

A-P scheme for the Vlasov-Poisson-Fokker-Planck equation

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The question at hand

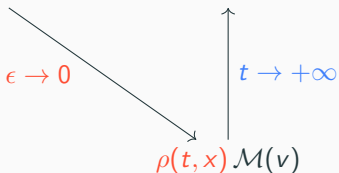
- Consider the **Vlasov-Poisson-Fokker-Planck** model

$$\left\{ \begin{array}{l} \epsilon \partial_t f^\epsilon + v \partial_x f^\epsilon - \underbrace{\partial_x \phi^\epsilon \partial_v f^\epsilon}_{\text{field interactions}} = \frac{1}{\epsilon} \underbrace{\partial_v [v f^\epsilon + \partial_v f^\epsilon]}_{\text{collisions}}, \\ -\partial_x^2 \phi^\epsilon = \rho^\epsilon - \rho_i, \quad \rho^\epsilon = \int_{\mathbb{R}} f^\epsilon dv. \end{array} \right.$$

- $f^\epsilon(t, x, v)$: density of particles at time t , position $x \in \mathbb{T}$, velocity $v \in \mathbb{R}$.

Possible dynamics:

$$f^\epsilon(t, x, v) \xrightarrow{t \rightarrow +\infty} \rho_\infty(x) \mathcal{M}(v)$$



- Gaussian** velocity distribution:

$$\mathcal{M}(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}|v|^2\right),$$

- $\rho(t, x)$ solves

$$\left\{ \begin{array}{l} \partial_t \rho = \nabla_x \cdot [\rho \nabla_x \phi + \nabla_x \rho], \\ -\Delta_x \phi = \rho - \rho_i. \end{array} \right.$$

Linear setting

- Case of a given electric field $\partial_x \phi$ and $(x, v) \in \mathbb{T} \times \mathbb{R}$

$$\epsilon \partial_t f^\epsilon + v \partial_x f^\epsilon - \partial_x \phi \partial_v f^\epsilon = \frac{1}{\epsilon} \partial_v [v f^\epsilon + \partial_v f^\epsilon].$$

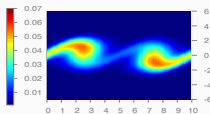
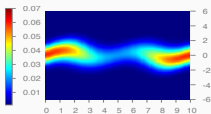
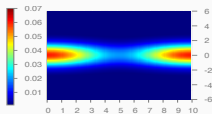
and $f^\epsilon(t, x, v) \rightarrow \rho(t, x) \mathcal{M}(v)$ as $\epsilon \rightarrow 0$ with

$$\partial_t \rho = \partial_x [\rho \partial_x \phi + \partial_x \rho].$$

We design discrete approximations (f_n^h, ρ_n^h) of (f^ϵ, ρ) such that

Theorem (with F. Filbet, 22')

$$\|f_n^h - \rho_n^h \mathcal{M}\| \lesssim \epsilon (1 + \kappa \Delta t)^{-\frac{n}{2}} + \left(1 + \frac{\Delta t}{2\epsilon^2}\right)^{-\frac{n}{2}}. \quad (1)$$



From key estimate to functional space

Dissipation of the L^2 -norm

$$\frac{d}{dt} \int \left| \frac{f^\epsilon}{\sqrt{\rho_\infty} \mathcal{M}} - \sqrt{\rho_\infty} \right|^2 dx \mathcal{M} dv = -\frac{2}{\epsilon^2} \int \left| \partial_v \left(\frac{f^\epsilon}{\sqrt{\rho_\infty} \mathcal{M}} \right) \right|^2 dx \mathcal{M} dv$$

→ Functional space :

$$\frac{f^\epsilon}{\sqrt{\rho_\infty} \mathcal{M}} \in L^2(dx \mathcal{M}(v) dv) .$$

- Spectral decomp. in Hermite basis $(H_k)_{k \in \mathbb{N}}$ of $L^2(\mathcal{M} dv)$

$$\frac{f^\epsilon}{\sqrt{\rho_\infty} \mathcal{M}}(t, x, v) = \sum_{k \in \mathbb{N}} D_k^\epsilon(t, x) H_k(v) .$$

- **No weight** with respect to dx so

$$D_k^\epsilon \in L^2(dx) .$$

Hermite decomposition

Vlasov-Fokker-Planck equation on $D^\epsilon = (D_k^\epsilon)_{k \in \mathbb{N}}$

$$\epsilon \partial_t D_k^\epsilon + \sqrt{k} \mathcal{A} D_{k-1}^\epsilon - \sqrt{k+1} \mathcal{A}^* D_{k+1}^\epsilon = -\frac{k}{\epsilon} D_k^\epsilon, \quad \forall k \in \mathbb{N},$$

with $\mathcal{A}u = \partial_x u + \frac{\partial_x \phi}{2} u$.

- Equilibrium is $D_{\infty,k} = \sqrt{\rho_\infty} \delta_{k=0}$.

Dissipation of the L^2 -norm in Hermite basis

$$\frac{d}{dt} \|D^\epsilon - D_\infty\|_{L^2}^2 = -\frac{2}{\epsilon^2} \sum_{k \in \mathbb{N}^*} k \|D_k^\epsilon\|_{L^2}^2.$$

Fully discrete scheme

$$\frac{D_k^{n+1} - D_k^n}{\Delta t} + \frac{1}{\epsilon} \left(\sqrt{k} \mathcal{A}_h D_{k-1}^{n+1} - \sqrt{k+1} \mathcal{A}_h^* D_{k+1}^{n+1} \right) = -\frac{k}{\epsilon^2} D_k^{n+1},$$

for all $k \in \mathbb{N}$ where discrete operators \mathcal{A}_h and \mathcal{A}_h^* verify

Properties	Preservation
$\langle \mathcal{A}_h u, v \rangle_{L^2} = \langle u, \mathcal{A}_h^* v \rangle_{L^2}$	duality structure
$\mathcal{A}_h \sqrt{\rho_\infty} = 0$	equilibrium state
$\sum_j \Delta x_j (\mathcal{A}_h^* u)_j \sqrt{\rho_{\infty,j}} = 0$	invariants
$\ u\ _{L^2} \leq C_d \ \mathcal{A}_h u\ _{L^2}$	macroscopic coercivity

for all $(u_j)_{j \in \mathcal{J}}, (v_j)_{j \in \mathcal{J}}$

We go back to the L^2 estimate

$$\frac{\|D^{n+1} - D_\infty\|_{L^2}^2 - \|D^n - D_\infty\|_{L^2}^2}{\Delta t} \leq -\frac{2}{\epsilon^2} \|D_\perp^{n+1}\|_{L^2}^2,$$

with $D_\perp^{n+1} = (0, D_1^{n+1}, D_2^{n+1}, D_3^{n+1}, \dots)$.

 Lack of coercivity¹

$$\boxed{\|D^{n+1} - D_\infty\|_{L^2}^2 \not\leq \|D_\perp^{n+1}\|_{L^2}^2}$$

¹Villani (2009)

Illuminating example

Consider

$$\begin{cases} \epsilon \frac{d}{dt} x^\epsilon = -y^\epsilon \\ \epsilon \frac{d}{dt} y^\epsilon = x^\epsilon - \frac{1}{\epsilon} y^\epsilon \end{cases} .$$

- Relative entropy estimate:

$$\frac{d}{dt} \left(|x^\epsilon(t)|^2 + |y^\epsilon(t)|^2 \right) = -\frac{2}{\epsilon^2} |y^\epsilon(t)|^2 .$$

- Modified entropy: $\mathcal{H}(f^\epsilon) = |x^\epsilon|^2 + |y^\epsilon|^2 - \alpha \epsilon x^\epsilon y^\epsilon$

$$\frac{d}{dt} \mathcal{H}(f^\epsilon) = -\frac{2}{\epsilon^2} |y^\epsilon(t)|^2 + \alpha \left(|y^\epsilon(t)|^2 - |x^\epsilon(t)|^2 + \frac{1}{\epsilon} x^\epsilon(t) y^\epsilon(t) \right) .$$

We deduce $\frac{d}{dt} \mathcal{H}(f^\epsilon) = -\kappa \mathcal{H}(f^\epsilon)$ and $|x^\epsilon(t)|^2 + |y^\epsilon(t)|^2 \lesssim e^{-\kappa t}$.

Discrete hypocoercivity

Define a modified entropy functional

$$\mathcal{H}_0^n = \|D^n - D_\infty\|_{L^2}^2 + \alpha \epsilon \langle D_1^n, \mathcal{A}_h u_h^n \rangle .$$

where u_h^n solves the elliptic problem

$$\begin{cases} (\mathcal{A}_h^* \mathcal{A}_h) u_h^n = D_0^n - D_{\infty,0} , \\ \sum_{j \in \mathcal{J}} \Delta x_j u_j \sqrt{\rho_{\infty,j}} = 0 , \end{cases}$$

Macroscopic coercivity \rightarrow we recover

$$\begin{cases} \|D^n - D_\infty\|_{L^2}^2 \lesssim \mathcal{H}_0^n \lesssim \|D^n - D_\infty\|_{L^2}^2 , \\ \frac{\mathcal{H}_0^{n+1} - \mathcal{H}_0^n}{\Delta t} \lesssim -\frac{2}{\epsilon^2} (1 - \alpha) \|D_\perp^{n+1}\|_{L^2}^2 - \alpha \|D_0^{n+1} - D_{\infty,0}\|_{L^2}^2 . \end{cases}$$

Numerical experiments

We take $\Delta t = 10^{-3}$, 200 Hermite modes, 64 points in space and

$$\phi(x) = 0.1 \cos(2\pi x) + 0.9 \cos(4\pi x) .$$

First Test: $\epsilon = 1$ and

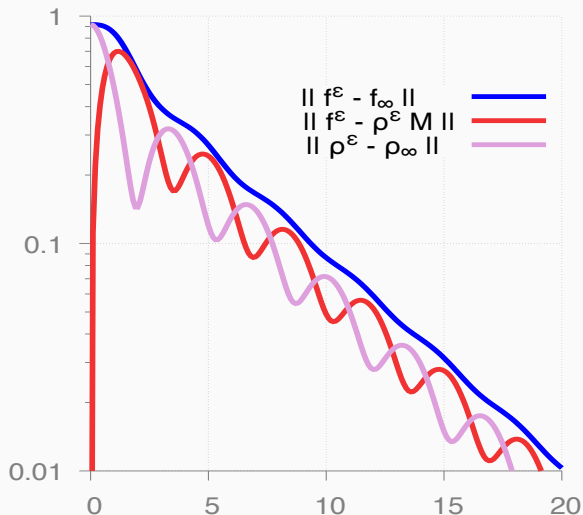
$$f_0(x, v) = (1 + 0.5 \cos(2\pi x)) \exp(-|v|^2/2) / \sqrt{2\pi} ,$$

Second Test: $\epsilon = 10^{-4}$ and

$$f_0(x, v) = (1 + 0.5 \cos(2\pi x)) \exp(-|v - 1|^2/2) / \sqrt{2\pi} .$$

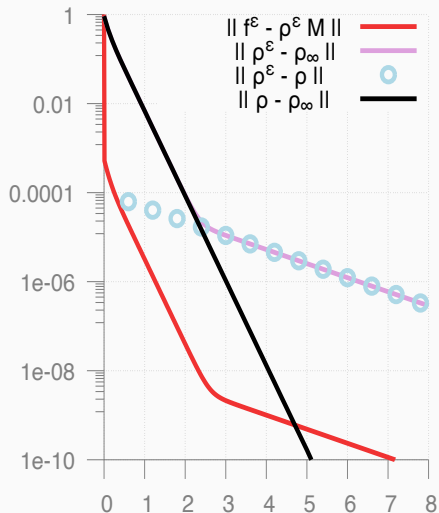
First Test, $\epsilon = 1$

Time evolution (log-scale): $\|f^\epsilon - f_\infty\|_{L^2(f_\infty^{-1})}$ (blue), $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink)



Second Test: $\epsilon = 10^{-4}$

Time evolution (log scale) of $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink), $\|\rho^\epsilon - \rho\|_{L^2(\rho_\infty^{-1})}$ (blue points) and $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (black)



We have proven that

$$\|D_\perp^n\| \leq \|D_\perp^0\| \left(1 + \frac{\Delta t}{2\epsilon^2}\right)^{-\frac{n}{2}} + \epsilon C \|D^0 - D_\infty\| (1 + \kappa \Delta t)^{-\frac{n}{2}},$$

and

$$\|D_0^n - \bar{D}_0^n\| \leq C\epsilon \|D^0 - D_\infty\| (1 + \kappa \Delta t)^{-\frac{n}{2}},$$

and

$$\|\bar{D}^n - D_\infty\| \leq \|\bar{D}^0 - D_\infty\| (1 + \tilde{\kappa} \Delta t)^{-\frac{n}{2}},$$

Linearized equation

Linear setting

- Consider the linearized equation

$$\begin{cases} \epsilon \partial_t f^\epsilon + v \partial_x f^\epsilon - \partial_x \phi_\infty \partial_v f^\epsilon - \partial_x \phi^\epsilon \partial_v f_\infty = \partial_v [v f^\epsilon + \partial_v f^\epsilon], \\ -\partial_x^2 \phi^\epsilon = \rho^\epsilon - \rho_\infty, \quad \rho^\epsilon = \int_{\mathbb{R}^d} f^\epsilon dv, \end{cases}$$

with $f_\infty(x, v) = \rho_\infty(x) \mathcal{M}(v)$ and

$$\begin{cases} \rho_\infty = e^{-\phi_\infty}, \\ -\partial_x^2 \phi_\infty = \rho_\infty - \rho_i. \end{cases}$$

We design discrete approximations (f_n^h, ρ_∞^h) of $(f^\epsilon, \rho_\infty)$ such that

Theorem (ongoing)

$$\|f_n^h - \rho_\infty^h \mathcal{M}\| \lesssim \left(1 + \kappa \frac{\Delta t}{\epsilon}\right)^{-\frac{n}{2}}. \quad (2)$$

Numerical experiments

Setting

Consider the fully non linear equation

$$\partial_t f + v \partial_x f - \partial_x \phi_\infty \partial_v f - \partial_x \phi \partial_v f_\infty - \partial_x \phi \partial_v (f - f_\infty) = \frac{1}{\tau} \partial_v [v f + \partial_v f],$$

$$-\partial_x^2 \phi = \rho - \rho_\infty, \quad \rho = \int_{\mathbb{R}^d} f \, dv.$$

Test 1: Landau damping ($\tau = 10^6$)

$$f(0, x, v) = f_\infty(x, v) + 0.01 \cos(x)$$

Test 2: Plasma echoes (variable τ)

$$\tilde{f}(0, x, v) = f(30, x, v) + 0.01 \cos(2x)$$

- derive the VPFP model as the mean-field limit of a particle system uniformly in the fluid limit²;
- quantitative numerical results for the non-linear model in a perturbative setting;
- quantitative long-time behavior of the non-linear model in non-perturbative setting;
- including collision operators closer to physics (ex: Landau³)

²D. Bresch, P.-E. Jabin, Z. Wang (19)

³S. Chaturvedi, J. Luk, T. Nguyen