Macroscopic limit of the Vlasov-Poisson-Fokker-Planck model

Alain Blaustein

February 7, 2023

Mathematical challenges in fluid dynamics and kinetic theory

The VPFP model

- We consider a **kinetic description** of electrons subjected to:

  (a) electric field ; (b) collisions with an ion background.

Vlasov-Fokker-Planck equation

$$\begin{cases} \epsilon\, \partial_t f^\epsilon + \underbrace{v \cdot \nabla_x f^\epsilon}_{\text{free transport}} + \underbrace{E^\epsilon \cdot \nabla_v f^\epsilon}_{\text{electric field}} = \frac{1}{\epsilon} \underbrace{\nabla_v \cdot [\, v\, f^\epsilon + \nabla_v f^\epsilon \,]}_{\text{collisions}}, \\ E^\epsilon = -\nabla_x \phi^\epsilon, \quad -\Delta_x \phi^\epsilon = \rho^\epsilon - \rho_i, \quad \rho^\epsilon = \int_{\mathbb{R}^d} f^\epsilon \, dv. \end{cases}$$

- $f^\epsilon(t,x,v)$: density of electrons at $(t,x,v) \in \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d$;
- $\rho_i(x)$: density of ions at position $x \in \mathbb{T}^d$.
- <u>GOAL</u>: analysis of the fluid regime $\epsilon \to 0$.

Formal argument: convergence towards local equilibrium

$$\epsilon\,\partial_t f^\epsilon + v\cdot\nabla_x f^\epsilon + E^\epsilon\cdot\nabla_v f^\epsilon = \frac{1}{\epsilon}\nabla_v\cdot\left[\,v\,f^\epsilon + \nabla_v f^\epsilon\,\right]$$

- Leading order in ϵ : we expect

$$\nabla_v\cdot\left[\,v\,f^\epsilon + \nabla_v f^\epsilon\,\right] \underset{\epsilon\to 0}{\sim} 0$$

- it holds

$$\nabla_v\cdot\left[\,v\,f^\epsilon + \nabla_v f^\epsilon\,\right] = \nabla_v\cdot\left(\mathcal{M}\,\nabla_v\left(\frac{f^\epsilon}{\mathcal{M}}\right)\right)\,,$$

where $\mathcal{M}$ is the standard Maxwellian distribution:

$$\mathcal{M}(v) = \frac{1}{\sqrt{2\,\pi}}\exp\left(-\frac{1}{2}|v|^2\right).$$

Therefore, we deduce

$$f^\epsilon(t,x,v) \underset{\epsilon\to 0}{\sim} \rho^\epsilon(t,x)\,\mathcal{M}(v) \text{ with } \rho^\epsilon(t,x) = \int_{\mathbb{R}^d} f^\epsilon(t,x,v)\mathrm{d}x\,.$$

- Next step : finding the **limit of** ρ^ϵ

Formal argument: convergence of ρ^ϵ

- Instead of $\rho^\epsilon(t,x) = \int_{\mathbb{R}^d} f^\epsilon(t,x,v)\mathrm{d}v$, we consider

$$\pi^\epsilon(t,x) = \int_{\mathbb{R}^d} f^\epsilon(t, x - \epsilon v, v)\,\mathrm{d}v\,,$$

which solves

$$\partial_t\,\pi^\epsilon \;-\; \nabla_x\cdot\left(\int_{\mathbb{R}^d} E^\epsilon\, f^\epsilon\,(t, x - \epsilon\, v, v)\;\mathrm{d}v \;+\; \nabla_x\,\pi^\epsilon\right) \;=\; 0\,.$$

- Therefore, we expect $f^\epsilon(t,x,v) \underset{\epsilon\to 0}{\longrightarrow} \rho(t,x)\,\mathcal{M}(v)$ with

Repulsive Keller-Segel equation

$$\begin{cases} \partial_t\,\rho + \nabla_x\cdot[\,\rho\,E - \nabla_x\,\rho\,] = 0\,, \\ E \;=\; -\nabla_x\,\phi\,, \quad -\Delta_x\,\phi \;=\; \rho - \rho_i\,. \end{cases}$$

State of art

Bibliography:

1) Compactness methods

- F. Poupaud, J. Soler; Math. Models Methods Appl. Sci.; 2000
- N. El Ghani, N. Masmoudi; IAENG Int. J. Appl. Math.; 2010
- M. Herda; J. Differential Equations; 2016

2) Perturbative methods

- F. Hérau, L. Thomann; J. Funct. Anal; 2016
- M. Herda, M. Rodrigues; J. Stat. Phys.; 2018
- L. Addala, J. Dolbeault, X. Li, M. L. Tayeb; J. Stat. Phys.; 2021

Goal:

Quantitative result in non-perturbative settings in any dimension d.

Main result

Theorem (22')

Supposing f_0^ϵ in weighted L^p space, it holds

$$\|f^\epsilon - \rho\,\mathcal{M}\|_{L^2} \lesssim \epsilon^\beta\,.$$

on bounded time intervals $[0, T^\epsilon]$, with (explicit) T^ϵ which verifies

$$T^\epsilon \underset{\epsilon \to 0}{\longrightarrow} +\infty$$

and where

$$\beta = \frac{p-d}{p-1}\,.$$

Main difficulty: appropriate functional framework

$$\epsilon\,\partial_t f^\epsilon + v\cdot\nabla_x f^\epsilon + E^\epsilon\cdot\nabla_v f^\epsilon = \frac{1}{\epsilon}\nabla_v\cdot\left[\,v\,f^\epsilon + \nabla_v f^\epsilon\,\right]$$

We should find a norm $|||\cdot|||$ such that:

(i) $|||\cdot|||$ is dissipated by the leading order in ϵ, that is

$$\epsilon\,\partial_t g = \frac{1}{\epsilon}\nabla_v\cdot\left[\,v\,g + \nabla_v g\,\right] \implies \epsilon^2\,\frac{\mathrm{d}}{\mathrm{d}t}|||g||| \leq 0\,,$$

$\rightarrow$ verified by : $\displaystyle\int_{\mathbb{R}^d}\varphi\left(\frac{g}{\mathcal{M}}\right)\mathcal{M}\,\mathrm{d}v$, for all convex φ

(ii) $|||\cdot|||$ controls E^ϵ : since $\nabla_x\cdot E^\epsilon = \rho^\epsilon$, for all $p > d$ we have

$$W^{1,p}\left(\mathbb{T}^d\right)\hookrightarrow L^\infty\left(\mathbb{T}^d\right) \implies \|E^\epsilon\|_{L^\infty(\mathbb{T}^d)} \leq \|\rho^\epsilon\|_{L^p(\mathbb{T}^d)}$$

$\rightarrow$ Poupaud & Soler(00') proposed $|||f^\epsilon|||_p^p = \displaystyle\int_{\mathbb{R}^d}\left|\frac{f^\epsilon}{\mathcal{M}}\right|^p\mathcal{M}\,\mathrm{d}v\,\mathrm{d}x$

Key step: estimate of the $|||\cdot|||_p$-norm

- Computations yield

$$\frac{\mathrm{d}}{\mathrm{d}\, t}|||f^\epsilon|||_p^p \;\; \leq \;\; C\; \|E^\epsilon\|^2_{L^\infty(\mathbb{T}^d)}\; |||f^\epsilon|||_p^p\,,$$

for C independent of ϵ. Poupaud & Soler deduce

$$\frac{\mathrm{d}}{\mathrm{d}\, t}|||f^\epsilon|||_p^p \;\; \leq \;\; C\, |||f^\epsilon|||_p^{p+2}\,,$$

which yields $|||f^\epsilon(t)|||_p < +\infty$ if $t \lesssim |||f_0^\epsilon|||_p^{-2}$ (blows up in finite time).

Key idea

- consider $\pi^\epsilon(t,x) = \int_{\mathbb{R}^d} f^\epsilon(t, x - \epsilon v, v)\,\mathrm{d}v$ and $I^\epsilon(t,x)$ defined by

$$I^\epsilon = -\nabla_x \psi^\epsilon\,, \quad -\Delta_x \psi^\epsilon = \pi^\epsilon - \rho_i\,,$$

and the following decomposition

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t} |||f^\epsilon|||_p^p &\leq C\, \|E^\epsilon\|^2_{L^\infty(\mathbb{T}^d)}\, |||f^\epsilon|||_p^p \\ &\leq C\left(\|E^\epsilon - I^\epsilon\|^2_{L^\infty(\mathbb{T}^d)} + \|I^\epsilon - E\|^2_{L^\infty(\mathbb{T}^d)} + \|E\|^2_{L^\infty(\mathbb{T}^d)}\right) |||f^\epsilon|||_p^p\,.\end{aligned}$$

- We focus on $\|E^\epsilon - I^\epsilon\|_{L^\infty(\mathbb{T}^d)}$. It holds

$$(E^\epsilon - I^\epsilon)(t,x) = \nabla_x\, \Delta_x^{-1} \int_{\mathbb{R}^d} f^\epsilon(t, x - \epsilon v, v) - f^\epsilon(t,x,v)\,\mathrm{d}v\,.$$

Therefore, for all $p > d$ it holds (with $\gamma = 1/(p-d)$)

$$W^{1,p}\left(\mathbb{T}^d\right) \hookrightarrow \mathcal{C}^\gamma\left(\mathbb{T}^d\right) \implies \|E^\epsilon - I^\epsilon\|_{L^\infty(\mathbb{T}^d)} \leq C\,\epsilon^\gamma\, |||f^\epsilon|||_p.$$

In the end we deduce : $\dfrac{\mathrm{d}}{\mathrm{d}\,t} |||f^\epsilon|||_p^p \leq C\,\epsilon^\gamma\, |||f^\epsilon|||_p^{p+2}$.

Perspectives

Some open problems:

• derive the VPFP model as the mean-field limit of a particle system uniformly in the fluid limit[1];

• quantitative long-time behavior of the non-linear model in non-perturbative setting;

• including collision operators closer to physics (ex: Landau[2])

---

[1] D. Bresch, P.-E. Jabin, Z. Wang (19)
[2] S. Chaturvedi, J. Luk, T. Nguyen