsilon}\left(D_{p'+1}[f^\varepsilon] + D_2[f^\varepsilon]\right),$$

where we also used assumption (6.17) to upper bound ρ_0^ε. Hence, we apply item (3) of Theorem 6.10 and deduce

$$\mathcal{A} \leq C\left[\left(D_2 + D_{p'+1}\right)[f^\varepsilon](0, \boldsymbol{x})\varepsilon^{-1}\exp\left(-2m_*\frac{t}{\varepsilon}\right) + 1\right],$$

for all $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ and $\varepsilon > 0$. To conclude this step, we use assumption (6.21b) which ensures that : $\left(D_2 + D_{p'+1}\right)[f^\varepsilon](0, \boldsymbol{x}) = O(\varepsilon)$ as $\varepsilon \to 0$. Therefore, we deduce $\mathcal{A} \leq C$, for some constant C independent of $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ and ε.

According to assumptions (6.12) and (6.13), $\mathcal{B}$ only displays moments of f^ε up to order $2(p + p')$, which are uniformly bounded with respect to both $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ and $\varepsilon > 0$, according to item (2) of Theorem 6.10. On top of that both $\mathcal{U}^\varepsilon$ and $\mathcal{E}\left(f^\varepsilon\right)$ are also uniformly bounded according to item (2) in Theorem 6.10. Furthermore, according to assumptions (6.15) and (6.17) on Ψ and ρ_0^ε, $\Psi * \rho_0^\varepsilon$ is uniformly bounded as well. Therefore, there exists a constant C such that

$$\mathcal{B} \leq C,$$

for all $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times K$ and $\varepsilon > 0$.

We obtain the expected result gathering the estimates obtained on $\mathcal{A}$ and $\mathcal{B}$. □

6.3 Proof of Theorem 6.6

In this section, we derive uniform L^∞ convergence estimates for the solution ϕ^ε to equation (6.10). To do so, our strategy consists in performing a Hilbert expansion of ϕ^ε with respect to ε and to prove that the higher order terms are uniformly bounded with respect to ε. Denote by ϕ_1^ε the correction of order 1 in the expansion of ϕ^ε with respect to ε

$$\phi^\varepsilon = -\frac{1}{2}\rho_0^\varepsilon \left|v - \mathcal{V}^\varepsilon\right|^2 + \varepsilon\phi_1^\varepsilon. \tag{6.26}$$

Plugging this *ansatz* in (6.10), we find that ϕ_1^ε solves the following equation

$$\partial_t\phi_1^\varepsilon + \nabla_{\boldsymbol{u}}\phi_1^\varepsilon \cdot \mathbf{b}^\varepsilon + \mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}^\varepsilon\right] - \partial_v^2\phi_1^\varepsilon - \left|\partial_v\phi_1^\varepsilon\right|^2 + \frac{1}{\varepsilon}\rho_0^\varepsilon\left(v - \mathcal{V}^\varepsilon\right)\partial_v\left(\phi_1^\varepsilon - \overline{\phi_1^\varepsilon}\right) = 0, \tag{6.27}$$

where $\overline{\phi_1^\varepsilon}$ is given by

$$\overline{\phi_1^\varepsilon}\,(t,\boldsymbol{x},\boldsymbol{u}) = n(v) - n(\mathcal{V}^\varepsilon) - (v-\mathcal{V}^\varepsilon)\left(N(\mathcal{V}^\varepsilon) + (w-\mathcal{W}^\varepsilon) + \mathcal{E}(f^\varepsilon) + \frac{1}{2}\Psi * \rho_0^\varepsilon\,(v-\mathcal{V}^\varepsilon)\right), \tag{6.28}$$

and where n is the primitive of N defined in Theorem 6.6. Keeping the leading order, we expect that ϕ_1^ε will look like $\overline{\phi_1^\varepsilon}$ as $\varepsilon \to 0$. Therefore, we look for sub and super-solutions to equation (6.27) with the form $\overline{\phi_1^\varepsilon} + \psi$, where ψ needs to be determined. This is done in the following lemma, which constitutes the keystone of our analysis.

Lemma 6.12. *Consider some positive constant α_0 and define ψ as follows*

$$\psi\,(t,\boldsymbol{x},\boldsymbol{u}) = \frac{\alpha_0}{2}\,|v-\mathcal{V}^\varepsilon(t,\boldsymbol{x})|^2 + \frac{\alpha(t)}{2}\,|w-\mathcal{W}^\varepsilon(t,\boldsymbol{x})|^2\,,$$

where α is given by

$$\alpha(t) = \alpha_0 e^{2(|a|+b)t} + \frac{1}{|a|+b}\left(e^{2(|a|+b)t} - 1\right),$$

where (a,b) are the coefficients in the definition (6.5) *of A. The functions*

$$\chi_+ = \overline{\phi_1^\varepsilon} + \psi + m \quad \textit{and} \quad \chi_- = \overline{\phi_1^\varepsilon} - \psi - m$$

*are respectively **super** and **sub**-solutions to equation* (6.27)*, where $(t \mapsto m(t))$ is given by*

$$m(t) = m_0 + C\,\exp\left(6(a+b)t\right),$$

for all $m_0 \in \mathbb{R}$ and where the constant C only depends on α_0, the constants in (6.17)-(6.19) *and* (6.21b)*, and the data of the problem N, A and Ψ.*

Proof. In this proof, we fix some $(t,\boldsymbol{x}) \in \mathbb{R}^+ \times K$ and denote by C a generic constant depending only on α_0, the constants in (6.17)-(6.19) and (6.21b), and the data of the problem N, A and Ψ. Furthermore we write $\mathcal{U}^\varepsilon$ instead of $\mathcal{U}^\varepsilon(t,\boldsymbol{x})$ for convenience.

The first step of the proof consists in proving that the following term, obtained when evaluating equation (6.27) in $\overline{\phi_1^\varepsilon}$, is of order 0 with respect to ε

$$\mathcal{A} \;=\; \partial_t\overline{\phi_1^\varepsilon} + \nabla_{\boldsymbol{u}}\overline{\phi_1^\varepsilon}\cdot\mathbf{b}^\varepsilon + \mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}^\varepsilon\right] - \partial_v^2\overline{\phi_1^\varepsilon} - \left|\partial_v\overline{\phi_1^\varepsilon}\right|^2. \tag{6.29}$$

To this aim, we compute the derivatives of $\overline{\phi_1^\varepsilon}$. To simplify the computations, we rewrite $\overline{\phi_1^\varepsilon}$ in a more convenient way : we consider some $(\tilde{v},w) \in \mathbb{R}^2$ and take the difference between $B^\varepsilon\,(t,\boldsymbol{x},\tilde{v},w)$ given in (6.5) and the first line of the system (6.6)

$$\left(B^\varepsilon - \partial_t\mathcal{V}^\varepsilon\right)(t,\boldsymbol{x},\tilde{v},w) = \left[N(\tilde{v}) - N(\mathcal{V}^\varepsilon) - (w-\mathcal{W}^\varepsilon) - (\tilde{v}-\mathcal{V}^\varepsilon)\;\Psi * \rho_0^\varepsilon - \mathcal{E}(f^\varepsilon)\right](t,\boldsymbol{x})\,.$$

Integrating the latter relation between $\mathcal{V}^\varepsilon$ and v with respect to $\tilde{v}$, we deduce that $\overline{\phi_1^\varepsilon}$ verifies

$$\overline{\phi_1^\varepsilon}\,(t,\boldsymbol{x},\boldsymbol{u}) = \int_{\mathcal{V}^\varepsilon}^{v} \left(B^\varepsilon - \partial_t \mathcal{V}^\varepsilon\right)(t,\boldsymbol{x},\tilde{v},w)\,\mathrm{d}\tilde{v}\,.$$

Using the last two relations, we deduce that for all $\boldsymbol{u} \in \mathbb{R}^2$, it holds

$$\left\{\begin{array}{lcl} \nabla_{\boldsymbol{u}}\overline{\phi_1^\varepsilon}(t,\boldsymbol{x},\boldsymbol{u}) & = & \begin{pmatrix} B^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) - \partial_t\mathcal{V}^\varepsilon \\ \\ -\left(v - \mathcal{V}^\varepsilon\right) \end{pmatrix}, \\ \partial_v^2\overline{\phi_1^\varepsilon}(t,\boldsymbol{x},\boldsymbol{u}) & = & \mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}^\varepsilon\right](\boldsymbol{x},\boldsymbol{u}) - \partial_w A(\boldsymbol{u})\,, \\ \partial_t\,\overline{\phi_1^\varepsilon}(t,\boldsymbol{x},\boldsymbol{u}) & = & \partial_t\mathcal{V}^\varepsilon\left(\partial_t\mathcal{V}^\varepsilon - B^\varepsilon\left(t,\boldsymbol{x},\mathcal{V}^\varepsilon,w\right)\right) - (v-\mathcal{V}^\varepsilon)\partial_t\left(\partial_t\mathcal{V}^\varepsilon - \Psi * \left(\rho_0^\varepsilon\,\mathcal{V}^\varepsilon\right)\right)\,. \end{array}\right.$$

Relying on the latter expressions for the derivatives of $\overline{\phi_1^\varepsilon}$ we deduce

$$\begin{aligned} \mathcal{A} \,=\, & \partial_t\mathcal{V}^\varepsilon\left(B^\varepsilon\left(t,\boldsymbol{x},\boldsymbol{u}\right) - B^\varepsilon\left(t,\boldsymbol{x},\left(\mathcal{V}^\varepsilon,w\right)\right)\right) \\ & - (v-\mathcal{V}^\varepsilon)\partial_t\left(\partial_t\mathcal{V}^\varepsilon - \Psi * \left(\rho_0^\varepsilon\,\mathcal{V}^\varepsilon\right)\right) + \partial_w A(\boldsymbol{u}) - (v-\mathcal{V}^\varepsilon)A(\boldsymbol{u})\,. \end{aligned}$$

On the one hand, since B^ε is given by (6.5), it holds

$$B^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) - B^\varepsilon(t,\boldsymbol{x},(\mathcal{V}^\varepsilon,w)) = N(v) - N(\mathcal{V}^\varepsilon) - \Psi * \rho_0^\varepsilon(v-\mathcal{V}^\varepsilon)\,.$$

On the other hand, taking the time derivative in the first line of equation (6.6) we obtain

$$\partial_t\left(\partial_t\mathcal{V}^\varepsilon - \Psi * \left(\rho_0^\varepsilon\,\mathcal{V}^\varepsilon\right)\right) = \partial_t\,\mathcal{V}^\varepsilon\left(N'\left(\mathcal{V}^\varepsilon\right) - \Psi * \rho_0^\varepsilon\right) - A\left(\mathcal{U}^\varepsilon\right) + \frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{E}(f^\varepsilon)\,.$$

Therefore, $\mathcal{A}$ may be expressed as follows

$$\begin{aligned} \mathcal{A} \,=\, & \partial_t\mathcal{V}^\varepsilon\left(N(v) - N\left(\mathcal{V}^\varepsilon\right)\right) \\ & - \left(v-\mathcal{V}^\varepsilon\right)\left(\partial_t\mathcal{V}^\varepsilon N'\left(\mathcal{V}^\varepsilon\right) + \left(A(\boldsymbol{u}) - A(\mathcal{U}^\varepsilon)\right) + \frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{E}(f^\varepsilon)\right) \,+\, \partial_w A(\boldsymbol{u})\,. \end{aligned}$$

On the one hand, $\partial_t\mathcal{V}^\varepsilon$ and $\mathcal{V}^\varepsilon$ are uniformly bounded according to item (2) in Theorem 6.10. On the other hand, we apply Corollary 6.11 which ensures that under the smallness assumption (6.21b), the time derivative of $\mathcal{E}\left(f^\varepsilon\right)$ is uniformly bounded as well. Consequently, for all positive ε, it holds

$$\left(|\partial_t\mathcal{V}^\varepsilon| + |\mathcal{U}^\varepsilon| + \left|\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{E}(f^\varepsilon)\right|\right)(s,\boldsymbol{y}) \,\leq\, C\,, \quad \forall\,(s,\boldsymbol{y}) \in \mathbb{R}^+ \times K\,,$$

for some constant C depending only on the constants in assumptions (6.17)-(6.19), (6.21b) and on the data of the problem N, A and Ψ. Therefore, we deduce the following bound for $\mathcal{A}$

$$|\mathcal{A}| \,\leq\, C\left(1 + |v-\mathcal{V}^\varepsilon| + |v-\mathcal{V}^\varepsilon|^2 + |N(v) - N(\mathcal{V}^\varepsilon)|\right) + b\,\left|(v-\mathcal{V}^\varepsilon)(w-\mathcal{W}^\varepsilon)\right|\,.$$

Then we apply Young's inequality to bound the crossed term between v and w and use assumption (6.12) to bound $N(v)$. In the end, it yields

$$|\mathcal{A}| \leq C\,(1 + |v - \mathcal{V}^\varepsilon|^p) + \frac{1}{2}\,|w - \mathcal{W}^\varepsilon|^2\,. \tag{6.30}$$

Building on this estimate, we can now pass to the heart of the proof and show that χ_+ and χ_- are respectively super- and sub-solutions to (6.27). We evaluate equation (6.27) in χ_+ and χ_- and obtain

$$\partial_t \chi_\pm + \nabla_{\boldsymbol{u}} \chi_\pm \cdot \mathbf{b}^\varepsilon + \mathrm{div}_{\boldsymbol{u}}\,[\mathbf{b}^\varepsilon] - \partial_v^2 \chi_\pm - |\partial_v \chi_\pm|^2 + \frac{1}{\varepsilon}\,\rho_0^\varepsilon\,(v - \mathcal{V}^\varepsilon)\,\partial_v \chi_\pm = \mathcal{A} \pm \mathcal{B} \pm m'(t) - |\partial_v \psi|^2\,,$$

where $\mathcal{A}$ is given by (6.29) and $\mathcal{B}$ is given by

$$\mathcal{B} = \partial_t \psi + \nabla_{\boldsymbol{u}} \psi \cdot \mathbf{b}^\varepsilon - \partial_v^2 \psi - 2\,\partial_v \psi\,\partial_v \overline{\phi_1^\varepsilon} + \frac{1}{\varepsilon}\,\rho_0^\varepsilon\,(v - \mathcal{V}^\varepsilon)\,\partial_v \psi\,. \tag{6.31}$$

In order to conclude that χ_+ and χ_- are respectively sub- and super-solutions to (6.27), it is sufficient to prove

$$\mathcal{B} + m'(t) - |\partial_v \psi|^2 - |\mathcal{A}| \geq 0\,.$$

Therefore, we focus on proving the latter inequality. To begin with, we have

$$\begin{aligned}
&\mathcal{B} - |\partial_v \psi|^2 \\
=& - \alpha_0\,(v - \mathcal{V}^\varepsilon)\,\partial_t \mathcal{V}^\varepsilon + \frac{\alpha'(t)}{2}\,|w - \mathcal{W}^\varepsilon|^2 - \alpha(t)\,(w - \mathcal{W}^\varepsilon)\,\partial_t \mathcal{W}^\varepsilon \\
&+ \alpha_0\,(v - \mathcal{V}^\varepsilon)\,B^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) + \alpha(t)\,(w - \mathcal{W}^\varepsilon)\,A(\boldsymbol{u}) - \alpha_0 - 2\alpha_0\,(v - \mathcal{V}^\varepsilon)\,(B^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) - \partial_t \mathcal{V}^\varepsilon) \\
&+ \frac{\rho_0^\varepsilon}{\varepsilon}\alpha_0\,|v - \mathcal{V}^\varepsilon|^2 - \alpha_0^2\,|v - \mathcal{V}^\varepsilon|^2 \\
=& \frac{\alpha'(t)}{2}\,|w - \mathcal{W}^\varepsilon|^2 - \alpha_0\,(v - \mathcal{V}^\varepsilon)\,(B^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) - \partial_t\,\mathcal{V}^\varepsilon) \\
&+ \alpha(t)\,(w - \mathcal{W}^\varepsilon)\,(A(\boldsymbol{u}) - A(\mathcal{U}^\varepsilon)) - \alpha_0 + \left(\frac{\rho_0^\varepsilon}{\varepsilon}\alpha_0 - \alpha_0^2\right)|v - \mathcal{V}^\varepsilon|^2 \\
=& \frac{\alpha'(t)}{2}\,|w - \mathcal{W}^\varepsilon|^2 - \alpha_0\,(v - \mathcal{V}^\varepsilon)\,(B^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) - B^\varepsilon(t, \boldsymbol{x}, \mathcal{U}^\varepsilon) - \mathcal{E}(f^\varepsilon)) \\
&+ \alpha(t)\,(w - \mathcal{W}^\varepsilon)\,(a\,(v - \mathcal{V}^\varepsilon) - b\,(w - \mathcal{W}^\varepsilon)) - \alpha_0 + \left(\frac{\rho_0^\varepsilon}{\varepsilon}\alpha_0 - \alpha_0^2\right)|v - \mathcal{V}^\varepsilon|^2 \\
\geq& \left(\frac{\alpha'(t)}{2} - \alpha(t)(|a| + b)\right)|w - \mathcal{W}^\varepsilon|^2 - \alpha_0\,(v - \mathcal{V}^\varepsilon)\,(B^\varepsilon(t, \boldsymbol{x}, \boldsymbol{u}) - B^\varepsilon(t, \boldsymbol{x}, \mathcal{U}^\varepsilon) - \mathcal{E}(f^\varepsilon)) \\
&- \alpha_0 + \left(\frac{\rho_0^\varepsilon}{\varepsilon}\alpha_0 - \alpha_0^2 - \frac{|a|}{4}\alpha(t)\right)|v - \mathcal{V}^\varepsilon|^2\,,
\end{aligned}$$

where we have used the Young inequality at the last line. Gathering the latter estimate and (6.30) we obtain

$$
\begin{aligned}
&\mathcal{B} + m'(t) - |\partial_v \psi|^2 - |\mathcal{A}| \\
\geq\, & m'(t) + \left(\frac{\alpha'(t)}{2} - \alpha(t)(|a|+b)\right)|w - \mathcal{W}^\varepsilon|^2 - \alpha_0\,(v - \mathcal{V}^\varepsilon)\,(B^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) - B^\varepsilon(t,\boldsymbol{x},\mathcal{U}^\varepsilon) - \mathcal{E}(f^\varepsilon)) \\
& - \alpha_0 + \left(\frac{\rho_0^\varepsilon}{\varepsilon}\alpha_0 - \alpha_0^2 - \frac{|a|}{4}\alpha(t)\right)|v - \mathcal{V}^\varepsilon|^2 - C\,(1 + |v - \mathcal{V}^\varepsilon|^p) - |w - \mathcal{W}^\varepsilon|^2\,. \\
\geq\, & m'(t) + \left(\frac{\alpha'(t)}{2} - \alpha(t)(|a|+b) - 1\right)|w - \mathcal{W}^\varepsilon|^2 - \alpha_0 - \alpha_0\,(v - \mathcal{V}^\varepsilon)\,(N(v) - N(\mathcal{V}^\varepsilon)) \\
& + \left(\alpha_0 \Psi * \rho_0^\varepsilon(\boldsymbol{x}) + \frac{\rho_0^\varepsilon}{\varepsilon}\alpha_0 - \frac{3}{2}\,\alpha_0^2 - \frac{|a|}{4}\alpha(t) - \left(|a| + \frac{b}{2} + \frac{1}{2}\right) - \frac{1}{2}\,\alpha_0\right)|v - \mathcal{V}^\varepsilon|^2 \\
& + \alpha_0\,(v - \mathcal{V}^\varepsilon)\,\mathcal{E}(f^\varepsilon) - C\,(1 + |v - \mathcal{V}^\varepsilon|^p)\,,
\end{aligned}
$$

where we have used Young inequality and the following relation

$$
B^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) - B^\varepsilon(t,\boldsymbol{x},\mathcal{U}^\varepsilon) = N(v) - N(\mathcal{V}^\varepsilon) - (w - \mathcal{W}^\varepsilon) - \Psi * \rho_0^\varepsilon(\boldsymbol{x})(v - \mathcal{V}^\varepsilon)\,.
$$

To control the contribution of the term $|w - \mathcal{W}^\varepsilon|^2$ in the latter expression, we choose $\alpha(t)$ such that $\dfrac{\alpha'(t)}{2} - \alpha(t)(|a|+b) - 1 = 0$, that is

$$
\alpha(t) = \alpha_0 \exp\left(2(|a|+b)t\right) + \frac{1}{|a|+b}\left(\exp\left(2(|a|+b)t\right) - 1\right).
$$

Furthermore, since $\mathcal{E}(f^\varepsilon)$, ρ_0^ε and $\Psi * \rho_0^\varepsilon$ are uniformly bounded according to (respectively) item (2) in Theorem 6.10, assumptions (6.17) and (6.15), we deduce

$$
\begin{aligned}
&\mathcal{B} + m'(t) - |\partial_v \psi|^2 - |\mathcal{A}| \\
\geq\, & m'(t) - \alpha_0\,(v - \mathcal{V}^\varepsilon)\,(N(v) - N(\mathcal{V}^\varepsilon)) - \,C\,\left(1 + \exp\left(2\,(|a|+b)\,t\right)|v - \mathcal{V}^\varepsilon|^2 + |v - \mathcal{V}^\varepsilon|^p\right),
\end{aligned}
$$

for some constant $C > 0$. To control the terms of order $|v - \mathcal{V}^\varepsilon|^2$ and $|v - \mathcal{V}^\varepsilon|^p$ in the latter expression, we rely on the confining property (6.12) of N, which ensures

$$
(v - \mathcal{V}^\varepsilon)\,(N(v) - N(\mathcal{V}^\varepsilon)) \leq C - \frac{1}{C}\,|v - \mathcal{V}^\varepsilon|^{p+1}\,,
$$

for some constant C great enough. Hence, we obtain

$$
\begin{aligned}
&\mathcal{B} + m'(t) - |\partial_v \psi|^2 - |\mathcal{A}| \\
\geq\, & m'(t) + \frac{1}{C}\,|v - \mathcal{V}^\varepsilon|^{p+1} \,-\, C\,\left(1 + \exp\left(2\,(|a|+b)\,t\right)|v - \mathcal{V}^\varepsilon|^2 + |v - \mathcal{V}^\varepsilon|^p\right).
\end{aligned}
$$

Then we find that

$$
\frac{1}{C}\,|v - \mathcal{V}^\varepsilon|^{p+1} - C\left(1 + \exp\left(2\,(|a|+b)\,t\right)|v - \mathcal{V}^\varepsilon|^2 + |v - \mathcal{V}^\varepsilon|^p\right) \,\geq\, -\,\tilde{C}\exp\left(\,6\,(|a|+b)\,t\right).
$$

Hence choosing the function $(t \mapsto m(t))$ such that

$$m'(t) \,=\, \tilde{C}\, \exp\left(\,6\,(|a|\,+\,b)\,t\,\right),$$

we obtain

$$\mathcal{B} + m'(t) - |\partial_v \psi|^2 - |\mathcal{A}| \,\geq\, 0,$$

which concludes the proof. □

We are now able to proceed to the proof of Theorem 6.6. Indeed, relying on Lemma 6.12 and applying a comparison principle to equation (6.27), we deduce convergence estimates for the Hopf-Cole transform ϕ^ε of f^ε.

Proof of Theorem 6.6. All along this proof, we consider some positive constants α_0 (to be determined later on) and we work with the associated quantities ψ, χ_+ and χ_- defined in Proposition 6.12. We proceed in three steps

1. we prove that under our set of assumptions, it holds uniformly in ε

$$\chi_-(0,\boldsymbol{x},\boldsymbol{u}) \leq \phi_1^\varepsilon(0,\boldsymbol{x},\boldsymbol{u}) \leq \chi_+(0,\boldsymbol{x},\boldsymbol{u}), \quad \forall (\boldsymbol{x},\boldsymbol{u}) \in K \times \mathbb{R}^2,$$

   where χ_+ and χ_- are defined in Lemma 6.12,
2. we apply Lemma 6.12 and prove a comparison principle to deduce that the latter inequality holds for all positive time, that is

$$\chi_-(t,\boldsymbol{x},\boldsymbol{u}) \leq \phi_1^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \leq \chi_+(t,\boldsymbol{x},\boldsymbol{u}), \quad \forall (t,\boldsymbol{x}) \in \mathbb{R}^+ \times K, \text{ a.e. in } \boldsymbol{u} \in \mathbb{R}^2,$$

3. we conclude that ϕ^ε converges locally uniformly to $-\frac{1}{2}\rho_0\,|v-\mathcal{V}|^2$.

We start with step (1). Dividing assumption (6.21a) by ε and replacing ϕ^ε with ϕ_1^ε according to (6.26) we obtain the following bound for all positive ε

$$n(v) - C\left(1+|\boldsymbol{u}|^2\right) \leq \left(\phi_1^\varepsilon + \frac{1}{\varepsilon}\left(-\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2 + \frac{1}{2}\rho_0\,|v-\mathcal{V}|^2\right)\right)(0,\boldsymbol{x},\boldsymbol{u}) \leq n(v) + C\left(1+|\boldsymbol{u}|^2\right),$$

for all $(\boldsymbol{x},\boldsymbol{u}) \in K \times \mathbb{R}^2$. On the one hand, according to assumptions (6.17), (6.18) and (6.20), it holds

$$\frac{1}{\varepsilon}\left|-\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}_0^\varepsilon(\boldsymbol{x})|^2 + \frac{1}{2}\rho_0\,|v-\mathcal{V}_0(\boldsymbol{x})|^2\right| \leq C\left(1+|\boldsymbol{u}-\mathcal{U}_0^\varepsilon(\boldsymbol{x})|^2\right),$$

for all $(\boldsymbol{x},\boldsymbol{u}) \in K \times \mathbb{R}^2$, for some constant C depending only on the initial condition f_0^ε (only through the constants appearing in assumptions (6.17), (6.18) and (6.20)). On the other hand, according to assumptions (6.17) and (6.15), $\Psi * \rho_0^\varepsilon$ is uniformly bounded with respect to both $\boldsymbol{x} \in K$ and $\varepsilon > 0$. On top of that, $\mathcal{U}_0^\varepsilon$ and $\mathcal{E}\left(f_0^\varepsilon\right)$ are also uniformly bounded with respect to both $\boldsymbol{x} \in K$ and $\varepsilon > 0$ according to assumptions (6.12) and (6.18). Therefore, according to the definition (6.28) of $\overline{\phi_1^\varepsilon}$, it holds

$$\left| n - \overline{\phi_1^\varepsilon} \right| (0, \boldsymbol{x}, \boldsymbol{u}) \leq C \left(1 + |\boldsymbol{u} - \mathcal{U}_0^\varepsilon(\boldsymbol{x})|^2 \right),$$

for all $(\boldsymbol{x}, \boldsymbol{u}) \in K \times \mathbb{R}^2$, for some constant C depending on the initial condition f_0^ε (only through the constants appearing in assumptions (6.17)-(6.18)) and N. Gathering these considerations and writing

$$\phi_1^\varepsilon - \overline{\phi_1^\varepsilon} = \frac{1}{\varepsilon} \left(\phi^\varepsilon + \frac{1}{2} \rho_0 \, |v - \mathcal{V}|^2 - \varepsilon n \right) + \frac{1}{\varepsilon} \left(-\frac{1}{2} \rho_0 \, |v - \mathcal{V}|^2 + \frac{1}{2} \rho_0^\varepsilon \, |v - \mathcal{V}^\varepsilon|^2 \right) + n - \overline{\phi_1^\varepsilon},$$

we deduce that according to assumption (6.21a), for all positive ε, it holds

$$\overline{\phi_1^\varepsilon}(0, \boldsymbol{x}, \boldsymbol{u}) - C \left(1 + |\boldsymbol{u} - \mathcal{U}_0^\varepsilon(\boldsymbol{x})|^2 \right) \leq \phi_1^\varepsilon(0, \boldsymbol{x}, \boldsymbol{u}) \leq \overline{\phi_1^\varepsilon}(0, \boldsymbol{x}, \boldsymbol{u}) + C \left(1 + |\boldsymbol{u} - \mathcal{U}_0^\varepsilon(\boldsymbol{x})|^2 \right),$$

for all $(\boldsymbol{x}, \boldsymbol{u}) \in K \times \mathbb{R}^2$. Therefore, taking $\alpha_0/2$ and $m(0)$ greater than C, we conclude step (1), indeed for all positive ε it holds

$$\chi_-(0, \boldsymbol{x}, \boldsymbol{u}) \leq \phi_1^\varepsilon(0, \boldsymbol{x}, \boldsymbol{u}) \leq \chi_+(0, \boldsymbol{x}, \boldsymbol{u}), \quad \forall (\boldsymbol{x}, \boldsymbol{u}) \in K \times \mathbb{R}^2,$$

where χ_+ and χ_- are given in Proposition 6.12.

Let us now turn to step (2), which consists in proving that the latter estimate holds true for all positive time by applying a comparison principle. For technical reasons detailed in Appendix D.1, we apply the comparison principle for the following linearized version of the kinetic equation (6.4), instead of working directly on equation (6.27)

$$\partial_t f + \mathrm{div}_{\boldsymbol{u}} \left[\mathbf{b}^\varepsilon f \right] - \partial_v^2 f = \frac{1}{\varepsilon} \rho_0^\varepsilon \partial_v \left[(v - \mathcal{V}^\varepsilon) f \right]. \tag{6.32}$$

Therefore, we define f_+ and f_- for all $(t, \boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$ as follows

$$f_\pm(t, \boldsymbol{x}, \boldsymbol{u}) = \sqrt{\frac{\rho_0(\boldsymbol{x})}{2\pi\varepsilon}} \exp \left(-\frac{1}{2\,\varepsilon} \, \rho_0^\varepsilon \, |v - \mathcal{V}^\varepsilon\,(t, \boldsymbol{x})|^2 + \chi_\pm\,(t, \boldsymbol{x}, \boldsymbol{u}) \right).$$

We prove that these quantities are classical super- and sub-solutions of (6.32) applying jointly Lemma 6.12, to ensure that χ_- and χ_+ are respectively sub and super-solutions to equation (6.27) and Lemma D.1, which ensures that under the regularity condition $f_\pm, \partial_t f_\pm, \nabla_{\boldsymbol{u}}^2 f_\pm \in \mathscr{C}^0 \left(\mathbb{R}^+ \times K \times \mathbb{R}^2 \right)$, f_- and f_+ are respectively sub and super-solutions to (6.32) if and only if χ_- and χ_+ are respectively sub and super-solutions to (6.27). To verify the regularity assumption, we apply Lemma 6.7 which ensures that $\mathcal{U}^\varepsilon$ and $\mathcal{E}(f^\varepsilon)$ are continuous and have continuous time derivative over $\mathbb{R}^+ \times K$. Therefore, $f_\pm$, $\partial_t f_\pm$ and $\nabla_{\boldsymbol{u}}^2 f_\pm$ lie in $\mathscr{C}^0 \left(\mathbb{R}^+ \times K \times \mathbb{R}^2 \right)$. To conclude, we rely on the previous step, which ensures

$$f_-(0, \boldsymbol{x}, \boldsymbol{u}) \leq f_0^\varepsilon(\boldsymbol{x}, \boldsymbol{u}) \leq f_+(0, \boldsymbol{x}, \boldsymbol{u}), \quad \forall (\boldsymbol{x}, \boldsymbol{u}) \in K \times \mathbb{R}^2.$$

Therefore, relying on the comparison principle proved in Lemma D.2, we deduce

$$f_-(t,\boldsymbol{x},\boldsymbol{u}) \leq f^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \leq f_+(t,\boldsymbol{x},\boldsymbol{u}), \quad \forall (t,\boldsymbol{x}) \in \mathbb{R}^+ \times K, \text{a.e. in } \boldsymbol{u} \in \mathbb{R}^2.$$

Taking the logarithm of the latter relation, we deduce that the bound obtained in step (1), propagates through time, that is, for all positive ε, it holds

$$\chi_-(t,\boldsymbol{x},\boldsymbol{u}) \leq \phi_1^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \leq \chi_+(t,\boldsymbol{x},\boldsymbol{u}), \quad \forall (t,\boldsymbol{x}) \in \mathbb{R}^+ \times K, \text{a.e. in } \boldsymbol{u} \in \mathbb{R}^2.$$

We can now turn to the last step and prove our main result. According to the definition of ϕ_1^ε and the result of step (2), it holds

$$-\frac{1}{2}\rho_0^\varepsilon \left|v-\mathcal{V}^\varepsilon\right|^2+\frac{1}{2}\rho_0 \left|v-\mathcal{V}\right|^2+\varepsilon\chi_- \leq \phi^\varepsilon+\frac{1}{2}\rho_0 \left|v-\mathcal{V}\right|^2 \leq -\frac{1}{2}\rho_0^\varepsilon \left|v-\mathcal{V}^\varepsilon\right|^2+\frac{1}{2}\rho_0 \left|v-\mathcal{V}\right|^2+\varepsilon\chi_+,$$

for all $\forall (t,\boldsymbol{x}) \in \mathbb{R}^+ \times K$, a.e. in $\boldsymbol{u} \in \mathbb{R}^2$. On the one hand, relying on item (1) in Theorem 6.10 and since the initial condition f_0^ε meets the compatibility assumption (6.20), there exists two positive constants C and ε_0 such that for all $\varepsilon \leq \varepsilon_0$, it holds

$$\left|-\frac{1}{2}\rho_0^\varepsilon \left|v-\mathcal{V}^\varepsilon\right|^2+\frac{1}{2}\rho_0 \left|v-\mathcal{V}\right|^2\right|(t,\boldsymbol{x},\boldsymbol{u}) \leq \varepsilon C e^{Ct}\left(1+|\boldsymbol{u}|^2\right),$$

for all $(t,\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$, where constants C and ε_0 only depends on the data of our problem : f_0^ε (only through the constants appearing in assumptions (6.17)-(6.19) and (6.20)), N, A and Ψ. On the other hand, according to assumption (6.17) and (6.15), $\Psi * \rho_0^\varepsilon$ is uniformly bounded with respect to both $\boldsymbol{x} \in K$ and $\varepsilon > 0$. On top of that, $\mathcal{U}^\varepsilon$ and $\mathcal{E}\left(f^\varepsilon\right)$ are also uniformly bounded with respect to both $(t,\boldsymbol{x}) \in \mathbb{R}^+ \times K$ and $\varepsilon > 0$ according item (2) in Theorem 6.10. Therefore, according to the definitions of χ_- and χ_+ (see Lemma 6.12) and since N is continuous, it holds

$$\left|\chi_\pm - n\right|(t,\boldsymbol{x},\boldsymbol{u}) \leq C e^{Ct}\left(1+|\boldsymbol{u}|^2\right),$$

for all $(t,\boldsymbol{x},\boldsymbol{u}) \in K \times \mathbb{R}^2$, where the constant C only depends on the data of our problem : f_0^ε (only through the constants appearing in assumptions (6.17)-(6.19)), N, A and Ψ. Hence, we deduce the result : for all $\varepsilon \leq \varepsilon_0$ it holds

$$\varepsilon\left(n(v)-Ce^{Ct}\left(1+|\boldsymbol{u}|^2\right)\right) \leq \left(\phi^\varepsilon+\frac{1}{2}\rho_0 \left|v-\mathcal{V}\right|^2\right)(t,\boldsymbol{x},\boldsymbol{u}) \leq \varepsilon\left(n(v)+Ce^{Ct}\left(1+|\boldsymbol{u}|^2\right)\right),$$

for all $\forall (t,\boldsymbol{x}) \in \mathbb{R}^+ \times K$, a.e. in $\boldsymbol{u} \in \mathbb{R}^2$, where constants C and ε_0 only depends on the data of our problem : f_0^ε (only through the constants appearing in assumptions (6.17)-(6.19) and (6.20)-(6.21b)), N, A and Ψ. □

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Annexe B

Annexe of Chapter 4

Contents

B.1 Proof of Theorem 4.3

In this section, we prove existence and uniqueness of solution for equation (4.3) in the sense of Definition 4.2. The main difficulty is to prove uniqueness and continuity of the solution

$$\mu^{\varepsilon} \in \mathscr{C}^0\left(\mathbb{R}^+ \times K\,,\, L^1\left(\mathbb{R}^2\right)\right)\,.$$

To this aim, we first establish *a priori* estimates on the solution by propagating exponential moments in $\boldsymbol{u} \in \mathbb{R}^2$. Then the key point of the proof is to define a modified relative entropy $H_\alpha\left[\,\mu\,|\,\nu\,\right]$, given for any α in $]0,1[$ as

$$H_\alpha\left[\,\mu\,|\,\nu\,\right] \,=\, \int_{\mathbb{R}^2} \mu\,\ln\left(\frac{\mu}{\alpha\mu + (1-\alpha)\nu}\right)\mathrm{d}\boldsymbol{u}\,, \tag{B.1}$$

for two non-negative functions μ, $\nu \in L^1\left(\mathbb{R}^2\right)$ with mass equal to one. Relative entropy has several technical advantages in comparison to the L^1 norm. On the one hand, it makes explicit the dissipation due to the Laplace operator, which we refer as the Fisher information in what follows. On the other hand, the latter modified relative entropy, where the denominator is a convex combination of μ & ν is well defined for all positive functions μ and ν with mass 1, with no further condition on the domain on which they vanish. Indeed, under these conditions, holds the following inequality

$$0\,\leq H_\alpha\left[\,\mu\,|\,\nu\,\right]\,\leq\,-\ln\left(\alpha\right)\,<\,+\infty.$$

We even prove the following stronger result.

Lemma B.1. *For any two non-negative functions μ, $\nu \in L^1(\mathbb{R}^2)$ with integral equal to one, the following estimate holds*

$$\frac{(1-\alpha)^2}{2}\,\|\mu-\nu\|^2_{L^1(\mathbb{R}^2)} \leq H_\alpha\,[\mu\,|\,\nu] \leq \frac{1-\alpha}{\alpha}\,\|\mu-\nu\|_{L^1(\mathbb{R}^2)}\,. \tag{B.2}$$

Proof. The first inequality is a straightforward consequence of the Csizár-Kullback inequality : for μ, $\nu \in L^1(\mathbb{R}^2)$ with integral equal to one

$$\|\mu-\nu\|^2_{L^1} \leq 2\,H_0[\mu\,|\,\nu].$$

The result is obtained applying the latter inequality with $\nu = \kappa^\alpha$, where κ^α is given by

$$\kappa^\alpha = \alpha\mu + (1-\alpha)\nu\,.$$

For the second inequality, we notice that $|\,\mu\,/\kappa^\alpha\,| \leq \alpha^{-1}$. Then we apply the convex inequality $\ln(1+x) \leq x$ to the following relation

$$H_\alpha\,[\,\mu\,|\,\nu\,] = \int_{\mathbb{R}^2} \mu \ln\left(1 + \frac{\mu-\kappa^\alpha}{\kappa^\alpha}\right) \mathrm{d}\boldsymbol{u}\,.$$

□

From this modified relative entropy functional, we prove the continuity of the solution with respect to both the time and the spatial variable in the functional space L^1. Now, we fix $\alpha = 1/2$ and denote by $I_{1/2}$ the associated relative Fisher information

$$I_{1/2}\,[\mu\,|\,\nu] := \int_{\mathbb{R}^2} \left|\partial_v \ln\left(\frac{2\mu}{\nu+\mu}\right)\right|^2 \mu\,\mathrm{d}\boldsymbol{u}\,. \tag{B.3}$$

To deal with existence issues, we provide an entropy estimate

$$H[\mu^\varepsilon_{t,\boldsymbol{x}}] := \int_{\mathbb{R}^2} \mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u}) \ln(\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\,,$$

and follow a classical regularization argument.

B.1.1 A priori estimates

In this section, we suppose that we have a smooth solution $\mu^\varepsilon_{t,\boldsymbol{x}}$ to equation (4.3) and provide exponential moment, entropy and continuity estimates for $\mu^\varepsilon_{t,\boldsymbol{x}}$.

We first define $J[\mu^\varepsilon_{t,\boldsymbol{x}}]$ as

$$J[\mu^\varepsilon_{t,\boldsymbol{x}}] := \int_{\mathbb{R}^2} e^{|\boldsymbol{u}|^2/2}\,\mu^\varepsilon_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,$$

and prove the following result.

Proposition B.2 (Exponential moments)**.** *For any $\varepsilon > 0$, suppose that assumptions* (4.9a) *and* (4.10)-(4.12) *are fulfilled whereas the initial condition μ_0^ε verifies the first condition of* (4.15)*. Then, for any solution μ^ε to equation* (4.3)*, there exists a positive constant C that may depend on ε such that*

$$J[\mu_{t,\boldsymbol{x}}^\varepsilon] \;\leq\; C\,, \quad \forall\,(t,\boldsymbol{x}) \,\in\, \mathbb{R}^+ \times K\,.$$

Remark B.3. *We emphasize that if we consider a simplified model with homogeneous adaptation variable w, it is possible to prove that exponential moment are not only propagated as in Proposition B.2 but also created, thanks to the super-linear confining properties* (4.9a)-(4.9b) *of the drift N. However, this approach does not work when considering the full model. Indeed, since the dynamics are linear with respect to w, one can not expect any gain with respect to w. In addition, due to the crossed terms between the v and the w variables displayed in equation* (4.3) *on μ^ε, it seems hard to estimate exponential moments with respect to the v and w variables separetely.*
On top of that, one can see in the proof of Proposition B.2 that super-linear (or at least linear with large enough coefficient) confining properties are required on N to control the term $\mathcal{J}_2$ (see below) and particularly the crossed terms and the convolution term that it displays.

Proof. We multiply equation (4.3) by $e^{|\boldsymbol{u}|^2/2}$ and integrate with respect to $\boldsymbol{u} \in \mathbb{R}^2$, after an integration by part, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t} J[\mu_{t,\boldsymbol{x}}^\varepsilon] \;=\; \mathcal{J}\,,$$

where $\mathcal{J}$ is split into $\mathcal{J} \;=\; \mathcal{J}_1 \,+\, \mathcal{J}_2 \,+\, \mathcal{J}_3$, with

$$\begin{cases} \mathcal{J}_1 \;=\; \left(\dfrac{1}{\varepsilon}\,(\rho_0^\varepsilon\,\mathcal{V}^\varepsilon) \,+\, \Psi *_r (\rho_0^\varepsilon\,\mathcal{V}^\varepsilon)\right)(t,\boldsymbol{x}) \displaystyle\int_{\mathbb{R}^2} v\, e^{|\boldsymbol{u}|^2/2}\,\mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,, \\ \mathcal{J}_2 \;=\; \displaystyle\int_{\mathbb{R}^2} \left(w\,A(\boldsymbol{u}) \,-\, v\,w \,-\, v^2\left(\frac{\rho_0^\varepsilon(\boldsymbol{x})}{\varepsilon} + \Psi *_r \rho_0^\varepsilon(\boldsymbol{x}) - 1\right) + 1\right) e^{|\boldsymbol{u}|^2/2}\,\mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,, \\ \mathcal{J}_3 \;=\; \displaystyle\int_{\mathbb{R}^2} v\,N(v)\,e^{|\boldsymbol{u}|^2/2}\,\mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,. \end{cases}$$

The proof follows the same lines as the one given in Proposition 4.10 on the moment estimates except that here the non-local term $\mathcal{J}_1$ can be roughly estimated. Indeed, applying Corollary 4.11 and assumptions (4.11) & (4.12), we first obtain

$$\mathcal{J}_1 \;\leq\; C \int_{\mathbb{R}^2} |v|\, e^{|\boldsymbol{u}|^2/2}\,\mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,$$

for some positive constant $C \,>\, 0$ that may depend on ε. Then we estimate $\mathcal{J}_2$ and $\mathcal{J}_3$ following the computations in the proof of Proposition 4.10 and taking advantage of the confinement property (4.9a) on N. In the end, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} J[\mu_{t,\boldsymbol{x}}^\varepsilon] \;\leq\; \int_{\mathbb{R}^2} \Big((C \,-\, \omega^-(v))\,|v|^2 \,-\, \alpha|w|^2 + C\Big)\, e^{|\boldsymbol{u}|^2/2}\,\mu_{t,\boldsymbol{x}}^\varepsilon(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,$$

where C and α are two positive constants that may depend on ε and where ω^- is defined as

$$\omega^-(v) \,=\, \Big(\omega(v)\ \mathbb{1}_{|v|\geq 1}\Big)^- \,,$$

where ω is given in (4.9a). Finally, according to (4.9a), we have

$$\lim_{|\boldsymbol{u}|\to+\infty} \left(C \,-\, \omega^-(v)\right)|v|^2 \,-\, \alpha|w|^2 + C \,=\, -\infty\,,$$

hence it gives

$$\frac{\mathrm{d}}{\mathrm{d}t}J[\mu^\varepsilon_{t,\boldsymbol{x}}] \,\leq\, C \,-\, \frac{1}{C}\,J[\mu^\varepsilon_{t,\boldsymbol{x}}]\,.$$

We conclude the proof applying Gronwall's lemma and using the first assumption in (4.15). □

Then we provide an entropy estimate of the solution $\mu^\varepsilon_{t,\boldsymbol{x}}$.

Proposition B.4 (Entropy estimates)**.** *For any $\varepsilon > 0$, suppose that assumptions* (4.9a)-(4.9b) *and* (4.10)-(4.12) *are fulfilled whereas the initial condition μ^ε_0 verifies the second condition in* (4.15)*. Then, for any solution μ^ε to equation* (4.3)*, there exists a positive constant $C > 0$ that may depend on ε such that*

$$H[\mu^\varepsilon_{t,\boldsymbol{x}}] \,\leq\, H[\mu^\varepsilon_{0,\boldsymbol{x}}] \,+\, C\,t\,, \quad \forall\,(t,\boldsymbol{x}) \,\in\, \mathbb{R}^+ \times K\,.$$

Remark B.5. *The latter result raises the natural question of wether the entropy estimate holds uniformly with respect to time. An answer was given in [168, Theorem 2.2] in a simplified setting : authors consider a spatially homogeneous network and treat the case where N is a cubic function. They provide uniform estimate for the norm of the solution to* (4.3) *in regular norms, namely in H^1 and $H^1 \cap H^2_v$ spaces. It might be possible to adapt their argument to a low regularity setting such as ours and to obtain some uniform in time estimates on the entropy. However, this is an interesting question on its own and since the present work focuses on the asymptotic $\varepsilon \to 0$ rather than on the long time behavior, we did not follow this path here.*

Proof. We multiply equation (4.3) by $\ln\left(\mu^\varepsilon_{t,\boldsymbol{x}}\right)$ and integrate with respect to $\boldsymbol{u} \in \mathbb{R}^2$. Then we use conservation of mass for (4.3) and integrate by part. It yields

$$\frac{\mathrm{d}}{\mathrm{d}t}H[\mu^\varepsilon_{t,\boldsymbol{x}}] \,+\, I\left[\mu^\varepsilon_{t,\,\boldsymbol{x}}\right] \,=\, \frac{\rho^\varepsilon_0}{\varepsilon} \,+\, \Psi *_r \rho^\varepsilon_0 \,+\, b \,+\, \int_{\mathbb{R}^2} N(v)\,\partial_v\left(\ln \mu^\varepsilon_{t,\boldsymbol{x}}\right)\,\mu^\varepsilon_{t,\boldsymbol{x}}\,\mathrm{d}\boldsymbol{u}\,,$$

where the Fisher information I is given by

$$I\left[\mu^\varepsilon_{t,\,\boldsymbol{x}}\right] \,=\, \int_{\mathbb{R}^2}\left|\partial_v\,\ln\left(\mu^\varepsilon_{t,\,\boldsymbol{x}}\right)\right|^2\,\mu^\varepsilon_{t,\,\boldsymbol{x}}\,\mathrm{d}\boldsymbol{u}\,.$$

According to Young's inequality, we have

$$\int_{\mathbb{R}^2} N(v)\,\partial_v\left(\ln \mu^\varepsilon_{t,\boldsymbol{x}}\right)\,\mu^\varepsilon_{t,\boldsymbol{x}}\,\mathrm{d}\boldsymbol{u} \,\leq\, \frac{C}{\eta}\int_{\mathbb{R}^2}|N(v)|^2\,\mu^\varepsilon_{t,\boldsymbol{x}}\,\mathrm{d}\boldsymbol{u} \,+\, \eta\,I\left[\mu^\varepsilon_{t,\boldsymbol{x}}\right]\,,$$

for all positive η. We choose $\eta = 1/2$, apply Proposition 4.10, and assumptions (4.9b), (4.11) & (4.12). This yields

$$\frac{\mathrm{d}}{\mathrm{d}t} H[\mu^{\varepsilon}_{t,\boldsymbol{x}}] \leq C \,,$$

for some $C > 0$ that may depend on ε. Then we integrate the former inequality with respect to time and obtain the result. □

We now turn to continuity estimates with respect to the spatial variable and close the estimates in L^1 making use of the modified relative entropy $H_\alpha\,[\,\mu\,|\,\nu\,]$ given in (B.1).

Proposition B.6 (Continuity with respect to $\boldsymbol{x}$)**.** *For any $\varepsilon > 0$, suppose that assumption* (4.9a) *and* (4.10)-(4.12) *are fulfilled whereas the initial condition μ^{ε}_0 verifies the first condition of* (4.15)*. Then, for any solution μ^{ε} to equation* (4.3)*, the following estimate holds*

$$\left\| \mu^{\varepsilon}_{t,\boldsymbol{x}} - \mu^{\varepsilon}_{t,\boldsymbol{y}} \right\|_{L^1(\mathbb{R}^2)} \leq C\, e^{Ct}\, \gamma(\boldsymbol{x}, \boldsymbol{y}), \;\; \forall\, (\boldsymbol{x}\,,\, \boldsymbol{y}) \in K^2 \,,$$

for some positive constant C which may depend on ε and where γ is the following continuous function

$$\gamma(\boldsymbol{x}, \boldsymbol{y}) \,=\, \left\| \mu^{\varepsilon}_{0,\boldsymbol{x}} - \mu^{\varepsilon}_{0,\boldsymbol{y}} \right\|^{1/2}_{L^1(\mathbb{R}^2)} + |\rho^{\varepsilon}_0(\boldsymbol{x}) - \rho^{\varepsilon}_0(\boldsymbol{y})| + \|\Psi(\boldsymbol{x}, \cdot) - \Psi(\boldsymbol{y}, \cdot)\|_{L^1(K)} \,.$$

Proof. Our strategy consists in estimating $H_{1/2}$ instead of the L^1 error in order to take advantage of the entropy dissipation, then Lemma B.1 will ensure the continuity in L^1.

We choose some $\boldsymbol{x}$ and $\boldsymbol{y}$ in K all along this proof and consider $\kappa = \left(\mu^{\varepsilon}_{t,\boldsymbol{x}} + \mu^{\varepsilon}_{t,\boldsymbol{y}}\right)/2$. It satisfies the following equation

$$\partial_t\, \kappa + \partial_v\left(\,(N(v) - w)\,\kappa\,\right) + \partial_w\left(\,A(\boldsymbol{u})\,\kappa\,\right) - \partial^2_v\,\kappa - \frac{1}{2}\partial_v\left(\mathcal{N}_{\boldsymbol{x}}\,\mu^{\varepsilon}_{t,\boldsymbol{x}} + \mathcal{N}_{\boldsymbol{y}}\,\mu^{\varepsilon}_{t,\boldsymbol{y}}\right) \,=\, 0\,,$$

where $\mathcal{N}$ gathers the non-linear terms and is given by

$$\mathcal{N}_{\boldsymbol{x}}(t,v) \,:=\, \frac{\rho^{\varepsilon}_0(\boldsymbol{x})}{\varepsilon}(v - \mathcal{V}^{\varepsilon}(t,\,\boldsymbol{x})) + \left(\Psi *_r \rho^{\varepsilon}_0(\boldsymbol{x})\, v \,-\, \Psi *_r \left(\rho^{\varepsilon}_0 \mathcal{V}^{\varepsilon}\right)(t,\,\boldsymbol{x})\right) \,.$$

We compute the time derivative of $H_{1/2}\left[\,\mu^{\varepsilon}_{t,\,\boldsymbol{x}}\,|\,\mu^{\varepsilon}_{t,\,\boldsymbol{y}}\,\right]$ using the former equation, equation (4.3) and conservation of mass for (4.3). After an integration by part, all the terms associated to linear transport cancel and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{1/2}\left[\,\mu^{\varepsilon}_{t,\,\boldsymbol{x}}\,|\,\mu^{\varepsilon}_{t,\,\boldsymbol{y}}\,\right] + I_{1/2}\left[\,\mu^{\varepsilon}_{t,\,\boldsymbol{x}}\,|\,\mu^{\varepsilon}_{t,\,\boldsymbol{y}}\,\right] \,=\, \mathcal{H}_1\,,$$

where $I_{1/2}\,[\mu\,|\,\nu]$ is given by (B.3) and corresponds to the dissipation due to the second order term whereas $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 \,=\, -\int_{\mathbb{R}^2} \partial_v\left(\frac{\mu^{\varepsilon}_{t,\boldsymbol{x}}}{\kappa}\right)\left(\mathcal{N}_{\boldsymbol{x}}\,\kappa - \frac{1}{2}\left(\mathcal{N}_{\boldsymbol{x}}\,\mu^{\varepsilon}_{t,\boldsymbol{x}} + \mathcal{N}_{\boldsymbol{y}}\,\mu^{\varepsilon}_{t,\boldsymbol{y}}\right)\right)\mathrm{d}\boldsymbol{u}\,.$$

Exact computations yield

$$\mathcal{H}_1 = -\frac{1}{2} \int_{\mathbb{R}^2} \partial_v \left(\ln \left(\frac{\mu^\varepsilon_{t,\boldsymbol{x}}}{\kappa} \right) \right) (\mathcal{N}_{\boldsymbol{x}} \; - \mathcal{N}_{\boldsymbol{y}} \,) \, \frac{\mu^\varepsilon_{t,\boldsymbol{y}}}{\kappa} \, \mu^\varepsilon_{t,\boldsymbol{x}} \, \mathrm{d}\boldsymbol{u} \, .$$

Then, we first notice that it holds

$$|\, \mathcal{V}^\varepsilon \, (t, \boldsymbol{x}) \; - \; \mathcal{V}^\varepsilon \, (t, \boldsymbol{y}) \, | \; \leq \; 2 \, W_1 \left(\mu^\varepsilon_{t,\boldsymbol{x}}, \kappa \right) \, .$$

Furthermore, from Proposition B.2, we get that $\mu^\varepsilon_{t,\boldsymbol{x}}$, $\mu^\varepsilon_{t,\boldsymbol{y}}$ and therefore κ have exponential moments, hence we can apply Corollary 2.4 in [27], which ensures

$$W_1 \left(\mu^\varepsilon_{t,\boldsymbol{x}}, \kappa \right) \; \leq \; C \, H_0 \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \kappa \right]^{1/2} \, .$$

Since by definition it holds $H_0 \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \kappa \right] \; = \; H_{1/2} \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \mu^\varepsilon_{t,\boldsymbol{y}} \right]$, we obtain

$$|\, \mathcal{V}^\varepsilon \, (t, \boldsymbol{x}) \; - \; \mathcal{V}^\varepsilon \, (t, \boldsymbol{y}) \, | \; \leq \; C \, H_{1/2} \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \mu^\varepsilon_{t,\boldsymbol{y}} \right]^{1/2} \, ,$$

for some positive constant $C > 0$ which may depend on ε. Consequently, using assumption (4.12) and Corollary 4.11, we obtain the following bound for $\mathcal{H}_1$

$$|\mathcal{H}_1| \; \leq \; C \left(\beta(\boldsymbol{x}, \boldsymbol{y}) \; + \; H_{1/2} \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \mu^\varepsilon_{t,\boldsymbol{y}} \right]^{1/2} \right) \int_{\mathbb{R}^2} \left| \partial_v \ln \left(\frac{\mu^\varepsilon_{t,\boldsymbol{x}}}{\kappa} \right) \right| \; (1 \; + \; |v|) \, \frac{\mu^\varepsilon_{t,\boldsymbol{y}}}{\kappa} \, \mu^\varepsilon_{t,\boldsymbol{x}} \, \mathrm{d}\boldsymbol{u} \, ,$$

where C is a positive constant that may depend on ε and where β is given by

$$\beta(\boldsymbol{x}, \boldsymbol{y}) \; = \; |\rho^\varepsilon_0(\boldsymbol{x}) \; - \; \rho^\varepsilon_0(\boldsymbol{y})| \; + \; \|\Psi(\boldsymbol{x}, \, \cdot) \; - \; \Psi(\boldsymbol{y}, \, \cdot)\|_{L^1(K)} \, .$$

Then we apply Young's inequality, Proposition 4.10 and the bound $\left| \mu^\varepsilon_{t,\boldsymbol{y}} \, / \, \kappa \right| \; \leq \; 2$. It yields

$$|\mathcal{H}_1| \; \leq \; C \, \eta^{-1} \left(\beta(\boldsymbol{x}, \boldsymbol{y})^2 \; + \; H_{1/2} \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \mu^\varepsilon_{t,\boldsymbol{y}} \right] \right) \; + \; \eta \, I_{1/2} \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \mu^\varepsilon_{t,\boldsymbol{y}} \right] \, ,$$

for all positive η and where C may depend on ε. Therefore, taking $\eta \; = \; 1/2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{1/2} \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \mu^\varepsilon_{t,\boldsymbol{y}} \right] \; + \; \frac{1}{2} \, I_{1/2} \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \mu^\varepsilon_{t,\boldsymbol{y}} \right] \; \leq \; C \left(\beta(\boldsymbol{x}, \boldsymbol{y})^2 \; + \; H_{1/2} \left[\mu^\varepsilon_{t,\boldsymbol{x}} \, | \, \mu^\varepsilon_{t,\boldsymbol{y}} \right] \right) \, .$$

We apply Gronwall's lemma and Lemma B.1. It yields

$$\left\| \mu^\varepsilon_{t,\boldsymbol{x}} \; - \; \mu^\varepsilon_{t,\boldsymbol{y}} \right\|_{L^1(\mathbb{R}^2)} \; \leq \; C \, e^{Ct} \left(\left\| \mu^\varepsilon_{0,\boldsymbol{x}} \; - \; \mu^\varepsilon_{0,\boldsymbol{y}} \right\|^{1/2}_{L^1(\mathbb{R}^2)} \; + \; \beta(\boldsymbol{x}, \boldsymbol{y}) \right) \, .$$

□

We now turn to continuity with respect to the time variable and split the proof into two steps. First we prove continuity at time $t = 0$ and then deduce continuity for all time $t > 0$.

Proposition B.7 (Continuity at time $t = 0$)**.** *Under the assumptions of Theorem 4.3, the following estimate holds*

$$\sup_{\boldsymbol{x}\in K} \left\| \mu^{\varepsilon}_{t,\boldsymbol{x}} - \mu^{\varepsilon}_{0,\boldsymbol{x}} \right\|_{L^1(\mathbb{R}^2)} \leq C\sqrt{t}\,, \ \ \forall\, t \in \mathbb{R}_+\,,$$

for some positive constant C which may depend on ε.

Remark B.8. *It is the only time that we use assumption* (4.16)*.*

Proof. All along this proof, we set $\kappa = \left(\mu^{\varepsilon}_{t,\boldsymbol{x}} + \mu^{\varepsilon}_{0,\boldsymbol{x}}\right)/2$. In order to simplify notation, we also define B^{ε} as follows

$$B^{\varepsilon}(t,\boldsymbol{x},\boldsymbol{u}) = N(v) - w - \mathcal{K}_{\Psi}[\rho^{\varepsilon}_0\,\mu^{\varepsilon}]\,(t,\boldsymbol{x},v) - \frac{1}{\varepsilon}\,\rho^{\varepsilon}_0\,(v - \mathcal{V}^{\varepsilon})\,,$$

and we point out that according to Corollary 4.11, assumptions (4.9b) on N, (4.11) & (4.12), there exists a positive constant $C > 0$ that may depend on ε such that

$$|B^{\varepsilon}(t,\boldsymbol{x},\boldsymbol{u})| \leq C\,(\,|\boldsymbol{u}|^p + 1\,)\,, \quad \forall\,(t,\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}_+ \times K \times \mathbb{R}^2\,.$$

We compute the derivative of $H_{1/2}\left[\,\mu^{\varepsilon}_{0,\boldsymbol{x}}\,|\,\mu^{\varepsilon}_{t,\boldsymbol{x}}\,\right]$ using equation (4.3). It yields

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{1/2}\left[\mu^{\varepsilon}_{0,\boldsymbol{x}}\,|\,\mu^{\varepsilon}_{t,\boldsymbol{x}}\right] = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3\,,$$

where

$$\begin{cases} \mathcal{H}_1 = -\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2} \partial^2_v\,\mu^{\varepsilon}_{t,\boldsymbol{x}}\,\frac{\mu^{\varepsilon}_{0,\boldsymbol{x}}}{\kappa}\,\mathrm{d}\boldsymbol{u}\,, \\[2ex] \mathcal{H}_2 = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2} \partial_v\left[B^{\varepsilon}\,\mu^{\varepsilon}_{t,\boldsymbol{x}}\right]\frac{\mu^{\varepsilon}_{0,\boldsymbol{x}}}{\kappa}\,\mathrm{d}\boldsymbol{u}\,, \\[2ex] \mathcal{H}_3 = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2} \partial_w\left[A\,\mu^{\varepsilon}_{t,\boldsymbol{x}}\right]\frac{\mu^{\varepsilon}_{0,\boldsymbol{x}}}{\kappa}\,\mathrm{d}\boldsymbol{u}\,. \end{cases}$$

We start with $\mathcal{H}_1$. First, we integrate by part and rewrite the term as follows

$$\mathcal{H}_1 = -\,I_{1/2}\left[\mu^{\varepsilon}_{0,\boldsymbol{x}}\,|\,\mu^{\varepsilon}_{t,\boldsymbol{x}}\right] + \int_{\mathbb{R}^2} \partial_v\left(\ln\frac{\mu^{\varepsilon}_{0,\boldsymbol{x}}}{\kappa}\right)\left(1 - \frac{1}{2}\frac{\mu^{\varepsilon}_{0,\boldsymbol{x}}}{\kappa}\right)\partial_v\left(\ln\mu^{\varepsilon}_{0,\boldsymbol{x}}\right)\mu^{\varepsilon}_{0,\boldsymbol{x}}\,\mathrm{d}\boldsymbol{u}\,.$$

Second, we apply the inequality $|\mu^{\varepsilon}_0\,/\,\kappa| \leq 2$, Young's inequality and assumption (4.16) on μ^{ε}_0. It yields for all positive η

$$\mathcal{H}_1 \leq -\left(1 - \frac{\eta}{2}\right)I_{1/2}\left[\mu^{\varepsilon}_{0,\boldsymbol{x}}\,|\,\mu^{\varepsilon}_{t,\boldsymbol{x}}\right] + 8\,\frac{m^{\varepsilon}}{\eta},$$

We turn to $\mathcal{H}_2$. After an integration by part, it rewrites as follows

$$\mathcal{H}_2 = -\frac{1}{2}\int_{\mathbb{R}^2} B^{\varepsilon}\,\partial_v\left(\ln\frac{\mu^{\varepsilon}_{0,\boldsymbol{x}}}{\kappa}\right)\frac{\mu^{\varepsilon}_{t,\boldsymbol{x}}}{\kappa}\,\mu^{\varepsilon}_{0,\boldsymbol{x}}\,\mathrm{d}\boldsymbol{u}\,.$$

Then we apply the inequality $|\mu^\varepsilon \,/\, \kappa| \leq 2$, Young's inequality, the bound on B^ε and the first assumption in (4.15). We obtain for some positive constant C and for all positive η

$$\mathcal{H}_2 \,\leq\, \eta\, I_{1/2}\left[\mu^\varepsilon_{0,\boldsymbol{x}} \,|\, \mu^\varepsilon_{t,\boldsymbol{x}}\right] \,+\, \frac{C}{\eta}\,.$$

To end with, we estimate $\mathcal{H}_3$. This term is a little trickier to estimate since we do not have dissipation with respect to the adaptation variable. We rewrite $\mathcal{H}_3 \,=\, \mathcal{H}_{31} \,+\, \mathcal{H}_{32}$, where

$$\begin{cases} \mathcal{H}_{31} \,=\, \displaystyle\int_{\mathbb{R}^2} \partial_w \left(A(\boldsymbol{u})\right)\, \psi\left(\frac{\mu^\varepsilon_{0,\boldsymbol{x}}}{\kappa}\right) \frac{\mu^\varepsilon_{0,\boldsymbol{x}}}{\kappa}\, \kappa \,\mathrm{d}\boldsymbol{u}\,, \\ \mathcal{H}_{32} \,=\, -\displaystyle\int_{\mathbb{R}^2} A(\boldsymbol{u})\, \psi\left(\frac{\mu^\varepsilon_{0,\boldsymbol{x}}}{\kappa}\right) \partial_w\left(\ln \mu^\varepsilon_{0,\boldsymbol{x}}\right)\, \mu^\varepsilon_{0,\boldsymbol{x}}\,\mathrm{d}\boldsymbol{u}\,, \end{cases}$$

where $\psi(x) \,=\, \ln(x) - x/2$. We notice that $\mu^\varepsilon_{0,\boldsymbol{x}}/\kappa$ lies in $[0,\,2]$ and that $(x \mapsto x\,\psi(x))$ is bounded on $[0,\,2]$. Hence, using conservation of mass for equation (4.3), we obtain

$$\mathcal{H}_{31} \,\leq\, b \sup_{x\in[0,2]} |x\,\psi(x)|\,.$$

We turn to $\mathcal{H}_{32}$. We apply Young's inequality and use assumption (4.16) on μ^ε_0. It yields

$$\mathcal{H}_{32} \,=\, \frac{1}{2}\int_{\mathbb{R}^2} |A(\boldsymbol{u})|^2 \left|\psi\left(\frac{\mu^\varepsilon_{0,\boldsymbol{x}}}{\kappa}\right)\right|^2 \frac{\mu^\varepsilon_{0,\boldsymbol{x}}}{\kappa}\,\kappa\,\mathrm{d}\boldsymbol{u} \,+\, 2\,m^\varepsilon\,.$$

We notice that $\left(x \mapsto x\,|\psi(x)|^2\right)$ is bounded on $[0,2]$ and that $\mu^\varepsilon_{0,\boldsymbol{x}}/\kappa$ lies in $[0,\,2]$. Furthermore, we apply Proposition 4.10 and assumption (4.13). It yields

$$\mathcal{H}_3 \,\leq\, C\,.$$

Gathering former computations and taking η small enough, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{1/2}\left[\mu^\varepsilon_{0,\boldsymbol{x}} \,|\, \mu^\varepsilon_{t,\boldsymbol{x}}\right] \,+\, \frac{1}{2}\, I_{1/2}\left[\mu^\varepsilon_{0,\boldsymbol{x}} \,|\, \mu^\varepsilon_{t,\boldsymbol{x}}\right] \,\leq\, C\,.$$

We integrate the former relation between 0 and t and apply Lemma B.1. Since the constant C does not depend on $\boldsymbol{x}$, we take the supremum over K. In the end, we obtain

$$\sup_{\boldsymbol{x}\in K} \left\| \mu^\varepsilon_{t,\boldsymbol{x}} \,-\, \mu^\varepsilon_{0,\boldsymbol{x}} \right\|_{L^1(\mathbb{R}^2)} \,\leq\, C\,\sqrt{t}\,.$$

□

Using Proposition B.7, we deduce strong continuity at all times for μ^ε.

Proposition B.9 (Continuity at time $t \,>\, 0$)**.** *For any* $\varepsilon > 0$*, suppose that assumptions* (4.9a) *and* (4.10)-(4.12) *are fulfilled whereas the initial condition* μ^ε_0 *verifies the first condition of* (4.15)*, the following estimate holds*

$$\sup_{\boldsymbol{x}\in K} \left\| \mu^\varepsilon_{t\,,\,\boldsymbol{x}} \,-\, \mu^\varepsilon_{t+h\,,\,\boldsymbol{x}} \right\|_{L^1(\mathbb{R}^2)} \,\leq\, e^{Ct} \sup_{\boldsymbol{x}\in K} \left\| \mu^\varepsilon_{0\,,\,\boldsymbol{x}} \,-\, \mu^\varepsilon_{h\,,\,\boldsymbol{x}} \right\|^{1/2}_{L^1(\mathbb{R}^2)}\,,\;\; \forall\,(t\,,\,h) \in (\mathbb{R}_+)^2\,,$$

for some positive constant C *which may depend on* ε*.*

Proof. All along this proof, we consider some $\boldsymbol{x} \in K$ and $h > 0$. We introduce the following notation $\kappa = \left(\mu^{\varepsilon}_{t\,,\boldsymbol{x}} + \mu^{\varepsilon}_{t+h\,,\boldsymbol{x}}\right)/2$, which satisfies the following equation

$$\begin{aligned}\partial_t \kappa = &- \partial_v \left(\left(N(v) - w - \left(\frac{\rho_0^{\varepsilon}}{\varepsilon} + \Psi *_r \rho_0^{\varepsilon}\right) v\right) \kappa \right) - \partial_w \left(A(\boldsymbol{u})\, \kappa \right) + \partial_v^2 \kappa \\ &- \frac{1}{2}\partial_v \left(\mathcal{N}_t\, \mu^{\varepsilon}_{t,\boldsymbol{x}} + \mathcal{N}_{t+h}\, \mu^{\varepsilon}_{t+h,\boldsymbol{x}} \right) ,\end{aligned}$$

where $\mathcal{N}$ gathers the non-linear terms with respect to the time variable and is given by

$$\mathcal{N}_t(\boldsymbol{x}) = \frac{\rho_0^{\varepsilon}(\boldsymbol{x})}{\varepsilon}\, \mathcal{V}^{\varepsilon}(t,\, \boldsymbol{x}) + \Psi *_r \left(\rho_0^{\varepsilon} \mathcal{V}^{\varepsilon}\right)(t,\, \boldsymbol{x})\,.$$

We compute the derivative of $H_{1/2}\left[\mu^{\varepsilon}_{t,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{t+h,\boldsymbol{x}}\right]$ using the former equation, equation (4.3) and conservation of mass for (4.3). After an integration by part, all the terms associated to linear transport cancel and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{1/2}\left[\mu^{\varepsilon}_{t,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{t+h,\boldsymbol{x}}\right] + I_{1/2}\left[\mu^{\varepsilon}_{t,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{t+h,\boldsymbol{x}}\right] = \mathcal{H}_1\,,$$

where $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = \left(\mathcal{N}_t - \mathcal{N}_{t+h}\right) \int_{\mathbb{R}^2} \partial_v \left(\ln \frac{\mu^{\varepsilon}_{t\,,\boldsymbol{x}}}{\kappa}\right) \frac{\mu^{\varepsilon}_{t\,,\boldsymbol{x}}\, \mu^{\varepsilon}_{t+h\,,\boldsymbol{x}}}{\kappa}\, \mathrm{d}\boldsymbol{u}\,.$$

We use Young's inequality, assumptions (4.11) & (4.12) and the inequality $|\,\mu^{\varepsilon} / \kappa\,| \leq 2$. It yields

$$|\mathcal{H}_1| \leq \eta\, I_{1/2}\left[\mu^{\varepsilon}_{t,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{t+h,\boldsymbol{x}}\right] + C\eta^{-1} \sup_{\boldsymbol{x} \in K} |\mathcal{V}(t,\boldsymbol{x}) - \mathcal{V}(t+h,\boldsymbol{x})|^2\,,$$

for all positive η and some constant C that may depend on ε. Then from Proposition B.2, we get exponentional moments on κ and apply Corollary 2.4 in [27], which yields

$$\sup_{\boldsymbol{x} \in K} |\mathcal{V}(t,\boldsymbol{x}) - \mathcal{V}(t+h,\boldsymbol{x})|^2 \leq C \sup_{\boldsymbol{x} \in K} H_{1/2}\left[\mu^{\varepsilon}_{t,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{t+h,\boldsymbol{x}}\right]\,,$$

for some constant $C > 0$ which may depend on ε. Gathering the former computations and taking $\eta = 1/2$, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{1/2}\left[\mu^{\varepsilon}_{t,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{t+h,\boldsymbol{x}}\right] + \frac{1}{2} I_{1/2}\left[\mu^{\varepsilon}_{t,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{t+h,\boldsymbol{x}}\right] \leq C \sup_{\boldsymbol{x} \in K} H_{1/2}\left[\mu^{\varepsilon}_{t,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{t+h,\boldsymbol{x}}\right]\,.$$

We integrate this relation between 0 and t and take the supremum over all $\boldsymbol{x}$ in K. It yields

$$\sup_{\boldsymbol{x} \in K} H_{1/2}\left[\mu^{\varepsilon}_{t\,,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{t+h\,,\boldsymbol{x}}\right] \leq \sup_{\boldsymbol{x} \in K} H_{1/2}\left[\mu^{\varepsilon}_{0\,,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{h\,,\boldsymbol{x}}\right] + C \int_0^t \sup_{\boldsymbol{x} \in K} H_{1/2}\left[\mu^{\varepsilon}_{s\,,\boldsymbol{x}} \,|\, \mu^{\varepsilon}_{s+h\,,\boldsymbol{x}}\right] \mathrm{d}s\,.$$

We apply Gronwall's lemma to the former inequality and use Lemma B.1. We obtain

$$\sup_{\boldsymbol{x} \in K} \left\|\mu^{\varepsilon}_{t\,,\boldsymbol{x}} - \mu^{\varepsilon}_{t+h\,,\boldsymbol{x}}\right\|_{L^1(\mathbb{R}^2)} \leq Ce^{Ct} \sup_{\boldsymbol{x} \in K} \left\|\mu^{\varepsilon}_{0\,,\boldsymbol{x}} - \mu^{\varepsilon}_{h\,,\boldsymbol{x}}\right\|^{1/2}_{L^1(\mathbb{R}^2)}\,.$$

□

B.1.2 Uniqueness

We turn to the proof of uniqueness for equation (4.3). We follow the same method as in the proof of Proposition B.9. We consider two solutions μ^1 and μ^2 to equation (4.3) in the sense of Definition 4.2 and with the same initial condition μ_0^ε. Then we take some $T > 0$ and prove that the following relative entropy

$$H_{1/2}\left[\,\mu^1_{t,\boldsymbol{x}}\mid\mu^2_{t,\boldsymbol{x}}\,\right],$$

is zero for all $(t,\boldsymbol{x}) \in [0\,,\,T] \times K$. All along this proof, we take some $\boldsymbol{x} \in K$ and omit the dependence with respect to $\boldsymbol{x}$ and t when the context is clear. Furthermore we write

$$\left(\mathcal{V}^1\,,\,\mathcal{V}^2\right) = \left(\int_{\mathbb{R}^2} v\,\mu^1\,\mathrm{d}\boldsymbol{u}\,,\,\int_{\mathbb{R}^2} v\,\mu^2\,\mathrm{d}\boldsymbol{u}\right),$$

and $\kappa = \left(\mu^1 + \mu^2\right)/2$. Since μ^1 and μ^2 are solution to (4.3), κ solves the following equation

$$\partial_t\kappa + \partial_v\left(\left(N(v) - w - \left(\frac{\rho_0^\varepsilon}{\varepsilon} + \Psi *_r \rho_0^\varepsilon\right)\right)\kappa\right) + \partial_w\left(A(\boldsymbol{u})\,\kappa\,\right) - \partial_v^2\,\kappa + \frac{1}{2}\partial_v\left(\mathcal{N}^1\,\mu^1 + \mathcal{N}^2\,\mu^2\right) = 0\,,$$

where $\mathcal{N}^i$, for $i \in \{1\,,\,2\}$, gathers the non-linear terms and is given by

$$\mathcal{N}^i(t,\,\boldsymbol{x}) = \frac{\rho_0^\varepsilon(\boldsymbol{x})}{\varepsilon}\mathcal{V}^i(t,\,\boldsymbol{x}) + \Psi *_r \left(\rho_0^\varepsilon\mathcal{V}^i\right)(t,\,\boldsymbol{x})\,.$$

We compute the derivative of $H_{1/2}\left[\,\mu^1\,|\,\mu^2\,\right]$ using the former equation, equation (4.3) and conservation of mass for (4.3). After an integration by part, all the terms associated to linear transport cancel and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{1/2}\left[\,\mu^1\,|\,\mu^2\,\right] + I_{1/2}\left[\,\mu^1\,|\,\mu^2\,\right] = \mathcal{H}_1\,,$$

where $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = \frac{\mathcal{N}^1 - \mathcal{N}^2}{2}\int_{\mathbb{R}^2}\partial_v\left(\ln\left(\frac{\mu^1}{\kappa}\right)\right)\frac{\mu^2}{\kappa}\,\mu^1\,\mathrm{d}\boldsymbol{u}\,.$$

According to Theorem 4.3, μ^1 and μ^2 both have uniformly bounded exponential moments on $[0,T]$, hence applying Corollary 2.4 in [27], we obtain that

$$\left|\mathcal{V}^1 - \mathcal{V}^2\right| \le W_1\left(\mu^1\,,\,\mu^2\right) \le C\,H_{1/2}\left[\,\mu^1\,|\,\mu^2\,\right]^{1/2}\,,$$

for some positive constant $C > 0$ that may depend on T. Consequently, we use Young's inequality, the estimate $\left|\mu^2/\kappa\right| \le 2$, assumptions (4.11) & (4.12) and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{1/2}\left[\,\mu^1_{t,\boldsymbol{x}}\,|\,\mu^2_{t,\boldsymbol{x}}\,\right] + \frac{1}{2}\,I_{1/2}\left[\,\mu^1_{t,\boldsymbol{x}}\,|\,\mu^2_{t,\boldsymbol{x}}\,\right] \le C\,\sup_{\boldsymbol{x}\in K}\left(H_{1/2}\left[\,\mu^1_{t,\boldsymbol{x}}\,|\,\mu^2_{t,\boldsymbol{x}}\,\right]\right).$$

We integrate the former relation between 0 and t and take the supremum over all $\boldsymbol{x}$ in K in the left-hand side. It yields

$$\sup_{\boldsymbol{x}\in K}\left(H_{1/2}\left[\mu^1_{t,\boldsymbol{x}}\,|\,\mu^2_{t,\boldsymbol{x}}\right]\right) \leq C\int_0^t \sup_{\boldsymbol{x}\in K}\left(H_{1/2}\left[\mu^1_{s,\boldsymbol{x}}\,|\,\mu^2_{s,\boldsymbol{x}}\right]\right)\mathrm{d}s\,.$$

We apply Gronwall's lemma and conclude the proof.

B.1.3 Existence

In this section, we outline the main ideas in order to construct the solution to equation (4.3) given by Theorem 4.3. Let us first point out that the continuity of the macroscopic quantities $\mathcal{V}^\varepsilon$ and $\mathcal{W}^\varepsilon$ may be deduced from the continuity and the exponential moments of the solution μ^ε. Indeed, according to [27] (see Corollary 2.4), we have

$$(1-\alpha)\,|\mathcal{V}^\varepsilon(t,\,\boldsymbol{x}) - \mathcal{V}^\varepsilon(s,\,\boldsymbol{y})| \leq C\,H_{1/2}\left[\mu^\varepsilon_{t,\boldsymbol{x}},\,\mu^\varepsilon_{s,\boldsymbol{y}}\right]^{1/2},$$

as soon as $\mu^\varepsilon_{t,\boldsymbol{x}}$ and $\mu^\varepsilon_{s,\boldsymbol{y}}$ both have exponential moments. Then we apply Lemma B.1 and deduce continuity.
Hence, it is sufficient to prove that μ^ε exists in order to complete the proof. For that matter, we consider the following regularized equation

$$\partial_t\,\mu^R + \partial_v\left(\left(N^R(v) - w - \frac{\rho_0^\varepsilon}{\varepsilon}(v-\mathcal{V}^R) - \mathcal{K}_\Psi\left[\rho_0^\varepsilon\,\mu^R\right]\right)\mu^R\right) + \partial_w\left(A(\boldsymbol{u})\,\mu^R\right) - \partial_v^2\,\mu^R = 0\,, \tag{B.4}$$

where

$$\mathcal{V}^R(t,\boldsymbol{x}) = \int_{\mathbb{R}^2} v\,\mu^R_{t,\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\,,$$

and where $\left(N^R\right)_{R>0}$ is a suitable sequence of globally Lipschitz functions. We construct solutions to (B.4) with an iterative scheme. In order to prove that the scheme converges, we use the exact same method as in the proof for uniqueness (see Section B.1.2).
Then we let the truncation parameter R grow to infinity. Making use of the continuity estimates in Section B.1.1, we check that Ascoli theorem applies here and prove that the sequence $\left(\mu^R\right)_{R>0}$ is relatively compact. Furthermore, we prove that the limit of any subsequence of $\left(\mu^R\right)_{R>0}$ which converges in $\mathscr{C}^0\left([0,T]\times K\,,\,L^1\left(\mathbb{R}^2\right)\right)$ and which has uniformly bounded exponential moments is a solution to (4.3).

B.2 Proof of Proposition 4.16

We provide an entropy estimate in order to ensure existence and a uniqueness estimate in order to ensure that

$$\pi^\varepsilon(t,\cdot) \in \Pi\left(\nu^\varepsilon_{t,\boldsymbol{x}},\,\mathcal{M}_{\rho_0^\varepsilon}\otimes\bar{\nu}_{t,\boldsymbol{x}}\right),\quad \forall\, t\in\mathbb{R}^+\,.$$

We start with the uniqueness result.

Lemma B.10. *Under the assumptions of Theorem 4.3, consider* $\boldsymbol{x} \in K$*,* $\varepsilon > 0$ *and a solution* π^ε *to equation* (4.18) *in the sense of Definition 4.15 and with initial condition* π_0^ε *lying in* $\Pi\left(\nu_{0,\boldsymbol{x}}^\varepsilon,\, \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}_{0,\boldsymbol{x}}\right)$*. Then we have*

$$\pi^\varepsilon(t,\cdot) \in \Pi\left(\nu_{t,\boldsymbol{x}}^\varepsilon,\, \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}_{t,\boldsymbol{x}}\right), \quad \forall\, t \in \mathbb{R}^+ .$$

Proof. We define π^1 (resp. π^2) the marginal of π^ε with respect to $\boldsymbol{u}$ (resp. $\boldsymbol{u}'$) and we drop the dependence with respect to time and space when the context is clear. We integrate equation (4.18) with respect to $\boldsymbol{u}'$. We obtain that the difference between π^1 and $\nu_{t,\,\boldsymbol{x}}^\varepsilon$ solves

$$\partial_t \left(\pi^1 - \nu^\varepsilon\right) + \mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}_0^\varepsilon \left(\pi^1 - \nu^\varepsilon\right)\right] = \frac{1}{\varepsilon}\,\partial_v \left[\,\rho_0^\varepsilon\, v\, \left(\pi^1 - \nu^\varepsilon\right) + \partial_v \left(\pi^1 - \nu^\varepsilon\right)\right].$$

Multiplying the former equation by $\mathrm{sign}\left(\pi^1 - \nu^\varepsilon\right)$, we obtain that $\left|\pi^1 - \nu^\varepsilon\right|$ is a sub-solution to the former equation. Then we integrate with respect to $\boldsymbol{u}$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\pi^1 \,-\, \nu^\varepsilon\right\|_{L^1(\mathbb{R}^2)} \leq 0 .$$

We follow the method same for π^2 and obtain the expected result. □

We end this section with an entropy estimate, which ensures existence for solutions to (4.18) in the sense of Definition 4.15. We define the Fisher information associated to (4.18)

$$I_{v,v'}\left[\,\pi\,\right] = \int_{\mathbb{R}^4} \left|\,(\,\partial_v \,+\, \partial_{v'}\,)\ln \pi\,\right|^2 \,\pi\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u}' .$$

Lemma B.11. *Under the assumptions of Theorem 4.3, consider* $\boldsymbol{x} \in K$*,* $\varepsilon > 0$ *and a solution* π^ε *to equation* (4.18) *with some initial condition* π_0^ε *lying in* $\Pi\left(\nu_{0,\boldsymbol{x}}^\varepsilon, \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}_{0,\boldsymbol{x}}\right)$*, there exists a positive constant* $C > 0$ *which may depend on* ε*,* m_**,* m_p *and* $\overline{m}_p$ *such that*

$$H\left[\,\pi_t^\varepsilon\,\right] \leq H\left[\,\pi_0^\varepsilon\,\right] + C\,t .$$

Proof. We integrate equation (4.18) with respect to $\boldsymbol{u}$ and $\boldsymbol{u}'$ and deduce that mass is conserved through time. Hence, we compute the derivative of $H\left[\,\pi_t^\varepsilon\,\right]$ multiplying equation (4.18) by $\ln\left(\pi_t^\varepsilon\right)$. After an integration by part, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t} H\left[\,\pi_t^\varepsilon\,\right] + \frac{1}{\varepsilon}\, I_{v,v'}\left[\,\pi_t^\varepsilon\,\right] = \Psi *_r \rho_0^\varepsilon(\boldsymbol{x}) + 2\left(b + \frac{\rho_0^\varepsilon}{\varepsilon}\right) + \mathcal{A},$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{1}{\sqrt{\varepsilon}} \int_{\mathbb{R}^4} N\left(\mathcal{V}^\varepsilon + \sqrt{\varepsilon}\, v\right)\, \partial_v\left(\ln \pi_t^\varepsilon\right)\, \pi_t^\varepsilon\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u}' .$$

Since the Fisher information $I_{v,v'}$ is somehow degenerate, we can not apply directly Young's inequality to $\mathcal{A}$. Hence, we re-write $\mathcal{A}$ as follows

$$\mathcal{A} \,=\, \frac{1}{\sqrt{\varepsilon}} \int_{\mathbb{R}^4} N\left(\mathcal{V}^\varepsilon + \sqrt{\varepsilon}\, v\right)\, \left(\,\partial_v \,+\, \partial_v'\,\right)\, \left(\ln \pi_t^\varepsilon\right)\, \pi_t^\varepsilon \,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u'}\,.$$

Then, we apply Young's inequality to $\mathcal{A}$ and obtain

$$\mathcal{A} \,\leq\, \frac{C\eta}{\varepsilon}\, I_{v,v'}\left[\,\pi_t^\varepsilon\,\right] \,+\, C\eta \int_{\mathbb{R}^4} \left|N\left(\mathcal{V}^\varepsilon + \sqrt{\varepsilon}\, v\right)\right|^2\, \pi_t^\varepsilon \,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u'}\,,$$

for all η in $]0,1[$. Applying Lemma B.10, assumption (4.9b), Proposition 4.12 and Corollary 4.11, it yields

$$\int_{\mathbb{R}^4} \left|N\left(\mathcal{V}^\varepsilon + \sqrt{\varepsilon}\, v\right)\right|^2\, \pi_t^\varepsilon \,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u'} \,\leq\, C\,.$$

Hence, taking η small enough, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H\left[\,\pi_t^\varepsilon\,\right] \,+\, \frac{1}{2\,\varepsilon}\, I_{v,v'}\left[\,\pi_t^\varepsilon\,\right] \,\leq\, C\,,$$

where C may depend on ε, m_p and $\overline{m}_p$. We integrate the former inequality between 0 and t and obtain the result. □

Annexe C

Annexe of Chapter 5

Contents

C.1 Proof of Lemma 5.6

Instead of estimating directly the L^1 norm of $f - g$, we estimate the following intermediate quantity, introduced in [24]

$$H_{1/2}\,[\,f\,|\,g\,] \;=\; \int_{\mathbb{R}^{d_1+d_2}} f\,\ln\left(\frac{2\,f}{f+g}\right) \mathrm{d}\boldsymbol{y}\,\mathrm{d}\xi\,,$$

which is easily comparable to the classical L^1 norm thanks to the following Lemma (see [24], Lemma A.1. for detailed proof)

Lemma C.1. *For any two non-negative functions f, $g \in L^1\left(\mathbb{R}^{d_1+d_2}\right)$ with integral equal to one, the following estimate holds*

$$\frac{1}{8}\,\|\,f\,-\,g\,\|^2_{L^1\left(\mathbb{R}^{d_1+d_2}\right)} \;\leq\; H_{1/2}\,[\,f\,|\,g\,] \;\leq\; \|\,f\,-\,g\,\|_{L^1\left(\mathbb{R}^{d_1+d_2}\right)}\,.$$

Unlike the L^1 norm, $H_{1/2}$ has an explicit dissipation with respect to the Laplace operator which is given by

$$I_{1/2}\,[\,f\,|\,g\,] \;:=\; \int_{\mathbb{R}^{d_1+d_2}} \left|\nabla_\xi \ln\left(\frac{2\,f}{f+g}\right)\right|^2 f\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\xi\,.$$

We consider the quantity $h \,:=\, (f+g)\,/\,2$, whose equation is obtained multiplying by $1/2$ the sum of the equations solved by f and g, that is

$$\partial_t\,h \,+\, \mathrm{div}_{\boldsymbol{y}}\,[\,\mathbf{a}\,h\,] \,+\, \frac{\lambda(t)}{2}\,\mathrm{div}_\xi\,[\,(\mathbf{b}_1 \,+\, \mathbf{b}_3)\;f \,+\, \mathbf{b}_2\,g\,] \,-\, \lambda(t)^2\,\Delta_\xi\,h \,=\, (\delta-1)\,\frac{\lambda(t)^2}{2}\,\Delta_\xi\,g\,.$$

Then, we compute the time derivative of the quantity $H_{1/2}\,[\,f\,|\,g\,]$ integrating with respect to both ξ and $\boldsymbol{y}$ the difference between the equation solved by f multiplied by $\ln\,(f\,/\,h)$ and the equation solved by h multiplied by $f\,/\,h$. After an integration by part, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,H_{1/2}\,[\,f\,|\,g\,]\,+\,\lambda(t)^2\,I_{1/2}\,[\,f\,|\,g\,]\,=\,\mathcal{A}\,+\,\mathcal{B}\,,$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{cases} \mathcal{A}\,=\,\dfrac{\lambda(t)}{2}\displaystyle\int_{\mathbb{R}^{d_1+d_2}}(\,\mathbf{b}_1\,-\,\mathbf{b}_2\,)\,\nabla_\xi\left(\ln\left(\frac{f}{h}\right)\right)\,\frac{g}{h}\,f\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\xi\,,\\[2ex] \mathcal{B}\,=\,-\,\dfrac{\lambda(t)}{2}\displaystyle\int_{\mathbb{R}^{d_1+d_2}}\mathrm{div}_\xi\,[\,\mathbf{b}_3\,g\,+\,(\delta\,-\,1)\,\lambda(t)\,\nabla_\xi\,g\,]\,\frac{f}{h}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\xi\,. \end{cases}$$

To estimate $\mathcal{A}$, we notice that $|\,g\,/\,h\,|\,\leq\,2$ and we apply Young's inequality. This yields

$$\mathcal{A}\,\leq\,\lambda(t)^2\,I_{1/2}\,[\,f\,|\,g\,]\,+\,\frac{1}{4}\int_{\mathbb{R}^{d_1+d_2}}|\,\mathbf{b}_1\,-\,\mathbf{b}_2\,|^2\,f\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\xi\,.$$

To estimate $\mathcal{B}$, we simply notice that $|\,f\,/\,h\,|\,\leq\,2$ and take the absolute value inside the integral. In the end, we obtain

$$\mathcal{A}\,+\,\mathcal{B}\,\leq\,\mathcal{R}(t)\,+\,\lambda(t)^2\,I_{1/2}\,[\,f\,|\,g\,]\,,$$

where $\mathcal{R}$ is defined as in Lemma 5.6. Therefore, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\,H_{1/2}\,[\,f(t)\,|\,g(t)\,]\,\leq\,\mathcal{R}(t)\,,\quad\forall\,t\in\mathbb{R}^+\,.$$

Then, we deduce (5.33) by integrating the latter inequality between 0 and t and applying Lemma C.1 in order to substitute $H_{1/2}$ with the L^1-norm.

Annexe D

Annexe of Chapter 6

Contents

D.1 Comparison principles

The object of this section is to prove a comparison principle for equation (6.27) in order to complete step (2) in the proof of Theorem 6.6. More precisely, we prove that if the quantity ϕ_1^ε defined by (6.26) verifies

$$\chi_-(0,\boldsymbol{x},\boldsymbol{u}) \leq \phi_1^\varepsilon(0,\boldsymbol{x},\boldsymbol{u}) \leq \chi_+(0,\boldsymbol{x},\boldsymbol{u}), \quad \forall (\boldsymbol{x},\boldsymbol{u}) \in K \times \mathbb{R}^2,$$

where χ_- an χ_+ are respectively sub and super-solutions to (6.27), then the latter estimate propagates through time, that is

$$\chi_-(t,\boldsymbol{x},\boldsymbol{u}) \leq \phi_1^\varepsilon(t,\boldsymbol{x},\boldsymbol{u}) \leq \chi_+(t,\boldsymbol{x},\boldsymbol{u}), \quad \forall (t,\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2.$$

Instead of working directly on equation (6.27), our strategy consists in proving a comparison principle for the linearized version (6.32) of the kinetic equation (6.4). Indeed, it is more convenient to work on equation (6.32) since we can rely on the decaying properties of solutions to (6.4) provided by Theorem 6.5. From the comparison principle on equation (6.32), we will easily deduce the expected result. This approach is made possible since, according to the following lemma, there is a direct link between sub- and super-solutions to equations (6.32) and (6.27)

Lemma D.1. *Consider some fixed $\varepsilon > 0$. Under the assumptions of Theorem 6.5, consider the solution f^ε to equation* (6.4) *and its associated macroscopic quantities* $(\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$

provided by Theorem 6.5. Furthermore, consider a strictly positive function f such that f, $\partial_t f$ and $\nabla^2_{\boldsymbol{u}} f$ lie in $\mathscr{C}^0\left(\mathbb{R}^+ \times K \times \mathbb{R}^2\right)$ and define χ as follows

$$f(t,\boldsymbol{x},\boldsymbol{u}) = \sqrt{\frac{\rho_0(\boldsymbol{x})}{2\pi\varepsilon}}\exp\left(\left(-\frac{1}{2\varepsilon}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2+\chi\right)(t,\boldsymbol{x},\boldsymbol{u})\right), \quad \forall (t,\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2.$$

Then the following statements are equivalent

1. *f is a super-solution (resp. sub-solution) to* (6.32).
2. *χ is a super-solution (resp. sub-solution) to* (6.27).

Proof. We consider f and χ as in Lemma D.1 and proceed in two steps. On the one hand, plugging f in equation (6.32), one has the following relation

$$\partial_t f + \mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}^\varepsilon f\right] - \partial_v^2 f - \frac{1}{\varepsilon}\rho_0^\varepsilon \partial_v\left[(v-\mathcal{V}^\varepsilon)f\right] = \frac{1}{\varepsilon} f\,\mathcal{A}\,,$$

for all $(t,\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$ where $\mathcal{A}$ gathers the terms obtained plugging $\phi := -\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2 + \varepsilon\chi$ into equation (6.10), that is

$$\mathcal{A} \,=\, \partial_t\phi + \nabla_{\boldsymbol{u}}\phi\cdot\mathbf{b}^\varepsilon + \varepsilon\mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}\right] - \partial_v^2\phi - \rho_0^\varepsilon - \frac{1}{\varepsilon}\left(\partial_v\left(\frac{1}{2}\rho_0^\varepsilon\,|v-\mathcal{V}^\varepsilon|^2+\phi\right)\partial_v\phi\right).$$

On the other hand, according to computations already detailed at the beginning of Section 6.3, $\mathcal{A}$ also corresponds to the terms obtained plugging χ into equation (6.27), that is

$$\mathcal{A} \,=\, \varepsilon\left(\partial_t\chi + \nabla_{\boldsymbol{u}}\chi\cdot\mathbf{b}^\varepsilon + \mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}^\varepsilon\right] - \partial_v^2\chi - |\partial_v\chi|^2 + \frac{1}{\varepsilon}\,\rho_0^\varepsilon\,(v-\mathcal{V}^\varepsilon)\,\partial_v\left(\chi-\overline{\phi_1^\varepsilon}\right)\right),$$

for all $(t,\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$. Therefore, we deduce

$$\begin{aligned}&\partial_t f + \mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}^\varepsilon f\right] - \partial_v^2 f - \frac{1}{\varepsilon}\rho_0^\varepsilon\partial_v\left[(v-\mathcal{V}^\varepsilon)f\right] = \\ &f\,\left(\partial_t\chi + \nabla_{\boldsymbol{u}}\chi\cdot\mathbf{b}^\varepsilon + \mathrm{div}_{\boldsymbol{u}}\left[\mathbf{b}^\varepsilon\right] - \partial_v^2\chi - |\partial_v\chi|^2 + \frac{1}{\varepsilon}\,\rho_0^\varepsilon\,(v-\mathcal{V}^\varepsilon)\,\partial_v\left(\chi-\overline{\phi_1^\varepsilon}\right)\right),\end{aligned}$$

which yields the result since f is strictly positive. □

It is now left to prove that a comparison principle holds for equation (6.32). It is the object of the following Lemma

Lemma D.2. *Consider some fixed $\varepsilon > 0$. Under the assumptions of Theorem 6.5 and Proposition 6.8, consider the solution f^ε to equation* (6.4) *and its associated macroscopic quantities $(\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$ provided by Theorem 6.5. Furthermore, consider a strictly positive function f such that f, $\partial_t f$ and $\nabla^2_{\boldsymbol{u}} f$ lie in $\mathscr{C}^0\left(\mathbb{R}^+ \times K \times \mathbb{R}^2\right)$. Suppose that at initial time, it holds*

$$f_0^\varepsilon(\boldsymbol{x},\boldsymbol{u}) \leq f(0,\boldsymbol{x},\boldsymbol{u}), \quad \forall (\boldsymbol{x},\boldsymbol{u}) \in K \times \mathbb{R}^2,$$

and that f is super-solution to equation (6.32)*, that is*

$$0 \leq \partial_t f + \mathrm{div}_{\boldsymbol{u}} \left[\mathbf{b}^{\varepsilon} f \right] - \partial_v^2 f - \frac{1}{\varepsilon} \rho_0^{\varepsilon} \partial_v \left[(v - \mathcal{V}^{\varepsilon}) f \right],$$

for all $(t, \boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$*. Then it holds*

$$f^{\varepsilon}(t, \boldsymbol{x}, \boldsymbol{u}) \leq f(t, \boldsymbol{x}, \boldsymbol{u}), \quad \forall (t, \boldsymbol{x}) \in \mathbb{R}^+ \times K, a.e. \text{ in } \boldsymbol{u} \in \mathbb{R}^2.$$

Furthermore, the latter statement is also true if we replace "super-solution" by "sub-solution" and the symbol "$\geq$" by "$\leq$".

Proof. Since the proof of the comparison principle for super and sub-solutions is the same, we only detail it for super-solutions. According to Proposition 6.8, f^{ε} is a classical solution to (6.4), hence it holds

$$\partial_t (f^{\varepsilon} - f) + \mathrm{div}_{\boldsymbol{u}} \left[\mathbf{b}^{\varepsilon} (f^{\varepsilon} - f) \right] - \partial_v^2 (f^{\varepsilon} - f) - \frac{1}{\varepsilon} \rho_0^{\varepsilon} \partial_v \left[(v - \mathcal{V}^{\varepsilon})(f^{\varepsilon} - f) \right] \leq 0$$

$a.e.$ with respect to $(t, \boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$. Therefore, multiplying the latter relation by $\mathbb{1}_{f^{\varepsilon} \geq f}$, we deduce that in the weak sense, it holds

$$\partial_t (f^{\varepsilon} - f)_+ + \mathrm{div}_{\boldsymbol{u}} \left[\mathbf{b}^{\varepsilon} (f^{\varepsilon} - f)_+ \right] - \partial_v^2 (f^{\varepsilon} - f)_+ - \frac{1}{\varepsilon} \rho_0^{\varepsilon} \partial_v \left[(v - \mathcal{V}^{\varepsilon})(f^{\varepsilon} - f)_+ \right] \leq 0,$$

where $(\cdot)_+$ stands for the positive part and is defined by $(\cdot)_+ = (|\cdot| + id_{\mathbb{R}})/2$. In order to derive the latter relation, we follow a classical procedure which we do not detail here and which consists in regularizing the positive part $(\cdot)_+$. Then we integrate the latter relation with respect to time and $\boldsymbol{u}$ and obtain

$$\int_{\mathbb{R}^2} (f^{\varepsilon} - f)_+(t, \boldsymbol{x}, \boldsymbol{u}) \mathrm{d}\boldsymbol{u} \leq \int_{\mathbb{R}^2} (f^{\varepsilon} - f)_+(0, \boldsymbol{x}, \boldsymbol{u}) \mathrm{d}\boldsymbol{u}.$$

The latter computations are justified since $0 \leq (f^{\varepsilon} - f)_+ \leq f^{\varepsilon}$ and since, according to Theorem 6.5, f^{ε} has moments up to any order with respect to $\boldsymbol{u}$. Since $f^{\varepsilon} \leq f$ at time $t = 0$, we deduce the result. □

D.2 Regularity estimates

In this section, we derive regularity estimates for the solution f^{ε} to equation (6.4) and therefore prove Proposition 6.8. The main difficulty here consists in dealing with the contribution due to the non-linear drift N. We bypass this difficulty by estimating the norm of f^{ε} and its derivatives in the following weighted L^1 spaces

$$L^1(\omega_q) = \left\{ f : \mathbb{R}^2 \mapsto \mathbb{R}, \int_{\mathbb{R}^2} |f| \omega_q(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} < +\infty \right\},$$

where $\omega_q(\boldsymbol{u}) = 1 + |\boldsymbol{u}|^q$, for $q \geq 2$. The first step consists in estimating the norm of f^ε in $L^1(\omega_q)$. This step relies on previous results obtained in [24]. Then we adapt these computations to evaluate the norm of the derivatives of f^ε in $L^1(\omega_q)$. Indeed, the derivatives of f^ε solve equation (6.4) with additional source terms whose contribution can be controlled thanks to the confining properties of the non-linear drift N. Let us outline the strategy in the case of the first order derivatives : equation (6.4) on f^ε reads as follows

$$\partial_t f^\varepsilon = \mathscr{A}^\varepsilon f^\varepsilon,$$

where operator $\mathscr{A}^\varepsilon$ is given by

$$\mathscr{A}^\varepsilon f = \partial_v^2 f + \frac{1}{\varepsilon}\rho_0^\varepsilon \partial_v \left[(v - \mathcal{V}^\varepsilon) f\right] - \mathrm{div}_{\boldsymbol{u}} \left[\mathbf{b}^\varepsilon f\right].$$

Relying the arguments developed in the proof of [24, Proposition 3.1] it holds

Lemma D.3. *Consider some fixed ε and some $q \geq 2$. Under the assumptions of Theorem 6.5, consider the operator $\mathscr{A}^\varepsilon$ associated to the solution f^ε to equation* (6.4)*. There exists a positive constant C such that for all $\boldsymbol{x} \in K$, it holds*

$$\int_{\mathbb{R}^2} \mathrm{sgn}\,(f) \mathscr{A}^\varepsilon\,(f)\,\omega_q(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} \leq C \|f\|_{L^1(\omega_q)} + q \int_{\mathbb{R}^2} \mathbb{1}_{|v|\geq 1} \frac{N(v)}{v} |v|^q f(\boldsymbol{u}) \mathrm{d}\boldsymbol{u},$$

for all function f lying in $W^{2,1}(\omega_{p+q})$.

As a direct consequence of Lemma D.3, we deduce that for any smooth initial data f_0^ε, the solution f^ε to equation (6.4) provided by Theorem 6.5 lies in $L^\infty_{loc}\left(\mathbb{R}^+ \times K, L^1\left(\omega_q\right)\right)$ for all exponent $q \geq 2$. Indeed, since N meets the confining assumption (6.12) we have : $\mathbb{1}_{|v|\geq 1} N(v)/v \leq C$ for some constant C. Therefore, multiplying equation (6.4) by sgn $(f^\varepsilon)\omega_q(\boldsymbol{u})$, integrating with respect to $\boldsymbol{u}$ and applying Lemma D.3, we obtain that for all exponent q, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f^\varepsilon\|_{L^1(\omega_q)} \leq C \|f^\varepsilon\|_{L^1(\omega_q)}.$$

Hence, applying Gronwall's lemma and taking the supremum over all $\boldsymbol{x} \in K$, we deduce

$$\|f^\varepsilon(t)\|_{L^\infty(K, L^1(\omega_q))} \leq e^{Ct} \|f_0^\varepsilon\|_{L^\infty(K, L^1(\omega_q))}, \tag{D.1}$$

for all $t \in \mathbb{R}^+$. We follow the same strategy for derivatives of f^ε : writing $(g, h) = (\partial_v f^\varepsilon, \partial_w f^\varepsilon)$ and differentiating equation (6.4) with respect to v and w we obtain

$$\begin{cases} \partial_t g = \mathscr{A}^\varepsilon g - \left(N' - \Psi * \rho_0^\varepsilon - \dfrac{1}{\varepsilon}\rho_0^\varepsilon\right) g - N'' f^\varepsilon - a h, \\[2ex] \partial_t h = \mathscr{A}^\varepsilon h + g + b h. \end{cases} \tag{D.2}$$

Therefore, g and h solve the same equation as f^ε with additional an high order term due to the non-linear drift N. We control this additional term thanks to the confining properties of N.

Proof of Proposition 6.8. Let us consider some initial data f_0^ε lying in $\mathscr{C}^0\left(K, \mathscr{C}_c^\infty\left(\mathbb{R}^2\right)\right)$ and the associated solution f^ε to (6.4) provided by Theorem 6.5. We start by proving that f^ε lies in $L^\infty_{loc}\left(\mathbb{R}^+ \times K, W^{1,1}\left(\mathbb{R}^2\right)\right)$. We fix some $\boldsymbol{x} \in K$, some exponent $q \geq 2$ and integrate with respect to $\boldsymbol{u}$ the sum between the first equation in (D.2) multiplied by sgn $(g)\omega_q(\boldsymbol{u})$ and the second multiplied by sgn $(h)\omega_q(\boldsymbol{u})$. According to Lemma D.3, assumptions (6.15) on Ψ, (6.17) on ρ_0^ε and (6.13) on N, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|g\|_{L^1(\omega_q)} + \|h\|_{L^1(\omega_q)}\right) \leq & C\left(\|g\|_{L^1(\omega_q)} + \|h\|_{L^1(\omega_q)} + \|f^\varepsilon\|_{L^1\left(\omega_{p'}\right)}\right) \\ & + \int_{\mathbb{R}^2}\left(q\mathbb{1}_{|v|\geq 1}\frac{N(v)}{v} - N'(v)\right)|v|^q g(\boldsymbol{u})\mathrm{d}\boldsymbol{u},\end{aligned}$$

where p' is given in assumption (6.13). Since N meets assumptions (6.12) and (6.24), we deduce that for q great enough, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|g\|_{L^1(\omega_q)} + \|h\|_{L^1(\omega_q)}\right) \leq C\left(\|g\|_{L^1(\omega_q)} + \|h\|_{L^1(\omega_q)} + \|f^\varepsilon\|_{L^1\left(\omega_{p'}\right)}\right).$$

Therefore, applying Gronwall's lemma, taking the supremum over all $\boldsymbol{x} \in K$ and replacing $\|f^\varepsilon\|_{L^1\left(\omega_{p'}\right)}$ according to estimate (D.1), we deduce

$$\begin{aligned}&\|g(t)\|_{L^\infty(K,L^1(\omega_q))} + \|h(t)\|_{L^\infty(K,L^1(\omega_q))} \leq \\ &\qquad e^{Ct}\left(\|g_0\|_{L^\infty(K,L^1(\omega_q))} + \|h_0\|_{L^\infty(K,L^1(\omega_q))} + \|f_0^\varepsilon\|_{L^\infty(K,L^1(\omega_q))}\right),\end{aligned}$$

for all time $t \in \mathbb{R}^+$. As a straightforward consequence, we obtain the expected result : $f^\varepsilon \in L^\infty_{loc}\left(\mathbb{R}^+ \times K, W^{1,1}\left(\mathbb{R}^2\right)\right)$.

We obtain that $f^\varepsilon \in L^\infty_{loc}\left(\mathbb{R}^+ \times K, W^{2,1}\left(\mathbb{R}^2\right)\right)$ iterating the same procedure as before but this time on the derivatives of g and h and using assumption (6.24), which ensures that N''' has polynomial growth.

To end with, we obtain $\partial_t f^\varepsilon \in L^\infty_{loc}\left(\mathbb{R}^+ \times K, L^1\left(\mathbb{R}^2\right)\right)$ noticing that according to the definition of $\mathscr{A}^\varepsilon$, it holds

$$\|\mathscr{A}^\varepsilon f^\varepsilon\|_{L^\infty(K,L^1(\mathbb{R}^2))} \leq C\|f^\varepsilon\|_{L^\infty(K,W^{2,1}(\omega_q))},$$

for q great enough. Then we apply the previous estimates to the relation

$$\partial_t f^\varepsilon = \mathscr{A}^\varepsilon f^\varepsilon.$$

□

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