# Kinetic and hyperbolic equations analysis, modeling and numerics

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# Motivations

We consider electrons subjected to:

(a) electric field $E = -\partial_x \phi$ , (b) collisions with motionless ions.

The system is described by the following equation

**Vlasov-Fokker-Planck equation**

$$\partial_t f^\epsilon + \frac{1}{\epsilon} v \, \partial_x f^\epsilon - \frac{1}{\epsilon} \partial_x \phi \, \partial_v f^\epsilon = \frac{1}{\epsilon^2} \partial_v \left[ v \, f^\epsilon + \partial_v f^\epsilon \right]. \qquad (1)$$

$f^\epsilon(t, x, v)$ is the probability of finding the electron at time $t \geq 0$, position $x \in \mathbb{T}$, with velocity $v \in \mathbb{R}$. For some parameter $\epsilon > 0$.

# hydrodynamic regime

We consider the hydrodynamic scaling $\epsilon \to 0$ in the equation

$$\partial_t f^\epsilon + \frac{1}{\epsilon} v \, \partial_x f^\epsilon - \frac{1}{\epsilon} \partial_x \phi \, \partial_v f^\epsilon = \frac{1}{\epsilon^2} \partial_v \left[ v f^\epsilon + \partial_v f^\epsilon \right] .$$

Considering the leading order, we expect

$$f^\epsilon(t,x,v) \underset{\epsilon \to 0}{\sim} \int_{\mathbb{R}} f^\epsilon(t,x,v) \, \mathrm{d}v \, \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{|v|^2}{2} \right) := \rho^\epsilon(t,x) \, \mathcal{M}(v) .$$

limit of $\rho^\epsilon$: consider $\pi^\epsilon = \int_{\mathbb{R}} f^\epsilon(t, x - \epsilon v, v) \, \mathrm{d}v$ , which solves

$$\partial_t \pi^\epsilon - \partial_x \left( \int_{\mathbb{R}} \partial_x \phi \, f^\epsilon \, (t, x - \epsilon \, v, v) \, \mathrm{d}v + \partial_x \pi^\epsilon \right) = 0 .$$

Therefore, we expect $f^\epsilon(t,x,v) \underset{\epsilon \to 0}{\sim} \rho(t,x) \, \mathcal{M}(v)$, with

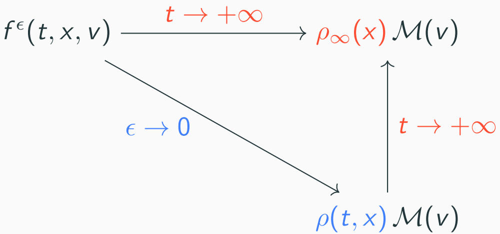
**Macroscopic equation**

$$\partial_t \rho - \partial_x \left[ \partial_x \phi \, \rho + \partial_x \rho \right] = 0 ,$$

whose stationary state is $\rho_\infty(x) = c \exp\left( -\phi(x) \right)$.

# Purposes

The situation is as follows




**GOAL**: Build a numerical method and prove **quantitative asymptotic preserving properties** for the limits $\epsilon \to 0$ and $t \to +\infty$ simultaneously

# Numerical analysis of the model

# From key estimate to functional space

## Dissipation of the $L^2$-norm

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left|\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}-\sqrt{\rho_\infty}\right|^2\mathrm{d}x\mathcal{M}\mathrm{d}v=-\frac{2}{\epsilon^2}\int\left|\partial_v\left(\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}\right)\right|^2\mathrm{d}x\mathcal{M}\mathrm{d}v$$

$\rightarrow$ Functional space :

$$\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}\in L^2\left(\mathrm{d}x\,\mathcal{M}(v)\,\mathrm{d}v\right)\,.$$

- Spectral decomp. in Hermite basis $(H_k)_{k\in\mathbb{N}}$ of $L^2\left(\mathcal{M}\,\mathrm{d}v\right)$

$$\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}(t,x,v)\;=\;\sum_{k\in\mathbb{N}}D_k(t,x)\,H_k(v)\,.$$

- **No weight** with respect to $\mathrm{d}x$ so

$$D_k\in L^2\left(\mathrm{d}x\right)\,.$$

# Reformulated equations

## Reformulated Vlasov-Fokker-Planck equation

$$\partial_t D_k^\epsilon + \frac{1}{\epsilon}\left(\sqrt{k}\,\mathcal{A}\,D_{k-1}^\epsilon - \sqrt{k+1}\,\mathcal{A}^\star\,D_{k+1}^\epsilon\right) = -\frac{k}{\epsilon^2}\,D_k^\epsilon\,,\quad k\in\mathbb{N}\,,$$

with $\mathcal{A}u = \partial_x u + \frac{\partial_x\phi}{2}\,u$ and $\langle \mathcal{A}u,\,v\rangle_{L^2} = \langle u,\,\mathcal{A}^\star v\rangle_{L^2}$. Equilibrium is

$$D_{\infty,0} = \sqrt{\rho_\infty}\,,\quad \text{and}\quad D_{\infty,k} = 0\ \text{ if }\ k\geq 1\,,$$

macroscopic equation reads

$$\partial_t\overline{D}_0 + \mathcal{A}^\star\mathcal{A}\,\overline{D}_0 = 0\,,\quad \text{and}\quad \overline{D}_k = 0\ \text{ if }\ k\geq 1\,,$$

and the $L^2$ estimate rewrites

## Dissipation of the $L^2$-norm in Hermite basis

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^\epsilon - D_\infty\right\|_{L^2}^2 = -\frac{2}{\epsilon^2}\sum_{k\in\mathbb{N}^*} k\,\left\|D_k^\epsilon\right\|_{L^2}^2\,.$$

# Fully discrete scheme

**Reformulated Vlasov-Fokker-Planck equation**

$$\frac{D_k^{n+1} - D_k^n}{\Delta t} + \frac{1}{\epsilon}\left(\sqrt{k}\,\mathcal{A}_h\, D_{k-1}^{n+1} - \sqrt{k+1}\,\mathcal{A}_h^\star\, D_{k+1}^{n+1}\right) = -\frac{k}{\epsilon^2}\, D_k^{n+1},$$

for all $k \in \mathbb{N}$ where discrete operators $\mathcal{A}_h$ and $\mathcal{A}_h^\star$ verify

| | |
|---|---|
| $\langle \mathcal{A}_h u,\, v\rangle_{L^2} = \langle u,\, \mathcal{A}_h^\star v\rangle_{L^2}$ | preservation of the key estimate |
| $\mathcal{A}_h \sqrt{\rho}_\infty = 0$ | preservation of equilibrium state |
| $\sum_j \Delta x_j\, (\mathcal{A}_h^\star u)_j\, \sqrt{\rho}_{\infty\,,\,j} = 0$ | conservation of mass |

for all $(u_j)_{j\in\mathcal{J}}$, $(v_j)_{j\in\mathcal{J}}$

# Result

Setting $D^n_{\perp,0} = 0$ and $D^n_{\perp,k} = D^n_k$ if $k \geq 1$, we prove

**Theorem (with F. Filbet, 22')**

*There exist $\tilde{\kappa} > \kappa > 0$ independent of the discretization such that*

$$\|D^n_\perp\| \leq \|D^0_\perp\| \left(1 + \frac{\Delta t}{2\,\epsilon^2}\right)^{-\frac{n}{2}} + \epsilon C \left\|D^0 - D_\infty\right\| (1 + \kappa\,\Delta t)^{-\frac{n}{2}},$$

*and*

$$\left\|D^n_0 - \overline{D}^n_0\right\| \leq C \left(\left\|D^0_0 - \overline{D}^0_0\right\| + \epsilon \left\|D^0 - D_\infty\right\|\right) (1 + \kappa \Delta t)^{-\frac{n}{2}},$$

*and*

$$\left\|\overline{D}^n - D_\infty\right\| \leq \left\|\overline{D}^0 - D_\infty\right\| (1 + \tilde{\kappa}\Delta t)^{-\frac{n}{2}},$$

*for all $n \geq 0$ and $\epsilon > 0$.*

# Key arguments of the proof: convergence in $L^2$

We go back to the $L^2$ estimate

$$\frac{\left\|D^{n+1} - D_\infty\right\|_{L^2}^2 - \left\|D^n - D_\infty\right\|_{L^2}^2}{\Delta t} \leq -\frac{2}{\epsilon^2} \left\|D_\perp^{n+1}\right\|_{L^2}^2 ,$$

⚠ Lack of coercivity w.r.t. $D_0 \rightarrow$ introduce the elliptic problem

$$\begin{cases} \left(\mathcal{A}_h^\star \mathcal{A}_h\right) u_h^n = D_0^n - D_{\infty,0} , \\ \displaystyle\sum_{j \in \mathcal{J}} \Delta x_j \, u_j \sqrt{\rho}_{\infty\,,\,j} = 0 , \end{cases}$$

and define a modified entropy functional

$$\mathcal{H}_0^n = \|D^n - D_\infty\|_{L^2}^2 + \alpha_0 \, \epsilon \, \langle D_1^n, \mathcal{A}_h u_h^n \rangle .$$

Relying on Poincaré inequality $\|\mathcal{A}_h u_h^n\|_{L^2} \leq C \|D_0^n - D_{\infty,0}\|_{L^2}$, we prove

$$\begin{cases} \|D^n - D_\infty\|_{L^2}^2 \lesssim \mathcal{H}_0^n \lesssim \|D^n - D_\infty\|_{L^2}^2 , \\ \dfrac{\mathcal{H}_0^{n+1} - \mathcal{H}_0^n}{\Delta t} \lesssim -\dfrac{2}{\epsilon^2}(1 - \alpha_0) \left\|D_\perp^{n+1}\right\|_{L^2}^2 - \alpha_0 \left\|D_0^{n+1} - D_{\infty,0}^{n+1}\right\|_{L^2}^2 . \end{cases}$$

# Key arguments of the proof: convergence in $H^1$

Introduce the following $H^1$ norm

$$\|\mathcal{B}_h\, D^n\|_{L^2}^2 = \sum_{k\in\mathbb{N}} \|\mathcal{B}_{h,k}\, D_k^n\|_{L^2}^2\,, \quad \text{where} \quad \mathcal{B}_{h,k} = \begin{cases} \mathcal{A}_h\,, \text{ if } k = 0\,, \\ \mathcal{A}_h^\star, \text{ else}\,. \end{cases}$$

For this special choice of $\mathcal{B}_h$, it holds

$$\frac{\|\mathcal{B}_h D^{n+1}\|_{L^2}^2 - \|\mathcal{B}_h D^n\|_{L^2}^2}{\Delta t} = -\frac{2}{\epsilon^2} \sum_{k\in\mathbb{N}^\star} k\left\|\mathcal{B}_{h,k} D_k^{n+1}\right\|_{L^2}^2 - \frac{2}{\epsilon}\sum_{k\geq 2}\sqrt{k}\,\left\langle\,[\mathcal{A}_h^\star, \mathcal{A}_h]\, D_{k-1}^{n+1},\, \mathcal{A}_h^\star D_k^{n+1}\right\rangle\,.$$

We use that $\|\,[\mathcal{A},\, \mathcal{A}^\star]\, D_{k-1}^{n+1}\|_{L^2} \leq \|\phi\|\,\|D_{k-1}^{n+1}\|_{L^2}$ and Young inequality

$$\frac{\|\mathcal{B}_h D^{n+1}\|_{L^2}^2 - \|\mathcal{B}_h D^n\|_{L^2}^2}{\Delta t} \leq -\frac{1}{\epsilon^2}\left\|\mathcal{B}_h D_\perp^{n+1}\right\|_{L^2}^2 + C\left\|D_\perp^{n+1}\right\|_{L^2}^2\,.$$

⚠ Lack of coercivity w.r.t. $D_0$ → we conclude as in the former step

# Numerical experiments

# Setting

We take $\Delta t = 10^{-3}$, 200 Hermite modes, 64 points in space and

$$\phi(x) = 0.1 \cos(2\pi x) + 0.9 \cos(4\pi x) .$$

Test 1: $\epsilon = 1$ and

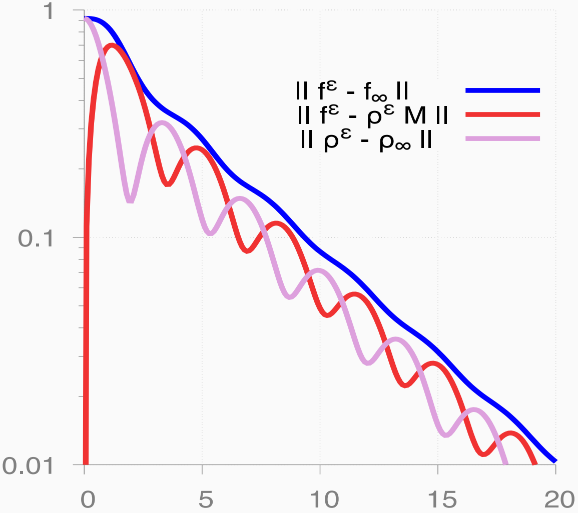
$$f_0(x, v) = (1 + 0.5 \cos(2\pi x)) \exp\left(-|v|^2/2\right) / \sqrt{2\pi} ,$$

Test 2: $\epsilon = 10^{-4}$ and

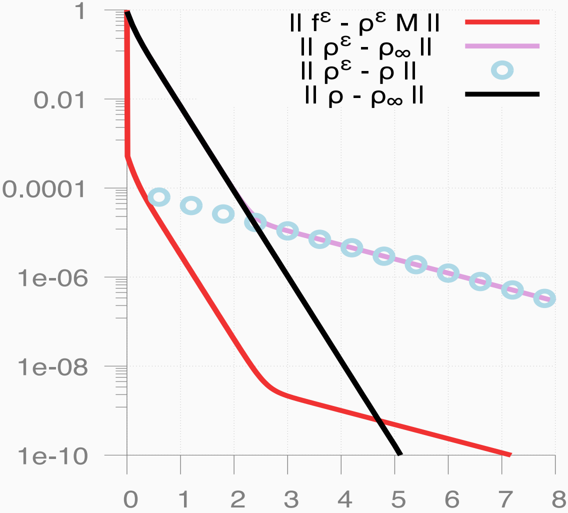
$$f_0(x, v) = (1 + 0.5 \cos(2\pi x)) \exp\left(-|v - 1|^2/2\right) / \sqrt{2\pi} .$$

# First Test, $\epsilon = 1$

Time evolution in log-scale of $\|f^\epsilon - f_\infty\|_{L^2(f_\infty^{-1})}$ (blue), $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2}(\rho_\infty^{-1})$ (pink)



Time evolution in log scale of $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink), $\|\rho^\epsilon - \rho\|_{L^2(\rho_\infty^{-1})}$ (blue points) and $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (black)



# Perspectives

• Analysis of the scheme with a Poisson non-linear coupling in perturbative setting and dimension 2;

• Spectral analysis of the model to quantitatively describe oscillations;

• analysis with a Poisson non-linear coupling in non-perturbative setting
$\rightarrow$ requires a better understanding of the model at the continuous level;