A-P scheme for the Vlasov-Poisson-Fokker-Planck equation

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The question at hand

- Consider the **Vlasov-Poisson-Fokker-Planck** model

$$\begin{cases} \epsilon\,\partial_t f^\epsilon + v\,\partial_x f^\epsilon - \underbrace{\partial_x \phi^\epsilon\,\partial_v f^\epsilon}_{\text{field interactions}} = \dfrac{1}{\epsilon}\,\underbrace{\partial_v\left[\,v\,f^\epsilon + \partial_v f^\epsilon\,\right]}_{\text{collisions}}, \\[2ex] -\,\partial_x^2 \phi^\epsilon = \rho^\epsilon - \rho_i\,, \quad \rho^\epsilon = \displaystyle\int_{\mathbb{R}} f^\epsilon\,dv\,. \end{cases}$$

- $f^\epsilon(t,x,v)$: density of particles at time t, position $x \in \mathbb{T}$, velocity $v \in \mathbb{R}$.

Possible dynamics:

$$\begin{tikzcd} f^\epsilon(t,x,v) \arrow[r, "t\to+\infty"] \arrow[dr, "\epsilon\to 0"'] & \rho_\infty(x)\,\mathcal{M}(v) \\ & \rho(t,x)\,\mathcal{M}(v) \arrow[u, "t\to+\infty"'] \end{tikzcd}$$

- **Gaussian** velocity distribution:

$$\mathcal{M}(v) = \frac{1}{\sqrt{2\,\pi}}\exp\left(-\frac{1}{2}|v|^2\right),$$

- $\rho(t,x)$ solves

$$\begin{cases} \partial_t\,\rho = \nabla_x\cdot\left[\rho\,\nabla_x\,\phi + \nabla_x\,\rho\right]\,, \\[2ex] -\,\Delta_x\,\phi = \rho - \rho_i\,. \end{cases}$$

Linear setting

- Case of a given electric field $\partial_x \phi$ and $(x, v) \in \mathbb{T} \times \mathbb{R}$

$$\epsilon\, \partial_t f^\epsilon + v\, \partial_x f^\epsilon - \partial_x \phi\, \partial_v f^\epsilon = \frac{1}{\epsilon} \partial_v \left[v\, f^\epsilon + \partial_v f^\epsilon \right].$$

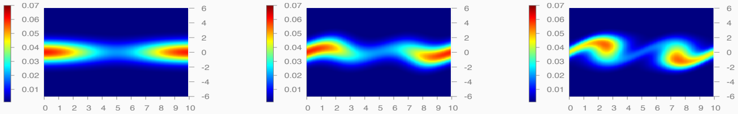
and $f^\epsilon(t, x, v) \to \rho(t, x)\, \mathcal{M}(v)$ as $\epsilon \to 0$ with

$$\partial_t \rho = \partial_x \left[\rho\, \partial_x \phi + \partial_x \rho \right].$$

We design discrete approximations (f_n^h, ρ_n^h) of (f^ϵ, ρ) such that

Theorem (with F. Filbet, 22')

$$\left\| f_n^h - \rho_n^h \mathcal{M} \right\| \lesssim \epsilon\, (1 + \kappa \Delta t)^{-\frac{n}{2}} + \left(1 + \frac{\Delta t}{2\, \epsilon^2} \right)^{-\frac{n}{2}}. \tag{1}$$



From key estimate to functional space

Dissipation of the L^2-norm

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left|\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}-\sqrt{\rho_\infty}\right|^2\mathrm{d}x\mathcal{M}\mathrm{d}v=-\frac{2}{\epsilon^2}\int\left|\partial_v\left(\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}\right)\right|^2\mathrm{d}x\mathcal{M}\mathrm{d}v$$

$\rightarrow$ Functional space :

$$\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}\in L^2\left(\mathrm{d}x\,\mathcal{M}(v)\,\mathrm{d}v\right).$$

- Spectral decomp. in Hermite basis $(H_k)_{k\in\mathbb{N}}$ of $L^2\left(\mathcal{M}\,\mathrm{d}v\right)$

$$\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}(t,x,v)\;=\sum_{k\in\mathbb{N}}D_k^\epsilon(t,x)\,H_k(v)\,.$$

- **No weight** with respect to $\mathrm{d}x$ so

$$D_k^\epsilon\in L^2\left(\mathrm{d}x\right).$$

Hermite decomposition

Vlasov-Fokker-Planck equation on $D^\epsilon = (D_k^\epsilon)_{k\in\mathbb{N}}$

$$\epsilon\,\partial_t D_k^\epsilon + \sqrt{k}\,\mathcal{A}\,D_{k-1}^\epsilon - \sqrt{k+1}\,\mathcal{A}^\star\,D_{k+1}^\epsilon = -\frac{k}{\epsilon}\,D_k^\epsilon\,, \quad \forall\,k\in\mathbb{N}\,,$$

with $\mathcal{A}u = \partial_x u + \frac{\partial_x\phi}{2}\,u$.

- Equilibrium is $D_{\infty,k} = \sqrt{\rho}_\infty\,\delta_{k=0}$.

Dissipation of the L^2-norm in Hermite basis

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^\epsilon - D_\infty\right\|_{L^2}^2 = -\frac{2}{\epsilon^2}\sum_{k\in\mathbb{N}^*} k\,\left\|D_k^\epsilon\right\|_{L^2}^2\,.$$

Discrete framework

Fully discrete scheme

$$\frac{D_k^{n+1} - D_k^n}{\Delta t} + \frac{1}{\epsilon}\left(\sqrt{k}\,\mathcal{A}_h\, D_{k-1}^{n+1} - \sqrt{k+1}\,\mathcal{A}_h^\star\, D_{k+1}^{n+1}\right) = -\frac{k}{\epsilon^2}\, D_k^{n+1}\,,$$

for all $k \in \mathbb{N}$ where discrete operators $\mathcal{A}_h$ and $\mathcal{A}_h^\star$ verify

Properties	Preservation
$\langle \mathcal{A}_h u,\, v\rangle_{L^2} = \langle u,\, \mathcal{A}_h^\star v\rangle_{L^2}$	duality structure
$\mathcal{A}_h\,\sqrt{\rho}_\infty = 0$	equilibrium state
$\sum_j \Delta x_j\, (\mathcal{A}_h^\star u)_j\, \sqrt{\rho}_{\infty\,,j} = 0$	invariants
$\lVert u\rVert_{L^2} \leq C_d\, \lVert \mathcal{A}_h\, u\rVert_{L^2}$	macroscopic coercivity

for all $(u_j)_{j\in\mathcal{J}}$, $(v_j)_{j\in\mathcal{J}}$

Hypocoercivity

We go back to the L^2 estimate

$$\frac{\left\| D^{n+1} - D_\infty \right\|_{L^2}^2 - \left\| D^n - D_\infty \right\|_{L^2}^2}{\Delta t} \le -\frac{2}{\epsilon^2} \left\| D_\perp^{n+1} \right\|_{L^2}^2 ,$$

with $D_\perp^{n+1} = \left(0\,,\ D_1^{n+1}\,,\ D_2^{n+1}\,,\ D_3^{n+1}, \dots\right)$.

⚠ Lack of coercivity[1]

$$\boxed{\left\| D^{n+1} - D_\infty \right\|_{L^2}^2 \not\lesssim \left\| D_\perp^{n+1} \right\|_{L^2}^2}$$

[1] Villani (2009)

Illuminating example

Consider

$$\begin{cases} \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t} x^\epsilon = -\, y^\epsilon \\ \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t} y^\epsilon = x^\epsilon - \dfrac{1}{\epsilon}\, y^\epsilon \end{cases} .$$

- Relative entropy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|x^\epsilon(t)|^2 + |y^\epsilon(t)|^2\right) = -\frac{2}{\epsilon^2}\,|y^\epsilon(t)|^2 .$$

- Modified entropy: $\mathcal{H}(f^\epsilon) = |x^\epsilon|^2 + |y^\epsilon|^2 - \alpha\,\epsilon\, x^\epsilon\, y^\epsilon$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f^\epsilon) = -\frac{2}{\epsilon^2}\,|y^\epsilon(t)|^2 + \alpha\left(|y^\epsilon(t)|^2 - |x^\epsilon(t)|^2 + \frac{1}{\epsilon} x^\epsilon(t) y^\epsilon(t)\right).$$

We deduce $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f^\epsilon) = -\kappa\,\mathcal{H}(f^\epsilon)$ and $|x^\epsilon(t)|^2 + |y^\epsilon(t)|^2 \lesssim e^{-\kappa\, t}$.

Discrete hypocoercivity

Define a modified entropy functional

$$\mathcal{H}_0^n \,=\, \|D^n - D_\infty\|_{L^2}^2 \,+\, \alpha\,\epsilon\,\langle D_1^n, \mathcal{A}_h u_h^n\rangle\,.$$

where u_h^n solves the elliptic problem

$$\begin{cases} (\mathcal{A}_h^\star\,\mathcal{A}_h)\,u_h^n \,=\, D_0^n - D_{\infty,0}\,, \\ \displaystyle\sum_{j\in\mathcal{J}} \Delta x_j\, u_j\,\sqrt{\rho}_{\infty\,,\,j} \,=\, 0\,, \end{cases}$$

Macroscopic coercivity $\rightarrow$ we recover

$$\begin{cases} \|D^n - D_\infty\|_{L^2}^2 \lesssim \mathcal{H}_0^n \lesssim \|D^n - D_\infty\|_{L^2}^2\,, \\ \dfrac{\mathcal{H}_0^{n+1} - \mathcal{H}_0^n}{\Delta t} \lesssim -\dfrac{2}{\epsilon^2}(1-\alpha)\,\left\|D_\perp^{n+1}\right\|_{L^2}^2 \,-\, \alpha\left\|D_0^{n+1} - D_{\infty,0}\right\|_{L^2}^2\,. \end{cases}$$

Numerical experiments

Setting

We take $\Delta t = 10^{-3}$, 200 Hermite modes, 64 points in space and

$$\phi(x) \, = \, 0.1 \, \cos\left(2\pi \, x\right) \, + \, 0.9 \, \cos\left(4\pi \, x\right) \, .$$

First Test: $\epsilon \, = \, 1$ and

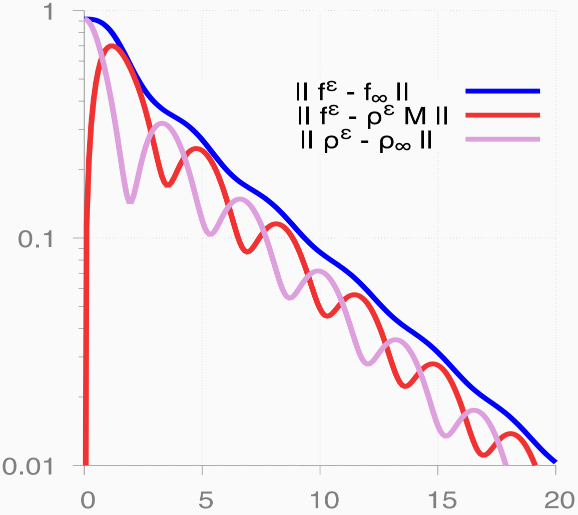
$$f_0(x, v) \, = \, \left(1 + 0.5 \cos\left(2\pi \, x\right)\right) \, \exp\left(-|v|^2/2\right) / \sqrt{2\pi} \, ,$$

Second Test: $\epsilon \, = \, 10^{-4}$ and

$$f_0(x, v) \, = \, \left(1 + 0.5 \cos\left(2\pi \, x\right)\right) \, \exp\left(-|v-1|^2/2\right) / \sqrt{2\pi} \, .$$

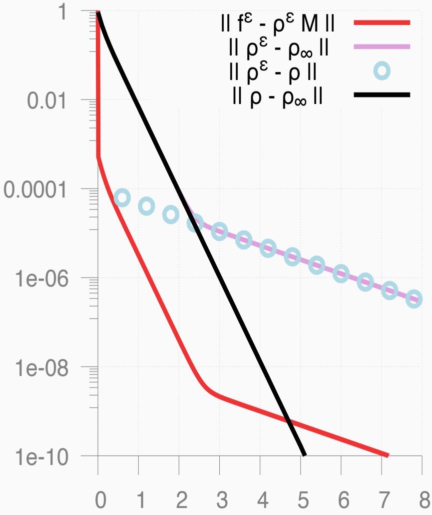
First Test, $\epsilon = 1$

Time evolution (log-scale): $\|f^\epsilon - f_\infty\|_{L^2(f_\infty^{-1})}$ (blue), $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink)



Second Test: $\epsilon = 10^{-4}$

Time evolution (log scale) of $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink), $\|\rho^\epsilon - \rho\|_{L^2(\rho_\infty^{-1})}$ (blue points) and $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (black)



We have proven that

$$\|D_\perp^n\| \le \|D_\perp^0\| \left(1 + \frac{\Delta t}{2\,\epsilon^2}\right)^{-\frac{n}{2}} + \epsilon C \left\|D^0 - D_\infty\right\| (1 + \kappa\,\Delta t)^{-\frac{n}{2}},$$

and

$$\left\|D_0^n - \overline{D}_0^n\right\| \le C\epsilon \left\|D^0 - D_\infty\right\| (1 + \kappa\Delta t)^{-\frac{n}{2}},$$

and

$$\left\|\overline{D}^n - D_\infty\right\| \le \left\|\overline{D}^0 - D_\infty\right\| (1 + \tilde{\kappa}\Delta t)^{-\frac{n}{2}},$$

Linearized equation

Linear setting

- Consider the linearized equation

$$\begin{cases} \epsilon\, \partial_t f^\epsilon + v\partial_x f^\epsilon - \partial_x \phi_\infty \partial_v f^\epsilon - \partial_x \phi^\epsilon \partial_v f_\infty = \partial_v \left[v\, f^\epsilon + \partial_v f^\epsilon \right] , \\ \\ -\partial_x^2 \phi^\epsilon = \rho^\epsilon - \rho_\infty , \quad \rho^\epsilon = \displaystyle\int_{\mathbb{R}^d} f^\epsilon \, dv , \end{cases}$$

with $f_\infty(x, v) = \rho_\infty(x)\, \mathcal{M}(v)$ and

$$\begin{cases} \rho_\infty = e^{-\phi_\infty} , \\ \\ -\partial_x^2 \, \phi_\infty = \rho_\infty - \rho_i \, . \end{cases}$$

We design discrete approximations (f_n^h, ρ_∞^h) of $(f^\epsilon, \rho_\infty)$ such that

Theorem (ongoing)

$$\left\| f_n^h - \rho_\infty^h \mathcal{M} \right\| \lesssim \left(1 + \kappa \frac{\Delta t}{\epsilon} \right)^{-\frac{n}{2}} . \tag{2}$$

Numerical experiments

Setting

Consider the fully non linear equation

$$\partial_t f + v\partial_x f - \partial_x\phi_\infty\partial_v f - \partial_x\phi\,\partial_v f_\infty - \partial_x\phi\partial_v(f - f_\infty) = \frac{1}{\tau}\partial_v\left[vf + \partial_v f\right],$$

$$-\partial_x^2\phi = \rho - \rho_\infty\,, \quad \rho = \int_{\mathbb{R}^d} f\,dv\,.$$

<u>Test 1:</u> Landau damping ($\tau = 10^6$)

$$f(0,x,v) = f_\infty(x,v) + 0.01\,cos(x)$$

<u>Test 2:</u> Plasma echoes (variable τ)

$$\tilde{f}(0,x,v) = f(30,x,v) + 0.01\,cos(2\,x)$$

Perspectives

• derive the VPFP model as the mean-field limit of a particle system uniformly in the fluid limit[2];

• quantitative numerical results for the non-linear model in a perturbative setting;

• quantitative long-time behavior of the non-linear model in non-perturbative setting;

• including collision operators closer to physics (ex: Landau[3])

[2] D. Bresch, P.-E. Jabin, Z. Wang (19)
[3] S. Chaturvedi, J. Luk, T. Nguyen