Concentration phenomena for a FitzHugh-Nagumo neural network

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Challenges in the Kinetic Modelling of Complex Systems - Part I of III

Introduction

Microscopic description

FitzHugh-Nagumo neural network of size n

For i between 1 and n :

$$\begin{cases} dv_t^i = \left(N(v_t^i) - w_t^i - \frac{1}{n} \sum_{j=1}^{n} \Phi(x_i, x_j)(v_t^i - v_t^j) \right) dt + \sqrt{2} dB_t^i, \\ dw_t^i = A(v_t^i, w_t^i) dt. \end{cases}$$

- v_t^i represents the tension of neuron i at time t.
- w_t^i is an adaptation variable added for modeling reasons.
- N (resp. A) is a non-linear drift (resp. an affinity) with confining property.
- Brownian motion B_t^i takes into account random fluctuation of the voltage.
- Neurons interact following Ohm's law. The conductance $\Phi(x_i, x_j)$ between neuron i and j depend on their spatial location x_i and x_j.

FitzHugh-Nagumo's mean-field equation

$$\partial_t f + \mathrm{div}_{(v,w)}\left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Phi(f) \\ A(v,w) \end{pmatrix} f\right] - \partial_v^2 f = 0,$$

- $f(t,x,v,w)$ is the probability of finding neurons at time $t \geq 0$ and position $x \in K$, with potential $v \in \mathbb{R}$ and adaptation variable $w \in \mathbb{R}$.
- $\mathcal{K}_\Phi(f)$ is the non-local term due to interactions between neurons

$$\mathcal{K}_\Phi(f)(x,v) = \int_{K\times\mathbb{R}^2} \Phi(x,x')(v - v')f(x',v',w')dx'dv'dw'.$$

We decompose the interaction kernel Φ as follows

$$\Phi(x,x') = \underbrace{\Psi(x,x')}_{\text{weak-long range interactions}} + \underbrace{\frac{1}{\epsilon}\delta_0(x - x')}_{\text{strong-short range interactions}}.$$

Regime of strong interactions

Weak-Long / Strong-Short decomposition

$$\Phi(x,x') = \Psi(x,x') + \frac{1}{\epsilon}\,\delta_0(x-x')\,.$$

The mean-field equation rewrites

$$\partial_t f^\epsilon + \mathrm{div}_{(v\,,\,w)}\left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v,w) \end{pmatrix} f^\epsilon\right] - \partial_v^2 f^\epsilon = \frac{\rho_0^\epsilon}{\epsilon}\,\partial_v\left[(v-\mathcal{V}^\epsilon)f^\epsilon\right],$$

where

$$\rho_0^\epsilon(x) = \int_{\mathbb{R}^2} f^\epsilon dvdw \quad \text{and} \quad \mathcal{V}^\epsilon(t,x) = \frac{1}{\rho_0^\epsilon}\int_{\mathbb{R}^2} vf^\epsilon dvdw.$$

Main goal

Analysis of the regime of Strong/Local interactions, that is when $\epsilon \to 0$.

Formal derivation

Strong interactions and concentration phenomenon

$$\partial_t f^\epsilon + \mathrm{div}_{(v\,,\,w)}\left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v,w) \end{pmatrix} f^\epsilon\right] - \partial_v^2 f^\epsilon \;=\; \frac{\rho_0^\epsilon}{\epsilon}\,\partial_v\left[(v-\mathcal{V}^\epsilon)f^\epsilon\right],$$

- Multiplying the equation by $|v-\mathcal{V}^\epsilon|^2$ and integrating yields

$$W_2^2\left(\frac{1}{\rho_0^\epsilon}\,f^\epsilon, \frac{1}{\rho_0^\epsilon}\,\delta_{\mathcal{V}^\epsilon}\otimes\int_{\mathbb{R}} f^\epsilon\,dv\right) \;=\; \int_{\mathbb{R}^2}|v-\mathcal{V}^\epsilon|^2\, f^\epsilon dvdw \underset{\epsilon\to 0}{=} O(\epsilon)\,,$$

where W_2 is the Wasserstein distance of order 2. In the end, we obtain [1]

$$f^\epsilon(t,x,v,w) \underset{\epsilon\to 0}{=} \delta_{\mathcal{V}(t,x)}(v)\otimes F(t,x,w) + O(\sqrt{\epsilon})\,,$$

where $(\mathcal{V}, F)$ satisfies

$$\begin{cases} \partial_t\mathcal{V} = N(\mathcal{V}) - \mathcal{W} - \left(\Psi *_r \rho_0(x)\mathcal{V} - \Psi *_r (\rho_0\mathcal{V})(x)\right), \\ \partial_t F + \partial_w\left(A(\mathcal{V},w)F\right) = 0, \\ \rho_0(x)\mathcal{W} = \displaystyle\int_{\mathbb{R}} wF\,dw. \end{cases} \tag{1}$$

1. Crevat, Faye, Filbet (19), Jabin, Rey (17), Kang, Vasseur (15)

Concentration profile

- To refine our description, we consider a re-scaled version g^ϵ of f^ϵ [2] :

$$f^\epsilon(t,x,v,w) = \frac{1}{\theta^\epsilon} g^\epsilon \left(t, x, \frac{v - \mathcal{V}^\epsilon}{\theta^\epsilon}, w - \mathcal{W}^\epsilon\right),$$

where $\theta^\epsilon > 0$ needs to be determined and

$$\mathcal{W}^\epsilon = \frac{1}{\rho_0^\epsilon} \int_{\mathbb{R}^2} w f^\epsilon dv dw \,.$$

Main goal

proving that g^ϵ converges and compute the limit.

2. Mouhot, Mischler (06), Rey (12)

Formal derivation

Suppose $\theta^\epsilon = \epsilon^\alpha$. Changing variables in the equation of f^ϵ it yields

$$\partial_t g^\epsilon + \mathrm{div}_{(v\,,\,w)}\left[\,\mathrm{b}_0^\epsilon\, g^\epsilon\,\right] = \frac{1}{\epsilon^{2\alpha}}\partial_v\left(\rho_0^\epsilon \epsilon^{2\alpha-1} v g^\epsilon + \partial_v g^\epsilon\right)\,,$$

Therefore $\theta^\epsilon = \sqrt{\epsilon}$ (*i.e.* $\alpha = 1/2$) is the only suitable choice

Equation on the profile

$$\partial_t g^\epsilon + \mathrm{div}_{(v\,,\,w)}\left[\,\mathrm{b}_0^\epsilon\, g^\epsilon\,\right] = \frac{1}{\epsilon}\partial_v\left[\,\rho_0^\epsilon\, v\, g^\epsilon + \partial_v g^\epsilon\,\right],$$

where b_0^ϵ depends on $1/\sqrt{\epsilon}$ and f^ϵ. Therefore, we expect

$$g^\epsilon(t,x,v,w) \underset{\epsilon\to 0}{=} \mathcal{M}_{\rho_0^\epsilon}(v) \otimes G^\epsilon(t,x,w) + O\left(\sqrt{\epsilon}\right).$$

Weak convergence result

Weak convergence result

Theorem (F. Filbet and A.B.[3])

Under suitable assumptions on f_0^ϵ, there exists $C > 0$ such that

$$\sup_{x\in K} W_2\left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0}\mathcal{M}_{\rho_{\mathbf{0}}/\epsilon}(\cdot - \mathcal{V}) \otimes F\right) \leq C\left(e^{Ct}\epsilon + e^{-\rho_{\mathbf{0}}^\epsilon\, t/\epsilon}\right),$$

for all $\epsilon > 0$ and $t \geq 0$.

Here, W_2 stands for the Wasserstein distance of order 2.

Key arguments of the proof :

- Uniform moment estimates.
- Analytic coupling method[4] in order to estimate the Wasserstein distance between g^ϵ and $\mathcal{M} \otimes G$ (G satisfies (1) after changing variables) .

3. Concentration phenomena in Fitzhugh-Nagumo's equations : A mesoscopic approach, arXiv :2201.02363
4. Fournier, Perthame (20).

Strong convergence results

Towards strong convergence : time dependent θ^ϵ

- $\theta^\epsilon = \sqrt{\epsilon}$ induces that at time $t = 0$, it holds

$$f_0^\epsilon(x, v, w) \, = \, \frac{1}{\sqrt{\epsilon}} \, g_0^\epsilon\left(x, \frac{v - \mathcal{V}_0^\epsilon}{\sqrt{\epsilon}}, w - \mathcal{W}_0^\epsilon\right) \, .$$

- Therefore, we impose $\theta^\epsilon(t = 0) \, = \, 1$. The only suitable choice is

$$\theta^\epsilon(t, x) \, = \, \sqrt{\epsilon} \left[1 \, + \, \underbrace{e^{-\,2\,\rho_0^\epsilon(x)\,t\,/\,\epsilon}\left(\epsilon^{-1} \, - \, 1\right)}_{\text{exponentially decaying remainder}} \right]^{\frac{1}{2}} \, .$$

The equation on g^ϵ rewrites

$$\partial_t g^\epsilon \, + \, \mathrm{div}_{(v\,,\,w)} \left[\mathrm{b}_0^\epsilon \, g^\epsilon \right] = \frac{1}{|\theta^\epsilon|^2} \partial_v \left[\rho_0^\epsilon \, v \, g^\epsilon + \partial_v g^\epsilon \right],$$

Strong convergence result

Theorem (A.B.[5])

Under suitable assumptions on f_0^ϵ, there exists $C > 0$ such that

$$\int_0^t \| f^\epsilon - f \|_{L^\infty_x L^1_{(v,w)}}(s)\, ds \leq C\, e^{C\,t} \sqrt{\epsilon}\,,$$

for all $\epsilon > 0$ and $t \geq 0$, where the limit f is given by

$$f(t,x,v,w) = \mathcal{M}_{\rho_0\, |\theta^\epsilon|^{-2}}(v - \mathcal{V}) \otimes F\,,$$

where $(\mathcal{V}, F)$ solves (1).

5. Large coupling in a FitzHug-Nagumo neural network : quantitative and strong convergence results (in preparation).

Key arguments

- Relative entropy estimate yields $g^\epsilon \underset{\epsilon\to 0}{\sim} \mathcal{M}_{\rho_0^\epsilon} \otimes G^\epsilon + O(\sqrt{\epsilon})$ in L^1 .
- Proving that G^ϵ converges towards G :

(*i*) We work on the re-scaled version H^ϵ

$$H^\epsilon = \int_{\mathbb{R}} g^\epsilon \left(t, x, v, w - \epsilon^{3/2} v \right) dv \,.$$

(*ii*) L^1-equicontinuity estimates for g^ϵ yield $H^\epsilon \underset{\epsilon\to 0}{=} G^\epsilon + O(\sqrt{\epsilon})$ in L^1.

(*iii*) Then we prove $H^\epsilon \underset{\epsilon\to 0}{=} G + O(\sqrt{\epsilon})$ in L^1.

H^ϵ converges towards G : simplified example

- We consider the diffusive scaling for Fokker-Planck equation

$$\partial_t g^\epsilon + \frac{1}{\epsilon} v \cdot \nabla_x g^\epsilon = \frac{1}{\epsilon^2} \mathrm{div}_v \left[v g^\epsilon + \nabla_v g^\epsilon \right] ,$$

and we prove

$$G^\epsilon = \int_{\mathbb{R}} g^\epsilon dv \underset{\epsilon \to 0}{\longrightarrow} G , \qquad \text{where} \qquad \partial_t G = \Delta_x G .$$

- Define

$$H^\epsilon(t,x) = \int_{\mathbb{R}} g^\epsilon(t, x + \epsilon v, v) dv .$$

- H^ϵ SOLVES the limiting equation

$$\partial_t H^\epsilon = \Delta_x H^\epsilon .$$

- Therefore, it is sufficient to prove $H^\epsilon \sim G^\epsilon$ (*i.e.* equicontinuity for g^ϵ).

Conclusion and perspectives

Conclusion :

• Weak convergence result with the (formal) optimal convergence rate.
• L^1 convergence result with deteriorated convergence rate.
• We also prove a convergence result in (inverse Gaussian) weighted L^2 spaces and recover the optimal rate by propagating regularity.

Perspective :

• Obtaining similar results (explicit convergence rates) following a Hamilton-Jacobi approach [6].

Thank you for your attention !

6. Quininao, Touboul (20)