Concentration phenomena for a FitzHugh-Nagumo neural network

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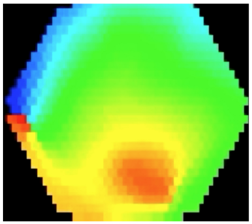
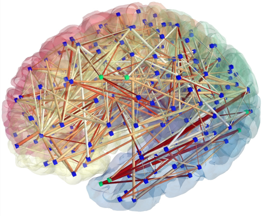
Introduction

Overview

We consider that neural networks are particle systems. It is possible to describe them through 3 distinct scales :



Macroscopic scale



Mesoscopic scale

Microscopic scale

GOAL : Quantitative analysis of the asymptotic regime

$$\text{" (mesoscopic scale)}_\epsilon \underset{\epsilon \to 0}{\longrightarrow} \text{macroscopic scale "}$$

Behavior of a *single* neuron

- We focus on the voltage through the membrane of a neuron.
- Hodgkin & Huxley (52') : precise but complicated model [gif] .
- FitzHugh-Nagumo : simplified model which captures the main features

FitzHugh-Nagumo's model

$$\begin{cases} dv_t = (N(v_t) - w_t + I_{\text{ext}})\, dt + \sqrt{2} dB_t, \\ dw_t = A\,(v_t, w_t)\, dt, \end{cases}$$

- 2 equations for periodic behavior (v_t : voltage, w_t : adaptation variable) .
- Confining assumptions to ensure spikes :

$$A(v, w) = a\,v - b\,w + c\,, \qquad \text{"}\,N(v) = v - v^3\,\text{"}\,.$$

- Noise to take into account random fluctuations.

Microscopic description

FitzHugh-Nagumo neural network of size n

For i between 1 and n :

$$\begin{cases} dv_t^i = \left(N(v_t^i) - w_t^i + I_{\text{ext}}^i\right) dt + \sqrt{2} dB_t^i, \\ dw_t^i = A(v_t^i, w_t^i) dt. \end{cases}$$

- Neurons interact following Ohm's law

$$I_{\text{ext}} = -\frac{1}{n} \sum_{j=1}^{n} \Phi(x_i, x_j)(v_t^i - v_t^j)\,.$$

- The conductance $\Phi(x_i, x_j)$ between neuron i and j depend on their spatial location x_i and x_j.

Mesoscopic description : $n \to +\infty$

FitzHugh-Nagumo's mean-field equation

$$\partial_t f \;+\; \mathrm{div}_{(v\,,\,w)}\left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Phi(f) \\ A(v,w) \end{pmatrix} f\right] - \partial_v^2 f = 0,$$

- $f(t,x,v,w)$ is the probability of finding neurons at time $t \geq 0$ and position $x \in K$, with potential $v \in \mathbb{R}$ and adaptation variable $w \in \mathbb{R}$.

- $\mathcal{K}_\Phi(f)$ is the non-local term due to interactions between neurons

$$\mathcal{K}_\Phi(f)(x,v) = \int_{K\times\mathbb{R}^2} \Phi(x,x')(v-v')f(x',v',w')dx'dv'dw'.$$

References

- E. Luçon and W. Stannat (14').
- S. Mischler, C. Quiñinao and J. Touboul (15').
- J. Crevat (19').

Regime of strong interactions

Weak-Long / Strong-Short decomposition

$$\Phi(x,x') = \underbrace{\Psi(x,x')}_{\text{weak-long range interactions}} + \underbrace{\frac{1}{\epsilon}\,\delta_0(x-x')}_{\text{strong-short range interactions}} .$$

The mean-field equation rewrites

$$\partial_t f^\epsilon + \mathrm{div}_{(v\,,\,w)}\left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v,w) \end{pmatrix} f^\epsilon\right] - \partial_v^2 f^\epsilon = \frac{\rho_0^\epsilon}{\epsilon}\,\partial_v\left[(v - \mathcal{V}^\epsilon) f^\epsilon\right],$$

where

$$\rho_0^\epsilon(x) = \int_{\mathbb{R}^2} f^\epsilon dvdw \quad \text{and} \quad \mathcal{V}^\epsilon(t,x) = \frac{1}{\rho_0^\epsilon}\int_{\mathbb{R}^2} v f^\epsilon dvdw.$$

Main goal

Analysis of the regime of Strong/Local interactions, that is when $\epsilon \to 0$.

Formal derivation

Strong interactions and concentration phenomenon

$$(1) : \partial_t f^\epsilon + \mathrm{div}_{(v,w)} \left[\begin{pmatrix} N(v) - w - \mathcal{K}_\Psi(f^\epsilon) \\ A(v,w) \end{pmatrix} f^\epsilon \right] - \partial_v^2 f^\epsilon = \frac{\rho_0^\epsilon}{\epsilon} \partial_v \left[(v - \mathcal{V}^\epsilon) f^\epsilon \right],$$

- f^ϵ converges to a local equilibrium : $\int |v - \mathcal{V}^\epsilon|^2 \times (1) \, \mathrm{d}v \, \mathrm{d}w$

$$W_2^2 \left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0^\epsilon} \delta_{\mathcal{V}^\epsilon} \otimes F^\epsilon \right) = \int_{\mathbb{R}^2} |v - \mathcal{V}^\epsilon|^2 f^\epsilon dv dw \underset{\epsilon \to 0}{=} O(\epsilon),$$

where W_2 is the Wasserstein distance of order 2. In the end, we obtain [1]

$$f^\epsilon(t,x,v,w) \underset{\epsilon \to 0}{=} \delta_{\mathcal{V}(t,x)}(v) \otimes F(t,x,w) + O(\sqrt{\epsilon}),$$

where $(\mathcal{V}, F)$ satisfies

$$\begin{cases} \partial_t \mathcal{V} = N(\mathcal{V}) - \mathcal{W} - \left(\Psi *_r \rho_0(x) \mathcal{V} - \Psi *_r (\rho_0 \mathcal{V})(x) \right), \\ \partial_t F + \partial_w \left(A(\mathcal{V}, w) F \right) = 0, \\ \rho_0(x) \mathcal{W} = \int_{\mathbb{R}} wF \, dw. \end{cases} \qquad (1)$$

1. Crevat, Faye, Filbet (19), Jabin, Rey (17), Kang, Vasseur (15)

Concentration profile

- To refine our description, we consider a re-scaled version g^ϵ of f^ϵ [2] :

$$f^\epsilon(t,x,v,w) = \frac{1}{\theta^\epsilon} g^\epsilon\left(t, x, \frac{v-\mathcal{V}^\epsilon}{\theta^\epsilon}, w - \mathcal{W}^\epsilon\right),$$

where $\theta^\epsilon > 0$ needs to be determined and

$$\mathcal{W}^\epsilon = \frac{1}{\rho_0^\epsilon} \int_{\mathbb{R}^2} w f^\epsilon dv dw\,.$$

Main goal

proving that g^ϵ converges and compute the limit.

2. Mouhot, Mischler (06), Rey (12)

Formal derivation

$$f^\epsilon(t,x,v,w) = \frac{1}{\theta^\epsilon} g^\epsilon\left(t, x, \frac{v-\mathcal{V}^\epsilon}{\theta^\epsilon}, w-\mathcal{W}^\epsilon\right).$$

Suppose $\theta^\epsilon = \epsilon^\alpha$. Changing variables in the equation on f^ϵ it yields

$$\partial_t g^\epsilon + \mathrm{div}_{(v,w)}\left[\mathrm{b}_0^\epsilon\, g^\epsilon\right] = \frac{1}{\epsilon^{2\alpha}}\partial_v\left(\rho_0^\epsilon \epsilon^{2\alpha-1} v g^\epsilon + \partial_v g^\epsilon\right),$$

Therefore $\theta^\epsilon = \sqrt{\epsilon}$ (*i.e.* $\alpha = 1/2$) is the only suitable choice

Equation on the profile

$$\partial_t g^\epsilon + \mathrm{div}_{(v,w)}\left[\mathrm{b}_0^\epsilon\, g^\epsilon\right] = \frac{1}{\epsilon}\partial_v\left[\rho_0^\epsilon\, v\, g^\epsilon + \partial_v g^\epsilon\right],$$

where b_0^ϵ depends on $1/\sqrt{\epsilon}$ and f^ϵ. Therefore, we expect

$$g^\epsilon(t,x,v,w) \underset{\epsilon\to 0}{=} \mathcal{M}_{\rho_0^\epsilon}(v) \otimes G^\epsilon(t,x,w) + O\left(\sqrt{\epsilon}\right).$$

Weak convergence result

Weak convergence result

Theorem (F. Filbet and A.B.[3])

Under suitable assumptions on f_0^ϵ, there exists $C > 0$ such that

$$\sup_{x \in K} W_2\left(\frac{1}{\rho_0^\epsilon} f^\epsilon, \frac{1}{\rho_0}\mathcal{M}_{\rho_{\mathbf{0}}/\epsilon}(\cdot - \mathcal{V}) \otimes F\right) \leq C\left(e^{Ct}\epsilon + e^{-\rho_{\mathbf{0}}^\epsilon t/\epsilon}\right),$$

for all $\epsilon > 0$ and $t \geq 0$.

Here, W_2 stands for the Wasserstein distance of order 2.

Key arguments of the proof :

- Uniform moment estimates.
- Analytic coupling method[4] in order to estimate the Wasserstein distance between g^ϵ and $\mathcal{M} \otimes G$ (G satisfies (1) after changing variables) .

3. Concentration phenomena in Fitzhugh-Nagumo's equations : A mesoscopic approach, arXiv :2201.02363
4. Fournier, Perthame (20).

Strong convergence results

Towards strong convergence : time dependent θ^ϵ

- $\theta^\epsilon = \sqrt{\epsilon}$ induces that at time $t = 0$, it holds

$$f_0^\epsilon(x, v, w) = \frac{1}{\sqrt{\epsilon}} \, g_0^\epsilon\left(x, \frac{v - \mathcal{V}_0^\epsilon}{\sqrt{\epsilon}}, w - \mathcal{W}_0^\epsilon\right) .$$

- Therefore, we impose $\theta^\epsilon(t = 0) = 1$. The only suitable choice is

$$\theta^\epsilon(t, x) = \sqrt{\epsilon}\left[1 + \underbrace{e^{-2\,\rho_0^\epsilon(x)\,t\,/\,\epsilon}\left(\epsilon^{-1} - 1\right)}_{\text{exponentially decaying remainder}}\right]^{\frac{1}{2}} .$$

The equation on g^ϵ rewrites

$$\partial_t g^\epsilon + \mathrm{div}_{(v\,,\,w)}\left[\mathrm{b}_0^\epsilon \, g^\epsilon\right] = \frac{1}{|\theta^\epsilon|^2}\partial_v\left[\rho_0^\epsilon \, v \, g^\epsilon + \partial_v g^\epsilon\right].$$

Strong convergence result

Theorem (A.B.[5])

Under suitable assumptions on f_0^ϵ, there exists $C > 0$ such that

$$\int_0^t \| f^\epsilon - f \|_{L^\infty_x L^1_{(v,w)}}(s)\, ds \leq C\, e^{C\,t} \sqrt{\epsilon}\,,$$

for all $\epsilon > 0$ and $t \geq 0$, where the limit f is given by

$$f(t,x,v,w) = \mathcal{M}_{\rho_0 |\theta^\epsilon|^{-2}}(v - \mathcal{V}) \otimes F\,,$$

where $(\mathcal{V}, F)$ solves (1).

5. Large coupling in a FitzHug-Nagumo neural network : quantitative and strong convergence results, arXiv :2203.14558v2.

Key arguments

- Relative entropy estimate yields $g^\epsilon \underset{\epsilon\to 0}{\sim} \mathcal{M}_{\rho_0^\epsilon} \otimes G^\epsilon + O(\sqrt{\epsilon})$ in L^1 .
- Proving that G^ϵ converges towards G :

(i) We work on the re-scaled version H^ϵ

$$H^\epsilon = \int_{\mathbb{R}} g^\epsilon \left(t, x, v, w - \epsilon^{3/2} v\right) \, dv \, .$$

(ii) L^1-equicontinuity estimates for g^ϵ yield $H^\epsilon \underset{\epsilon\to 0}{=} G^\epsilon + O(\sqrt{\epsilon})$ in L^1.

(iii) Then we prove $H^\epsilon \underset{\epsilon\to 0}{=} G + O(\sqrt{\epsilon})$ in L^1.

H^ϵ converges towards G : simplified example

- We consider the diffusive scaling for Fokker-Planck equation

$$\partial_t g^\epsilon + \frac{1}{\epsilon} v \cdot \nabla_x g^\epsilon = \frac{1}{\epsilon^2} \mathrm{div}_v \left[v g^\epsilon + \nabla_v g^\epsilon \right] ,$$

and we prove

$$G^\epsilon = \int_{\mathbb{R}} g^\epsilon dv \underset{\epsilon \to 0}{\longrightarrow} G , \qquad \text{where} \qquad \partial_t G = \Delta_x G .$$

- Define

$$H^\epsilon(t,x) = \int_{\mathbb{R}} g^\epsilon(t, x - \epsilon v, v) dv .$$

- H^ϵ SOLVES the limiting equation

$$\partial_t H^\epsilon = \Delta_x H^\epsilon .$$

- Therefore, it is sufficient to prove $H^\epsilon \sim G^\epsilon$ (*i.e.* equicontinuity for g^ϵ).

Hamilton-Jacobi approach

Hopf-Cole transform of f^ϵ

- We consider a re-scaled version ϕ^ϵ of f^ϵ

$$f^\epsilon(t,x,v,w) = \exp\left(\frac{1}{\epsilon}\,\phi^\epsilon(t,x,v,w)\right),$$

Goal

proving that ϕ^ϵ converges uniformly to

$$\psi_0(t,x,v,w) = -\frac{\rho_0}{2}\,|\,v - \mathcal{V}(t\,,\,x)\,|^2$$

with explicit convergence rates .

Strategy : Hilbert expansion of ϕ^ϵ

- ϕ^ϵ solves

$$H_0^\epsilon [\, \phi^\epsilon \,] + \frac{1}{\epsilon} J_0^\epsilon [\, \phi^\epsilon \,] = 0\,, \tag{2}$$

where J_0^ϵ is defined as follows

$$J_0^\epsilon [\, \phi \,] = -\partial_v \phi \, [\, \partial_v \phi + \rho_0^\epsilon \, (v - \mathcal{V}^\epsilon) \,] \,,$$

- Therefore, we write

$$\phi^\epsilon(t, x, v, w) = -\frac{\rho_0^\epsilon}{2} \, | \, v - \mathcal{V}^\epsilon(t\,,\, x) \, |^2 + \epsilon \, \phi_1^\epsilon(t, x, v, w)\,,$$

where the correction ϕ_1^ϵ solves

$$H_1^\epsilon [\, \phi_1^\epsilon \,] + \frac{1}{\epsilon} J_1^\epsilon [\, \phi_1^\epsilon \,] = 0\,. \tag{3}$$

Then we construct sub/super-solution for the operator $H_1^\epsilon + \frac{1}{\epsilon} J_1^\epsilon$.

Conclusion and perspectives

Conclusion :

- Weak convergence result with the (formal) optimal convergence rate.
- L^1 convergence result with deteriorated convergence rate.
- We also prove a convergence result in (inverse Gaussian) weighted L^2 spaces and recover the optimal rate by propagating regularity.
- L^∞ convergence estimates following a Hamilton-Jacobi approach.