Concentration phenomena for a FitzHugh-Nagumo neural network

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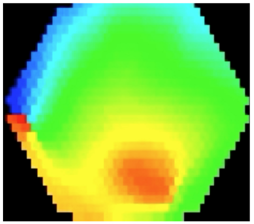
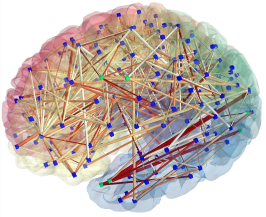
Overview

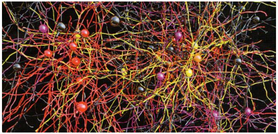
Models for neural networks generally focus on the dynamics of the electrical potential in neurons membrane throughout the network. We consider models describing 3 distinct scales :



Macroscopic scale : Rough description ; describes quantities we can measure.



Mesoscopic scale : Also called Mean-field model. Intermediate scale.



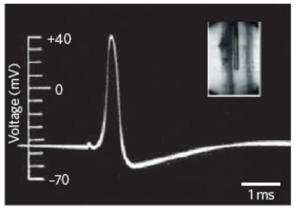
Microscopic scale : Exhaustive description of the system.

Contents

Physical model & motivations

Behavior of a neuron

• We focus on the dynamics of the voltage through the membrane of a neuron. Experiments showcase 2 main features



Hodgkin & Huxley, '52.

(*i*) Delay when responding to an external input.

(*ii*) Self-regulation.

• Hodgkin & Huxley obtained a precise but mathematically complicated model.

• We will use a simplified version that captures its main features : FitzHugh-Nagumo's model for a neuron

FitzHugh-Nagumo's neuron

FitzHugh-Nagumo's equation for one neuron

$$\begin{cases} dv_t = (N(v_t) - w_t + I_{\text{ext}})\, dt + \sqrt{2} dB_t, \\ dw_t = A\,(v_t, w_t)\, dt, \end{cases}$$

where $v_t \in \mathbb{R}$ stands for the potential of the membrane and $w_t \in \mathbb{R}$ is an adaptation variable which captures delay.

- A and N have confining properties to capture self-regulation

$$A(v, w) = a\, v - b\, w + c,$$

where $a, c \in \mathbb{R}$ and $b > 0$. N is non-linear, the canonical example is

$$N(v) = v - v^3.$$

- Brownian motion B_t captures uncertainty in our description.
- I_{ext} is the external input (artificial stimulation ; interaction with other neurons).

Microscopic scale

- When the neural network consists in n neurons, the i^{th} neuron receives current from other neurons :

$$I_{\text{ext}} = -\frac{\psi}{n}\sum_{j=1}^{n}(v^i - v^j), \tag{1}$$

where coefficient $\psi \in \mathbb{R}_+$ describes the strength of interactions between neurons ($\psi \geq 0 \rightarrow$ attractive behavior).

- We obtain the following microscopic model, where $1 \leq i \leq n$

$$\begin{cases} dv_t^i = \left(N(v_t^i) - w_t^i - \frac{\psi}{n}\sum_{j=1}^{n}(v_t^i - v_t^j)\right) dt + \sqrt{2} dB_t^i, \\ dw_t^i = A(v_t^i, w_t^i) dt. \end{cases} \tag{2}$$

Mean field limit

It was proved that in the mean field limit $n \to +\infty$ the microscopic system is described by

FitzHugh-Nagumo's mean field equation

$$\begin{cases} \partial_t f = -\partial_v \left(\left(N(v) - w - \psi(v - \mathcal{V}) \right) f \right) - \partial_w \left(A(v,w) f \right) + \partial_v^2 f, \\ f(0,\cdot) = f_0, \end{cases}$$

where $f(t,v,w)$ is the probability of finding neurons with a potential $v \in \mathbb{R}$ and an adaptation variable $w \in \mathbb{R}$ at time $t \geq 0$ within the network, and the macroscopic quantities $\mathcal{V}$ and $\mathcal{W}$ are given by :

$$(\mathcal{V}(t), \mathcal{W}(t)) = \int_{\mathbb{R}^2} (v', w') f(t, v', w') dv' dw'.$$

Mean field limit

References

- J. Baladron, D. Fasoli, O. Faugeras and J. Touboul. *Mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons* (2012).
- M. Bossy, O. Faugeras and D. Talay. *Clarification and complement to "mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons"* (2015).
- S. Mischler, C. Quiñinao and J. Touboul. *On a kinetic FitzHugh-Nagumo model of neuronal network* (2015).
- E. Luçon and W. Stannat. *Mean-field limit for disordered diffusions with singular interactions* (2014).

Strong interactions & concentration phenomenon

- Let us now focus on the regime of strong interactions, that is when $\psi \to +\infty$. We set $\psi = \frac{1}{\epsilon}$ and get

$$\partial_t f^\epsilon = -\partial_v \left(\left(N(v) - w - \frac{1}{\epsilon}(v - \mathcal{V}^\epsilon) \right) f^\epsilon \right) - \partial_w \left(A(v,w) f^\epsilon \right) + \partial_v^2 f^\epsilon.$$

- Multiplying the equation by $|v - \mathcal{V}^\epsilon|^2$ and integrating yields

$$\int_{\mathbb{R}^2} \left| v - \mathcal{V}^\epsilon(t) \right|^2 f^\epsilon dv dw \underset{\epsilon \to 0}{=} O(\epsilon). \tag{3}$$

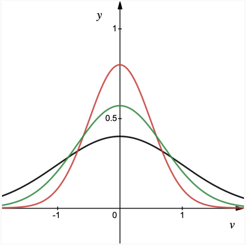
Hence, f^ϵ is expected to concentrate around $\mathcal{V}^\epsilon$

$$f^\epsilon(t,v,w) \underset{\epsilon \to 0}{\sim} \delta_{\mathcal{V}^\epsilon(t)}(v) \otimes F^\epsilon(t,w), \tag{4}$$

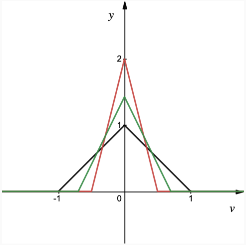
where $F^\epsilon(t,w) = \int_{\mathbb{R}} f^\epsilon(t,v,w) dv$.

Concentration's profile

Our goal is to determine the profile of this concentration.



Concentration with Gaussian profile



Concentration with triangular profile

Here are plots of

$$y = \frac{1}{\theta^\epsilon} g\left(\frac{v}{\theta^\epsilon}\right),$$

for $\theta^\epsilon = 1; 0.7; 0.5$ and g a gaussian profile (fig. 1) and triangular profile (fig. 2).

Goal of the presentation

- Consider the following re-scaled version g^ϵ of f^ϵ :

$$f^\epsilon(t,v,w) = \frac{1}{\theta^\epsilon(t)} g^\epsilon \left(t, \frac{v - \mathcal{V}^\epsilon(t)}{\theta^\epsilon(t)}, w - \mathcal{W}^\epsilon(t) \right), \tag{5}$$

where θ^ϵ is the concentration rate of f^ϵ around its mean value $\mathcal{V}^\epsilon$. We expect

$$\theta^\epsilon(t) \underset{\epsilon \to 0}{\longrightarrow} 0.$$

- GOAL : proving that g^ϵ converges and compute the limit.

Formal derivation of the concentration's profile

We obtain the equation on g^ϵ changing variables in the equation on f^ϵ

$$\begin{aligned}\partial_t g^\epsilon = & -\frac{1}{\theta^\epsilon}\partial_u \left[\left(N(\mathcal{V}^\epsilon + \theta^\epsilon u) - N(\mathcal{V}^\epsilon) - \mathcal{E}(f^\epsilon(t,\cdot)) - x\right) g^\epsilon\right] \\ & -\partial_x \left[A_0\left(\theta^\epsilon u, x\right) g^\epsilon\right] + \frac{1}{(\theta^\epsilon)^2}\partial_u \left(\alpha^\epsilon u g^\epsilon + \partial_u g^\epsilon\right),\end{aligned}$$

where $\alpha^\epsilon = (\theta^\epsilon)'\theta^\epsilon + \frac{(\theta^\epsilon)^2}{\epsilon}$, $A_0 = A - c$ and $\mathcal{E}$ is an error term.

- It is natural to take $\theta^\epsilon(t) = \theta^\epsilon = \sqrt{\epsilon}$. Indeed, we obtain

$$\begin{aligned}\partial_t g^\epsilon = & -\frac{1}{\sqrt{\epsilon}}\partial_u \left[\left(N(\mathcal{V}^\epsilon + \sqrt{\epsilon}u) - N(\mathcal{V}^\epsilon) - \mathcal{E}(f^\epsilon(t,\cdot)) - x\right) g^\epsilon\right] \\ & -\partial_x \left[A_0\left(\sqrt{\epsilon}u, x\right) g^\epsilon\right] + \frac{1}{\epsilon}\partial_u \left(u g^\epsilon + \partial_u g^\epsilon\right).\end{aligned} \tag{6}$$

Formal derivation

- Considering the stiffer term in the former equation, we expect

$$g^\epsilon(t,u,x) \underset{\epsilon\to 0}{\sim} M(u) \otimes G^\epsilon(t,x),$$

where $M(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$ and $G^\epsilon(t,x) = \int_{\mathbb{R}} g^\epsilon(t,u,x)\,du$.

- Furthermore, since G^ϵ solves

$$\partial_t G^\epsilon + a\sqrt{\epsilon}\partial_x \left(\int_{\mathbb{R}} u g^\epsilon du\right) - b\partial_x (xG^\epsilon) = 0, \qquad (7)$$

it is expected that

$$g^\epsilon(t,u,x) \underset{\epsilon\to 0}{\longrightarrow} M(u) \otimes G(t,x),$$

where G solves

$$\partial_t G - b\partial_x (xG) = 0.$$

Weak convergence

Setting

- In this section, we will prove the following weak convergence result

$$g^{\epsilon}(t,u,x) \underset{\epsilon\to 0}{\rightharpoonup} M(u)\otimes G(t,x),$$

in $\mathcal{P}^2(\mathbb{R}^2)$, the set of probability distributions with moments up to order 2

$$\mathcal{P}^2\left(\mathbb{R}^2\right) = \left\{ g\in\mathcal{P}(\mathbb{R}^2) | \int_{\mathbb{R}^2} \left(|u|^2+|x|^2\right) g(u,x)dudx \right\}.$$

- Weak convergence in $\mathcal{P}^2(\mathbb{R}^2)$ is induced by the Wasserstein metric W_2

$$W_2^2\left(g,h\right) = \inf_{\pi\in\Pi(g,h)} \int_{\mathbb{R}^4} |u-\nu|^2 + |x-\chi|^2 \pi(u,x,\nu,\chi) dudxd\nu d\chi,$$

where $\Pi(g,h)$ stands for the set of all couplings between g and h.

Weak convergence result

Théorème

g^ϵ converges to $M \otimes G$ in $\mathcal{C}^0\left(\mathbb{R}_+^*, \left(\mathcal{P}^2(\mathbb{R}^2), W_2\right)\right)$. Furthermore, for all $(\delta_u, \delta_x) \in]0,1[\times]0,b[$, there exists $C > 0$ such that

$$W_2\left(g^\epsilon, M \otimes G\right) \le W_2\left(G_0^\epsilon, G_0\right) e^{-\delta_x t} + \frac{C}{\sqrt{\epsilon}} e^{-\frac{\delta_u}{\epsilon} t} + C\sqrt{\epsilon},$$

for all $\epsilon > 0$ and $t \ge 0$.

- The following confining property is required

$$\limsup_{|v| \to +\infty} \sup_{\mathcal{V} \in K} \frac{N(\mathcal{V} + v) - N(\mathcal{V})}{v} = -\infty. \tag{8}$$

- Recall that coefficient b is the confining part of A

$$A(v, w) = av - bw + c.$$

- Standard assumptions are required (moments on f_0^ϵ ; polynomial growth and local Lipschitz regularity for N).

Key arguments

- Uniform estimates (in time and ϵ) on f^ϵ's moments using confining properties of A and N (uniformity in time) and concentrating properties of the stiffer term $-\frac{1}{\epsilon}(v - \mathcal{V}^\epsilon)$ (uniformity in ϵ).

- Regularization effects for g^ϵ's moments with respect to the re-scaled voltage variable u using concentration properties of the stiffer term $\frac{1}{\epsilon}\partial_u \left(ug^\epsilon + \partial_u g^\epsilon\right)$.

- Coupling method in order to estimate the Wasserstein distance between g^ϵ and $M \otimes G$

Strong convergence

Additional difficulties

- For now, we only achieved weak convergence. In this section, we will obtain a strong convergence result.
- In the strong convergence result, we will require uniform control over g_0^ϵ. Together with our homogeneous in time concentration rate $\theta^\epsilon = \sqrt{\epsilon}$, this yields

$$f_0^\epsilon(v,w) = \frac{1}{\sqrt{\epsilon}} g_0^\epsilon\left(\frac{v - \mathcal{V}_0^\epsilon}{\sqrt{\epsilon}}, w - \mathcal{W}_0^\epsilon\right). \tag{9}$$

- Consequently, if we suppose for instance the following control

$$\sup_{\epsilon>0} \int_{\mathbb{R}^2} |u|^2 g^\epsilon du dx < +\infty,$$

then we implicitly consider well prepared initial data f_0^ϵ :

$$W_2^2(f_0^\epsilon, \delta_{\mathcal{V}_0^\epsilon} \otimes F_0^\epsilon) \underset{\epsilon\to 0}{=} O(\epsilon).$$

Scaling for strong convergence

Since we do not want to prepare the initial data f_0^ϵ, we suppose

$$\theta_0^\epsilon = \theta_0 > 0.$$

Another constraint on θ^ϵ arises from the equation on g^ϵ

$$\begin{aligned} \partial_t g^\epsilon = \quad & -\frac{1}{\theta^\epsilon}\partial_u\left[\left(N(\mathcal{V}^\epsilon + \theta^\epsilon u) - N(\mathcal{V}^\epsilon) - \mathcal{E}(f^\epsilon(t,\cdot)) - x\right) g^\epsilon\right] \\ & -\partial_x\left[A_0\left(\theta^\epsilon u, x\right) g^\epsilon\right] \\ & +\frac{1}{(\theta^\epsilon)^2}\partial_u\left(\left((\theta^\epsilon)'\theta^\epsilon + \frac{(\theta^\epsilon)^2}{\epsilon}\right) u g^\epsilon + \partial_u g^\epsilon\right). \end{aligned}$$

Using the weak convergence result, we know that the concentration profile is Gaussian. Hence, it is natural to impose

$$(\theta^\epsilon)'\theta^\epsilon + \frac{(\theta^\epsilon)^2}{\epsilon} = 1.$$

Equation on g^ϵ

- Hence, we obtain the following equation

$$\begin{aligned}\partial_t g^\epsilon = \quad & -\frac{1}{\theta^\epsilon}\partial_u \left[\left(N(\mathcal{V}^\epsilon + \theta^\epsilon u) - N(\mathcal{V}^\epsilon) - \mathcal{E}(f^\epsilon(t,\cdot)) - x \right) g^\epsilon \right] \\ & -\partial_x \left[A_0 \left(\theta^\epsilon u, x \right) g^\epsilon \right] + \frac{1}{(\theta^\epsilon)^2}\partial_u \left(u g^\epsilon + \partial_u g^\epsilon \right).\end{aligned} \tag{10}$$

- The convergence rate is given by

$$\theta^\epsilon(t) = \sqrt{\theta_0^2 e^{-\frac{2t}{\epsilon}} + \epsilon \left(1 - e^{-\frac{2t}{\epsilon}} \right)}. \tag{11}$$

- We recover the case of well-prepared initial data taking $\theta_0 = \sqrt{\epsilon}$.

Théorème

g^ϵ converges to MG in $L^\infty_{loc}\left(\mathbb{R}^+_*, L^2\left(m_{1,\kappa}\right)\right)$ for some $\kappa > 0$ great enough. Furthermore, for all $\delta \in]0,1[$, there exists $C > 0$ such that the following estimate holds true :

$$\|g^\epsilon - MG\|^2_{L^2(m_{1,\kappa})} \leq e^{Ct}\|G_0^\epsilon - G_0\|^2_{L^2\left(M_\kappa^{-1}\right)}$$

$$+ Ce^{Ct}\left(\epsilon + e^{-\frac{2\delta t}{\epsilon}}\epsilon^{-\delta}\right)\left(\|(1+x)g_0^\epsilon\|^2_{L^2(m_{1,\kappa})} + \|\partial_x g_0^\epsilon\|^2_{L^2(m_{1,\kappa})}\right),$$

for all $\epsilon > 0$ and $t \geq 0$.

- The weight $m_{1,\kappa}$ is given by

$$m_{1,\kappa}(u,x) = M^{-1}(u)M_\kappa^{-1}(x),$$

where

$$M_\kappa(u) = \sqrt{\frac{\kappa}{2\pi}}\exp\left(-\frac{\kappa}{2}u^2\right).$$

Key arguments : a priori estimates

• Using confining properties of N and A and "good" contribution of the Fokker-Planck dissipation D_{FP}, we obtain for $0 < \delta < 1$

$$\frac{1}{2}\frac{d}{dt}\|g^\epsilon\|^2_{L^2(m_{1,\kappa})} + \frac{\delta}{(\theta^\epsilon)^2} D_{FP}(g^\epsilon) \leq C\|g^\epsilon\|^2_{L^2(m_{1,\kappa})}. \qquad (12)$$

• Similar results stand for $\partial_x g^\epsilon$ and $x g^\epsilon$ since they solve equation similar to g^ϵ.

• Hence, we obtain uniform bounds on

$$\left|\frac{d}{dt}\|G^\epsilon\|^2_{L^2(M_\kappa^{-1})}\right|,$$

using the former estimates.

Key Argument : convergence

- We use the following relation

$$\|g^\epsilon - MG\|^2_{L^2(m_{1,\kappa})} = \|g^\epsilon - MG^\epsilon\|^2_{L^2(m_{1,\kappa})} + \|G^\epsilon - G\|^2_{L^2(M_\kappa^{-1})}. \quad (13)$$

- We estimate $\|G^\epsilon - G\|^2_{L^2(M_\kappa^{-1})}$ using that $G^\epsilon - G$ solves a transport equation with source term controlled by $\partial_x g^\epsilon$.
- We estimate $\|g^\epsilon - MG^\epsilon\|^2_{L^2(m_{1,\kappa})}$ using that

$$\|g^\epsilon - MG^\epsilon\|^2_{L^2(m_{1,\kappa})} = \|g^\epsilon\|^2_{L^2(m_{1,\kappa})} - \|G^\epsilon\|^2_{L^2(M_\kappa^{-1})}, \quad (14)$$

coupled with (12) and Gauss-Poincare's inequality

$$\|g^\epsilon - MG^\epsilon\|^2_{L^2(m_{1,\kappa})} \leq D_{FP}(g^\epsilon).$$

Current work

- Obtaining similar results following a Hamilton-Jacobi method.
- Adding a space variable to the model.
- Numerical analysis of concentration phenomena.

Thanks

Thank you for your attention !