# Concentration phenomena for a FitzHugh-Nagumo neural network

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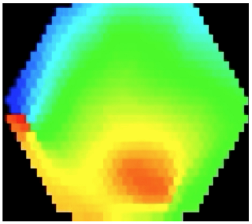
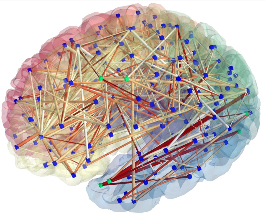
# Introduction

# Overview

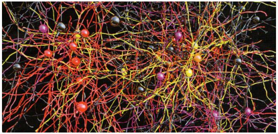
3 distinct scales to describe neural networks :



Macroscopic scale



Mesoscopic scale



Microscopic scale

GOAL : Quantitative analysis of the asymptotic regime

$$\text{" } (\text{mesoscopic scale})_\epsilon \underset{\epsilon \to 0}{\longrightarrow} \text{macroscopic scale "}$$

# Behavior of a *single* neuron

- We focus on the voltage through the membrane of a neuron.
- Hodgkin & Huxley (52') : precise but complicated model [gif] .
- FitzHugh-Nagumo : simplified model which captures the main features

**FitzHugh-Nagumo's model**

$$\begin{cases} dv_t = (N(v_t) - w_t + I_{\text{ext}})\, dt + \sqrt{2} dB_t, \\ dw_t = A\,(v_t, w_t)\, dt, \end{cases}$$

- 2 equations for periodic behavior ($v_t$ : voltage, $w_t$ : adaptation variable) .
- Confining assumptions to ensure spikes :

$$A(v, w) = a\, v - b\, w \;+\; c\,, \qquad \text{"}\, N(v) = v - v^3 \,\text{"}\,.$$

- Noise to take into account random fluctuations.

# Microscopic description

**FitzHugh-Nagumo neural network of size $n$**

For $i$ between 1 and $n$ :

$$\begin{cases} dv_t^i = \left(N(v_t^i) - w_t^i + I_{\text{ext}}^i\right) dt + \sqrt{2} dB_t^i, \\ dw_t^i = A(v_t^i, w_t^i) dt. \end{cases}$$

- Neurons interact following Ohm's law

$$I_{\text{ext}}^i = -\frac{1}{n} \sum_{j=1}^{n} \Phi(x_i, x_j)(v_t^i - v_t^j) \,.$$

- The conductance $\Phi(x_i, x_j)$ between neuron $i$ and $j$ depend on their spatial location $x_i$ and $x_j$.

# Mesoscopic description : $n \to +\infty$

## FitzHugh-Nagumo's mean-field equation

$$\partial_t f + \nabla_{(v,w)} \cdot \left[ \begin{pmatrix} N(v) - w - \mathcal{K}_\Phi(f) \\ A(v,w) \end{pmatrix} f \right] - \partial_v^2 f = 0,$$

- $f(t,x,v,w)$ is the probability of finding neurons at time $t \geq 0$ and position $x \in K$, with potential $v \in \mathbb{R}$ and adaptation variable $w \in \mathbb{R}$.

- $\mathcal{K}_\Phi(f)$ is the non-local term due to interactions between neurons

$$\mathcal{K}_\Phi(f)(x,v) = \int_{K \times \mathbb{R}^2} \Phi(x,x')(v - v')f(x',v',w')dx'dv'dw'.$$

## References


- E. Luçon and W. Stannat (14').
- S. Mischler, C. Quiñinao and J. Touboul (15').
- J. Crevat (19').

# Regime of strong interactions

## Weak-Long / Strong-Short decomposition

$$\Phi(x,x') = \underbrace{\Psi(x,x')}_{\text{weak-long range interactions}} + \underbrace{\frac{1}{\epsilon}\delta_0(x-x')}_{\text{strong-short range interactions}} .$$

The mean-field equation rewrites

$$\partial_t f^\epsilon + \nabla_{(v,w)} \cdot \left[ \begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v,w) \end{pmatrix} f^\epsilon \right] - \partial_v^2 f^\epsilon = \frac{\rho_0^\epsilon}{\epsilon} \partial_v \left[ (v - \mathcal{V}^\epsilon) f^\epsilon \right],$$

where

$$\rho_0^\epsilon(x) = \int_{\mathbb{R}^2} f^\epsilon dvdw \quad \text{and} \quad \mathcal{V}^\epsilon(t,x) = \frac{1}{\rho_0^\epsilon} \int_{\mathbb{R}^2} v f^\epsilon dvdw.$$

## Main goal

Analysis of the regime of Strong/Local interactions, that is when $\epsilon \to 0$.

# Formal derivation

# Strong interactions and concentration phenomenon

$$(1) : \partial_t f^\epsilon + \nabla_{(v,w)} \cdot \left[ \begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v,w) \end{pmatrix} f^\epsilon \right] - \partial_v^2 f^\epsilon = \frac{\rho_0^\epsilon}{\epsilon} \partial_v \left[ (v - \mathcal{V}^\epsilon) f^\epsilon \right],$$

- We expect : $(v - \mathcal{V}^\epsilon) f^\epsilon \underset{\epsilon \to 0}{\sim} 0$, that is

$$f^\epsilon \underset{\epsilon \to 0}{\sim} \delta_{\mathcal{V}^\epsilon(t,x)}(v) \otimes F^\epsilon(t,x,w),$$

where $F^\epsilon = \int_{\mathbb{R}} f^\epsilon \, dv$. In the end, we obtain [1]

$$f^\epsilon(t,x,v,w) \underset{\epsilon \to 0}{\longrightarrow} \delta_{\mathcal{V}(t,x)}(v) \otimes F(t,x,w),$$

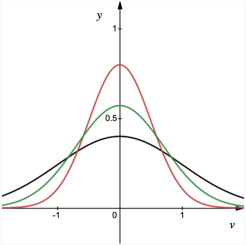
where $(\mathcal{V}, F)$ satisfies

$$\begin{cases} \partial_t \mathcal{V} = N(\mathcal{V}) - \mathcal{W} - \left( \Psi *_r \rho_0(x) \mathcal{V} - \Psi *_r (\rho_0 \mathcal{V})(x) \right), \\ \partial_t F + \partial_w \left( A(\mathcal{V}, w) F \right) = 0, \\ \rho_0(x) \mathcal{W} = \int_{\mathbb{R}} wF \, dw. \end{cases} \quad (1)$$

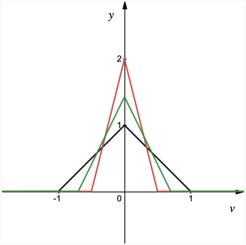
1. Crevat, Faye, Filbet (19)

# Concentration's profile

What is the profile of concentration ?




Concentration with Gaussian profile




Concentration with triangular profile

Here are plots of

$$y = \frac{1}{\sqrt{\epsilon}} g\left(\frac{v}{\sqrt{\epsilon}}\right),$$

for $\sqrt{\epsilon} = 1; 0.7; 0.5$ and $g$ a gaussian profile (fig. 1) and triangular profile (fig. 2).

# Formal derivation of the profile

- It is driven by diffusion term with respect to the voltage variable $v$

$$\partial_t f^\epsilon + \nabla_{(v,w)} \cdot \left[ \begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v,w) \end{pmatrix} f^\epsilon \right] = \partial_v \left[ \frac{\rho_0^\epsilon}{\epsilon}(v - \mathcal{V}^\epsilon) f^\epsilon + \partial_v f^\epsilon \right],$$

- $f^\epsilon$ converges to the local equilibrium :

$$f^\epsilon(t,x,v,w) \underset{\epsilon \to 0}{\sim} \mathcal{M}_{\rho_0^\epsilon/\epsilon}(v - \mathcal{V}^\epsilon) \otimes F^\epsilon(t,x,w),$$

where $\mathcal{M}_{\rho_0^\epsilon/\epsilon}(v - \mathcal{V}^\epsilon) = \sqrt{\frac{\rho_0^\epsilon}{2\pi\epsilon}} \exp\left( -\frac{\rho_0^\epsilon}{2\epsilon}(v - \mathcal{V}^\epsilon)^2 \right)$.

**Goal**

Rigorously prove that the profile is Gaussian with quantitative estimates.

# Hamilton-Jacobi approach

# Hopf-Cole transform of $f^\epsilon$

- Consider the Hopf-Cole transform [2] $\phi^\epsilon$ of $f^\epsilon$

$$f^\epsilon(t,x,v,w) = \sqrt{\frac{\rho_0}{2\pi\epsilon}} \exp\left(\frac{1}{\epsilon}\phi^\epsilon(t,x,v,w)\right).$$

We prove

**Theorem (E. Bouin and A.B. [3])**

*Suppose that in* $L^\infty_{loc}(K \times \mathbb{R}^2)$

$$\phi_0^\epsilon(x,v,w) \underset{\epsilon\to 0}{=} -\frac{\rho_0(x)}{2}\left|v - \mathcal{V}_0(x)\right|^2 + O(\epsilon),$$

*Then it holds in* $L^\infty_{loc}(\mathbb{R}^+ \times K \times \mathbb{R}^2)$

$$\phi^\epsilon(t,x,v,w) \underset{\epsilon\to 0}{=} -\frac{\rho_0(x)}{2}\left|v - \mathcal{V}(t,x)\right|^2 + O(\epsilon)^{4}.$$

2. Barles, Mirrahimi, Perthame (09)
3. In preparation
4. Mirrahimi, Roquejoffre (15)

# Strategy

Strategy : we write

$$\phi^\epsilon(t,x,v,w) = -\frac{\rho_0^\epsilon}{2}\,|\,v - \mathcal{V}^\epsilon(t\,,\,x)\,|^2 + \epsilon\,\phi_1^\epsilon(t,x,v,w)\,.$$

The correction $\phi_1^\epsilon$ solves

$$H_1^\epsilon\,[\,\phi_1^\epsilon\,] + \frac{1}{\epsilon}\,J_1^\epsilon\,[\,\phi_1^\epsilon\,] = 0\,.$$

We look for sub/super-solution for the operator $H_1^\epsilon + \frac{1}{\epsilon}\,J_1^\epsilon$ under the form $\phi_\pm = \phi_1 \pm \phi$, where $J_1^\epsilon\,[\,\phi_1\,] = 0$ then apply a comparison principle.

# Kinetic approach

# Formal derivation

- We consider a re-scaled version $g^\epsilon$ of $f^\epsilon$

$$f^\epsilon(t,x,v,w) = \frac{1}{\theta^\epsilon}\, g^\epsilon\left(t, x, \frac{v-\mathcal{V}^\epsilon}{\theta^\epsilon}, w - \mathcal{W}^\epsilon\right).$$

Suppose $\theta^\epsilon = \sqrt{\epsilon}$. Changing variables in the equation on $f^\epsilon$ it yields

**Equation on the profile**

$$\partial_t g^\epsilon + \nabla_{(v,w)} \cdot [\, \mathrm{b}_0^\epsilon\, g^\epsilon \,] = \frac{1}{\epsilon}\partial_v \left[\, \rho_0^\epsilon\, v\, g^\epsilon + \partial_v g^\epsilon \,\right],$$

where $\mathrm{b}_0^\epsilon$ depends on $1/\sqrt{\epsilon}$ and $f^\epsilon$. Therefore, we expect

$$g^\epsilon(t,x,v,w) \underset{\epsilon\to 0}{=} \mathcal{M}_{\rho_0^\epsilon}(v) \otimes G^\epsilon(t,x,w).$$

**Strategy**

proving that $g^\epsilon$ converges to $\mathcal{M}_{\rho_0} \otimes G$.

# Weak convergence result

# Weak convergence result

**Theorem (F. Filbet and A.B.[5])**

*Under suitable assumptions on $f_0^\epsilon$, there exists $C > 0$ such that*

$$\sup_{x\in K} W_2\left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0}\mathcal{M}_{\rho_0/\epsilon}(\cdot - \mathcal{V}) \otimes F\right) \leq C\left(e^{Ct}\epsilon + e^{-\rho_0^\epsilon t/\epsilon}\right),$$

*for all $\epsilon > 0$ and $t \geq 0$.*

Here, $W_2$ stands for the Wasserstein distance of order 2.

**Key arguments of the proof :**

- Uniform moment estimates.
- Analytic coupling method[6] in order to estimate the Wasserstein distance between $g^\epsilon$ and $\mathcal{M} \otimes G$ ($G$ satisfies (1) after changing variables) .

5. Concentration phenomena in Fitzhugh-Nagumo's equations : A mesoscopic approach, arXiv :2201.02363
6. Fournier, Perthame (20).

# Conclusion and perspectives

Conclusion :

- $L^\infty$ convergence estimates following a Hamilton-Jacobi approach.
- Weak convergence result with the (formal) optimal convergence rate.
- $L^1$ convergence result with deteriorated convergence rate.
- We also prove a convergence result in (inverse Gaussian) weighted $L^2$ spaces and recover the optimal rate by propagating regularity.