A-P scheme for the Vlasov-Poisson-Fokker-Planck equation

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February 15, 2023

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A general framework for kinetic theory

- Statistical description: distribution $f(t, x, v)$ of particles at time t, position x, velocity v;

- dynamics of f driven by an evolution equation

$$\partial_t f \;+\; T f \;=\; L f \,,$$

- operator T represents transport of particles (non-dissipating);

- operator L represents collisions between particles (dissipating).

Fundamental example

The Vlasov-Poisson system; $L = 0$ and

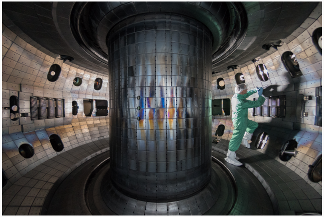
$$T\,f \;=\; v\cdot\nabla_x f - \nabla_v\phi\cdot\nabla_x f \quad \text{with} \quad -\Delta\phi = \rho - 1 \quad \text{and} \quad \rho = \int f\,\mathrm{d}v\,.$$

Main question: Long time behavior of the model. Why is it difficult:

- ∞-many equilibrium states;
- field interactions $\rightarrow$ non-linear equation.

Landau damping: non-linear stability for a class of equilibriums[1,2,3].

Application:
Plasma physics



[1] C. Mouhot, C. Villani; Acta Math. 11'
[2] J. Bedrossian, N. Masmoudi, C. Mouhot; Ann. PDE 2 16'
[3] E. Grenier, T. Nguyen, I. Rodnianski; Math. Res. Lett 21'

Macroscopic limits: from kinetic theory to fluid dynamics

Example: limit $\epsilon \to 0$ for the Vlasov-Fokker-Planck model

$$\epsilon\,\partial_t f^\epsilon + T\,f^\epsilon = \frac{1}{\epsilon}\,L\,f^\epsilon\,,$$

with $T\,f^\epsilon = v\cdot\nabla_x f^\epsilon$ and $L\,f^\epsilon = \nabla_v\cdot[v\,f^\epsilon + \nabla_v f^\epsilon]$ which verifies

$$L\,f^\epsilon = 0 \iff f^\epsilon(t,x,v) = \rho^\epsilon(t,x)\mathcal{M}(v)\,,\text{ where } \begin{cases} \mathcal{M}(v) = \dfrac{1}{\sqrt{2\,\pi}}e^{-\frac{1}{2}|v|^2}, \\ \rho^\epsilon = \displaystyle\int_{\mathbb{R}^d} f^\epsilon\,dv\,. \end{cases}$$

Hilbert expansion: $f^\epsilon = f_0 + \epsilon\,f_1 + \epsilon^2\,f_2 + O(\epsilon^3)$

$$\begin{cases} \boxed{\epsilon^{-1}:}\quad L\,f_0 = 0 \implies f_0(t,x,v) = \rho_0(t,x)\,\mathcal{M}(v)\,, \\ \left.\begin{array}{l} \boxed{\epsilon^{0}\;:}\quad T\,f_0 = L\,f_1 \\ \boxed{\epsilon^{1}\;:}\quad \partial_t f_0 + T\,f_1 = L\,f_2 \end{array}\right\} \implies \partial_t\rho_0 - \Delta_x\,\rho_0 = 0\,. \end{cases}$$

We deduce $f^\epsilon \underset{\epsilon\to 0}{\longrightarrow} \rho_0\,\mathcal{M}$, with $\partial_t\rho_0 - \Delta_x\,\rho_0 = 0$

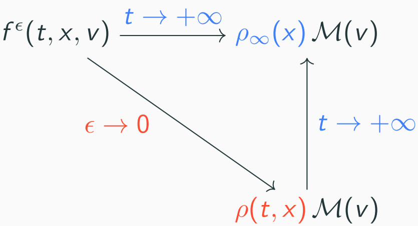
The question at hand

- Consider the **Vlasov-Poisson-Fokker-Planck** model

$$\begin{cases} \epsilon\, \partial_t f^\epsilon + v \cdot \nabla_x f^\epsilon - \underbrace{\nabla_x \phi^\epsilon \cdot \nabla_v f^\epsilon}_{\text{field interactions}} = \dfrac{1}{\epsilon} \underbrace{\nabla_v \cdot \left[v f^\epsilon + \nabla_v f^\epsilon \right]}_{\text{collisions}}, \\ -\Delta_x \phi^\epsilon = \rho^\epsilon - \rho_i\,, \quad \rho^\epsilon = \displaystyle\int_{\mathbb{R}^d} f^\epsilon\, dv\,. \end{cases}$$

- $f^\epsilon(t,x,v)$: density of particles at time t, position x, velocity v.

Possible dynamics:



- **Gaussian** velocity distribution:

$$\mathcal{M}(v) = \frac{1}{\sqrt{2\,\pi}} \exp\left(-\frac{1}{2} |v|^2 \right),$$

- $\rho(t,x)$ solves

$$\begin{cases} \partial_t \rho = \nabla_x \cdot \left[\rho \nabla_x \phi + \nabla_x \rho \right], \\ -\Delta_x \phi = \rho - \rho_i\,. \end{cases}$$

Contributions

- Case of a given electric field $\partial_x \phi$ and $(x, v) \in \mathbb{T} \times \mathbb{R}$

$$\epsilon\, \partial_t f^\epsilon + v\, \partial_x f^\epsilon - \partial_x \phi\, \partial_v f^\epsilon = \frac{1}{\epsilon} \partial_v \left[v\, f^\epsilon + \partial_v f^\epsilon \right] .$$

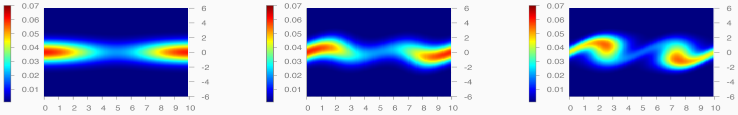
and

$$\partial_t \rho = \partial_x \left[\rho\, \partial_x \phi + \partial_x \rho \right] .$$

We design discrete approximations (f_n^h, ρ_n^h) of (f^ϵ, ρ) such that

Theorem (with F. Filbet, 22')

$$\left\| f_n^h - \rho_n^h \mathcal{M} \right\| \lesssim \epsilon \left(1 + \kappa \Delta t\right)^{-\frac{n}{2}} + \left(1 + \frac{\Delta t}{2\,\epsilon^2}\right)^{-\frac{n}{2}} . \qquad (1)$$



From key estimate to functional space

Dissipation of the L^2-norm

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left|\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}-\sqrt{\rho_\infty}\right|^2 \mathrm{d}x\mathcal{M}\mathrm{d}v=-\frac{2}{\epsilon^2}\int\left|\partial_v\left(\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}\right)\right|^2 \mathrm{d}x\mathcal{M}\mathrm{d}v$$

$\rightarrow$ Functional space :

$$\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}\in L^2\left(\mathrm{d}x\,\mathcal{M}(v)\,\mathrm{d}v\right).$$

- Spectral decomp. in Hermite basis $(H_k)_{k\in\mathbb{N}}$ of $L^2\left(\mathcal{M}\,\mathrm{d}v\right)$

$$\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}(t,x,v)=\sum_{k\in\mathbb{N}}D_k^\epsilon(t,x)\,H_k(v).$$

- **No weight** with respect to $\mathrm{d}x$ so

$$D_k^\epsilon\in L^2\left(\mathrm{d}x\right).$$

Hermite decomposition

Vlasov-Fokker-Planck equation on $D^\epsilon = (D_k^\epsilon)_{k\in\mathbb{N}}$

$$\begin{cases} \epsilon\,\partial_t D^\epsilon + T\,D^\epsilon = -\dfrac{1}{\epsilon}\,L\,D^\epsilon\,, \\ (T\,D^\epsilon)_k = \sqrt{k}\,\mathcal{A}\,D_{k-1}^\epsilon - \sqrt{k+1}\,\mathcal{A}^\star\,D_{k+1}^\epsilon\,, \text{ and } (L\,D^\epsilon)_k = k\,D_k^\epsilon\,, \end{cases}$$

with $\mathcal{A}u = \partial_x u + \frac{\partial_x \phi}{2}\,u$. Equilibrium is $D_{\infty,k} = \sqrt{\rho_\infty}\,\delta_{k=0}$ and macroscopic equation reads

$$\partial_t \overline{D} + \Pi\,T^2\,\overline{D} = 0\,, \quad \text{with} \quad \left(\Pi\,\overline{D}\right)_k = \overline{D}_k\,\delta_{k=0}\,.$$

and the L^2 estimate rewrites

Dissipation of the L^2-norm in Hermite basis

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|D^\epsilon - D_\infty\|_{L^2}^2 = -\frac{2}{\epsilon^2}\sum_{k\in\mathbb{N}^*} k\,\|D_k^\epsilon\|_{L^2}^2\,.$$

Discrete framework

Fully discrete scheme

$$\epsilon \frac{D^{n+1} - D^n}{\Delta t} + T^h D^{n+1} = -\frac{1}{\epsilon} L^h D^{n+1},$$

T^h,L^h discrete versions of T and L which verify

Properties	Preservation
$\left(T^h\right)^\star = -T^h$ and $\left(L^h\right)^\star = L^h$	duality structure
$T^h D_\infty = L^h D_\infty = 0$	equilibrium state
$\int T^h u^h = \int L^h u^h = 0\,,\ \forall u^h$	invariants
$\lVert u^h \rVert_{L^2} \leq C_d \lVert \mathcal{A}_h u^h \rVert_{L^2}$	macroscopic coercivity

Hypocoercivity

We go back to the L^2 estimate

$$\frac{\left\|D^{n+1} - D_\infty\right\|_{L^2}^2 - \left\|D^{n} - D_\infty\right\|_{L^2}^2}{\Delta t} \leq -\frac{2}{\epsilon^2} \left\|(1-\Pi)\left(D^{n+1} - D_\infty\right)\right\|_{L^2}^2 ,$$

⚠ Lack of coercivity[4]

$$\boxed{\left\|D^{n+1} - D_\infty\right\|_{L^2}^2 \not\lesssim \left\|(1-\Pi)\left(D^{n+1} - D_\infty\right)\right\|_{L^2}^2}$$

[4] Villani (2009)

Illuminating example

Consider $f^\epsilon = (x^\epsilon(t), y^\epsilon(t))^T$, $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

$$\epsilon \frac{\mathrm{d}}{\mathrm{d}t} f^\epsilon + T f^\epsilon = \frac{1}{\epsilon} L f^\epsilon \iff \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x^\epsilon = -\frac{1}{\epsilon} y^\epsilon \\ \frac{\mathrm{d}}{\mathrm{d}t} y^\epsilon = \frac{1}{\epsilon} x^\epsilon - \frac{1}{\epsilon^2} y^\epsilon \end{cases} .$$

- Relative entropy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(|x^\epsilon(t)|^2 + |y^\epsilon(t)|^2 \right) = -\frac{2}{\epsilon^2} |y^\epsilon(t)|^2 .$$

- Modified entropy: $\mathcal{H}(f^\epsilon) = |x^\epsilon|^2 + |y^\epsilon|^2 - \alpha \, \epsilon \, x^\epsilon \, y^\epsilon$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}(f^\epsilon) = -\frac{2}{\epsilon^2} |y^\epsilon(t)|^2 + \alpha \left(|y^\epsilon(t)|^2 - |x^\epsilon(t)|^2 + \frac{1}{\epsilon} x^\epsilon(t) y^\epsilon(t) \right) .$$

We deduce $\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}(f^\epsilon) = -\kappa \, \mathcal{H}(f^\epsilon)$ and $|x^\epsilon(t)|^2 + |y^\epsilon(t)|^2 \lesssim e^{-\kappa \, t}$.

Discrete hypocoercivity

Define a modified entropy functional

$$\mathcal{H}_0^n \;=\; \|D^n - D_\infty\|_{L^2}^2 \;+\; \alpha\,\epsilon\,\langle D_1^n, \mathcal{A}_h u_h^n\rangle\;.$$

where u_h^n solves the elliptic problem

$$\begin{cases} (\mathcal{A}_h^\star\,\mathcal{A}_h)\,u_h^n \;=\; D_0^n - D_{\infty,0}\,, \\ \displaystyle\sum_{j\in\mathcal{J}} \Delta x_j\,u_j\,\sqrt{\rho}_{\infty\,,\,j} \;=\; 0\,, \end{cases}$$

Macroscopic coercivity $\rightarrow$ we recover

$$\begin{cases} \|D^n - D_\infty\|_{L^2}^2 \lesssim \mathcal{H}_0^n \;\lesssim\; \|D^n - D_\infty\|_{L^2}^2\,, \\ \dfrac{\mathcal{H}_0^{n+1} - \mathcal{H}_0^n}{\Delta t} \lesssim -\dfrac{2}{\epsilon^2}(1-\alpha)\,\left\|(1-\Pi)\,D^{n+1}\right\|_{L^2}^2 \,-\, \alpha\left\|\Pi\left(D^{n+1} - D_\infty\right)\right\|_{L^2}^2\;. \end{cases}$$

Numerical experiments

Setting

We take $\Delta t = 10^{-3}$, 200 Hermite modes, 64 points in space and

$$\phi(x) \,=\, 0.1\,\cos\left(2\pi\, x\right) \,+\, 0.9\,\cos\left(4\pi\, x\right)\,.$$

First Test: $\epsilon \,=\, 1$ and

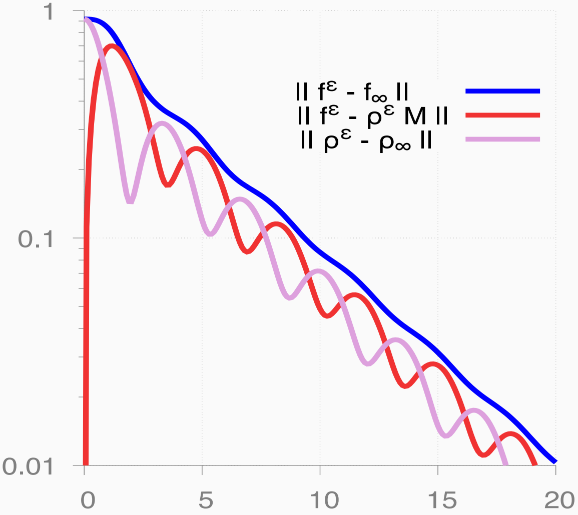
$$f_0(x,v) \,=\, \left(1 + 0.5\cos\left(2\pi\, x\right)\right)\,\exp\left(-|v|^2/2\right)/\sqrt{2\pi}\,,$$

Second Test: $\epsilon \,=\, 10^{-4}$ and

$$f_0(x,v) \,=\, \left(1 + 0.5\cos\left(2\pi\, x\right)\right)\,\exp\left(-|v-1|^2/2\right)/\sqrt{2\pi}\,.$$

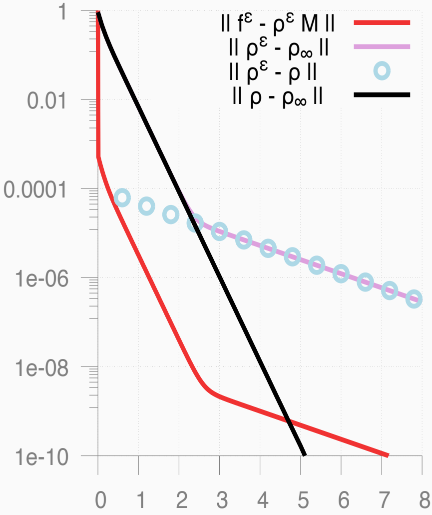
First Test, $\epsilon = 1$

Time evolution (log-scale): $\|f^\epsilon - f_\infty\|_{L^2(f_\infty^{-1})}$ (blue), $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink)



Second Test: $\epsilon = 10^{-4}$

Time evolution in log scale of $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink), $\|\rho^\epsilon - \rho\|_{L^2(\rho_\infty^{-1})}$ (blue points) and $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (black)



We have proven that

$$\|D_\perp^n\| \leq \|D_\perp^0\| \left(1 + \frac{\Delta t}{2\,\epsilon^2}\right)^{-\frac{n}{2}} + \epsilon C \left\|D^0 - D_\infty\right\| (1 + \kappa\,\Delta t)^{-\frac{n}{2}}\,,$$

and

$$\left\|D_0^n - \overline{D}_0^n\right\| \leq C\epsilon \left\|D^0 - D_\infty\right\| (1 + \kappa \Delta t)^{-\frac{n}{2}}\,,$$

and

$$\left\|\overline{D}^n - D_\infty\right\| \leq \left\|\overline{D}^0 - D_\infty\right\| (1 + \tilde{\kappa}\Delta t)^{-\frac{n}{2}}\,,$$

Perspectives

• derive the VPFP model as the mean-field limit of a particle system uniformly in the fluid limit[5];

• quantitative numerical results for the non-linear model in a perturbative setting;

• quantitative long-time behavior of the non-linear model in non-perturbative setting;

• including collision operators closer to physics (ex: Landau[6])

[5] D. Bresch, P.-E. Jabin, Z. Wang (19)
[6] S. Chaturvedi, J. Luk, T. Nguyen